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THE BROWNIAN MOTION OF STRINGS AND ELASTIC RODS

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ABSTRACT

The method introduced by Ornstein is applied to calculate the Brownian-motion mean-square deviation for strings and for elastic rods, the surrounding medium being a gas. For the string, a varying tension and elastic binding at the ends are supposed, and a formula is obtained for the mean-square deviation of any point at time t, having started with a given deviation of that point; the result contains infinite series. This result is specialized to the string with fixed endsand constant tension. For the mid-point, and for a limited time interval, the series are summed; for $t\rightarrow\infty$, the result is given for all points, agreeing with that given by Ornstein for the mid-point. Elastic rods are treated similarly, and similar results are obtained. The effect of gravity, when the rod is vertical, is introduced by a simple and consequent preturbation method, and a formula is obtained for the mean-square deviation of the lower end; this agrees closely with Houdijk's experimental results. The time dependence given by the complete formula cannot yet be tested, for Houdijk gives only long-time mean values in his publication.

INTRODUCTION

1. Most of the theoretical work on Brownian motion has been concerned with only one degree of freedom, and it is the purpose of this paper to apply the method first introduced by Ornstein[1] to cases in which there are more (in fact, an infinite number) of degrees of freedom. The systems to be treated are: I. the stretched string, and II. the elastic rod. The first has been treated by Ornstein[2] and the second by Houdijk,[3] but in each case results were obtained only for $t\rightarrow\infty$ (that is, only "equipartition" values were found); our extensions will consist principally in giving the dependence of the mean-square deviation on the initial deviation and on the time, but we also treat a more general string before specializing to a particular case, and give a better treatment of the effect of gravity in the case of the rod.

In general our treatment will follow the lines of that used by Uhlenbeck and Ornstein[4] in their second (exact) calculation for the mean-square deviation of a harmonically-bound particle in Brownian motion, where they use the method of Ornstein. In this method one starts from the equation of motion and certain assumptions about the influence of the surrounding medium on the particle, and calculates directly the mean values sought, making use of certain properties of a canonical ensemble (a large number of similar, but

[1] L. S. Ornstein, K. Akad. Amsterdam Proc. **21,** 96 (1919), (in English).

[2] L. S. Ornstein, Zeits. f. Physik **41,** 848 (1927).

[3] A. Houdijk, Archives Neerlandaises des Sciences Exactes et Naturelles, Series III A, **11,** 212 (1928).

[4] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36,** 823 (1930).

independent, particles) in evaluating constants which appear in the calculation. It will be of interest to record their result here, for it is of the same type as those which we shall obtain; with a slight change of notation, it is:

$$\overline{s^2}^{s_0} = \frac{kT}{m\omega^2} + \left(s_0^2 - \frac{kT}{m\omega^2}\right)e^{-\beta t}\left(\cos\omega_1 t + \frac{\beta}{2\omega_1}\sin\omega_1 t\right)^2 \tag{1}$$

where s is the displacement from equilibrium position, t the time, m the mass, and ω the (undamped) natural frequency in 2π sec., $\beta = f/m$, where f is the damping force per unit velocity, and $\omega_1^2 = \omega^2 - \beta^2/4$. This result, like all those obtained by this method, is not subject to the restriction (which must be imposed on results obtained by certain other methods) that t must be large compared to β^{-1}, hence the term "exact."

Now it is well known that the vibrating string may be treated by means of its normal vibrations, and that each normal vibration obeys the equation of the harmonic oscillator; also, we shall show the same to be true of the rod. Thus we may, so to say, consider in each case the system to be an assembly of damped harmonic oscillators, each having for its frequency that of the corresponding normal vibration. The deviation of any point will then be determined by the displacements of all the component "oscillators," and similarly for the squared displacement, velocity, etc. Thus one can foresee that the treatment will not be essentially different from that for the harmonic oscillator itself, the additional elements necessary being: (1) the deduction from our assumptions (about the influence of the surrounding medium on the system) of results which will take the places of analogous assumptions for the individual "oscillators," (2) the treatment of the sums which result, and (3) a certain averaging process which is more complicated than that necessary for a single oscillator, and which will arise when we wish to make the known initial conditions apply to only one point of the string or rod.

I. The Brownian Motion of a Homogeneous String

2. We consider the Brownian motion of a homogeneous string (i.e., one with constant linear density ρ) of length L, surrounded by a gas. The string is bound elastically at its ends and is under a tension $\tau(x)$ which may vary with the distance x along it. The equation of motion of the string is:

$$\rho\frac{\partial^2 s}{\partial t^2} + f\frac{\partial s}{\partial t} = \frac{\partial}{\partial x}\left(\tau\frac{\partial s}{\partial x}\right) + F(x, t)$$

where f is the friction coefficient and $F(x,t)$ is the fluctuating force. This may be written:

$$\frac{\partial^2 s}{\partial t^2} + \beta\frac{\partial s}{\partial t} = p'\frac{\partial s}{\partial x} + p\frac{\partial^2 s}{\partial x^2} + A(x, t) \tag{2}$$

where $\beta = f/\rho$, $p = \tau/\rho$, $A = F/\rho$. The conditions to be satisfied at the ends are:

$$\begin{aligned} h_0 s(0, t) - \left(\frac{\partial s}{\partial x}\right)_{x=0} &= 0 \\ - h_L s(L, t) - \left(\frac{\partial s}{\partial x}\right)_{x=L} &= 0 \end{aligned} \tag{3}$$

where h_0 and h_L are the ratios of elastic constant of binding to tension at the two ends.

The assumptions which we make for this case are the natural generalizations of Ornstein's assumptions for the simpler cases, namely that the influence of the surrounding gas may be split up into two parts; (1) a systematic frictional force, $-f(\partial s/\partial t)$ per unit length, where f is a constant depending on the nature of the gas and its pressure, and (2) a fluctuating force, $F(x,t)$ per unit length, about which we make further assumptions in terms of $A(x,t)$:

$$\overline{A(x, t)} = 0 \tag{4}$$

$$\overline{A(x_1, t_1)A(x_2, t_2)} = \phi(x_1 - x_2, t_1 - t_2) \tag{5}$$

where $\phi(x,t)$ is even in both x and t and has a sharp maximum at (0,0). The mean is taken over a subensemble, each member of which started at $t=0$ with a given shape and distribution of velocity along it.

If we consider the homogeneous equation obtained by omitting the last term from (2), the usual treatment by separation of variables gives rise to two differential equations, of which the "space" one is a slightly-specialized case of the Sturm-Liouville equation. This equation, together with the boundary conditions, defines in the usual way the orthogonal eigenfunctions X_n, which we shall assume normalized, and which satisfy the following differential equation and boundary conditions:

$$pX_n'' + p'X_n' + \lambda_n X_n = 0 \tag{6}$$

and

$$\begin{aligned} h_0 X_n(0) - X_n'(0) &= 0 \\ - h_L X_n(L) - X_n'(L) &= 0 \end{aligned} \tag{7}$$

where λ_n is the corresponding eigenvalue.

We now return to Eq. (2), and expand $s(x,t)$ and $A(x,t)$ each in a series of eigenfunctions with coefficients which depend on the time:

$$s(x, t) = \sum_{n=1}^{\infty} S_n(t) X_n(x) \tag{8}$$

$$A(x, t) = \sum_{n=1}^{\infty} A_n(t) X_n(x). \tag{9}$$

Substituting these expansions into (2), equating to zero each term of the sum which results if we transfer all terms to the left-hand member, and using (6), we obtain as the equation for $S_n(t)$:

$$S_n'' + \beta S_n' + \lambda_n S_n = A_n(t). \tag{10}$$

We can write down at once the general solution of this, which is:

$$\begin{aligned} S_n(t) &= \left[\frac{\beta S_n(0) + 2S_n'(0)}{2\omega_n} \sin \omega_n t + S_n(0) \cos \omega_n t\right] e^{-\beta t/2} \\ &\quad + \frac{1}{\omega_n}\int_0^t A_n(v) e^{-\beta(t-v)/2} \sin \omega_n(t-v)dv \end{aligned} \tag{11}$$

where

$$\omega_n^2 = \lambda_n - \tfrac{1}{4}\beta^2 \tag{12}$$

and we restrict ourselves to the case in which all the eigenvibrations are periodic by assuming

$$\lambda_n > \tfrac{1}{4}\beta^2.$$

$S_n(0)$ and $S_n'(0)$ are clearly the n-th coefficients of the expansions of the initial displacement and velocity, respectively.

3. As a preliminary step, which will furnish us with results to be needed later, we shall calculate the mean-square deviation of any point for simple initial conditions, supposing that at $t=0$ the string is at rest in its equilibrium position. This means that $S_n(0) = S_n'(0) = 0$, and denoting by $\overline{S^2}^0$ the mean-square deviation for these initial conditions, we obtain

$$\overline{s^2}^0 = \sum_1^\infty \frac{1}{\omega_n^2} \overline{I_n^2} X_n^2(x) \tag{13}$$

where

$$I_n = \int_0^t A_n(v) e^{-\beta(t-v)/2} \sin \omega_n(t-v)dv \tag{14}$$

because the double sum which results directly from (8) is reduced to a single sum by the fact (proved in Note 1, on the basis of assumption (5)) that $\overline{I_n I_m} = 0$ if $m \neq n$; this has the meaning that the different eigenvibrations are uncorrelated, in so far as they arise solely from the fluctuating force.

For $\overline{I_n^2}$ we get an integral similar to those occurring in the paper of Uhlenbeck and Ornstein;[4] using the results of Note 1, we find

$$\overline{I_n^2(t)} = \frac{\alpha_n}{2\beta}(1 - e^{-\beta t}) - \frac{\gamma_n}{4\lambda_n}\left\{\tfrac{1}{2}\beta + e^{-\beta t}(\omega_n \sin 2\omega_n t - \tfrac{1}{2}\beta \cos 2\omega_n t)\right\} \tag{15}$$

where

$$\gamma_n = \int_{-\infty}^{+\infty} \psi_n(v)dv, \quad \alpha_n = \int_{-\infty}^{+\infty} \psi_n(v) \cos \omega_n v dv,$$

$$\psi_n(v - w) = \overline{A_n(v)A_n(w)};$$

thus

$$\overline{I_n^2(\infty)} = \frac{\alpha_n}{2\beta} - \frac{\gamma_n \beta}{8\lambda_n}. \tag{16}$$

To determine the constants α_n and γ_n, we make use of the same devices as were used by Uhlenbeck and Ornstein for the harmonic oscillator; that is, we equate the mean potential energy of each eigenvibration (the mean being taken over a canonical ensemble) to $kT/2$, and the correlation between deviation and velocity to zero. In this way we obtain the result:

$$\alpha_n = \gamma_n = \frac{2\beta kT}{\rho}\cdot \tag{17}$$

The calculations are given in Note 2.

One might fear that the assigning of the potential (and kinetic) energy $kT/2$ to each eigenvibration would give rise to convergence difficulties, since it makes the total energy of the system infinite, but this is not the case. We inquire only about the mean-square deviation, and the series which result are all amply convergent. However, there is a further point which may seem strange, namely that we found α_n and γ_n to be equal and independent of n. Their equality (which follows from only our initial assumptions and the lack of correlation between deviation and velocity, the use of the equipartition energy not being involved) would indicate that the correlation function, $\phi(x,t)$, is infinitely sharp in both x and t, and this one cannot believe.[5] However this difficulty becomes serious only at high frequencies, and seems to indicate that our assumptions must not be used where one is concerned with times that are too small, say of the order of magnitude of the time between successive molecular impacts on any small length considered. However, this limitation is much less stringent than that which had to be imposed on certain earlier work, where the results were not valid for times of the order of magnitude of β^{-1} or less, and on account of the convergence properties of our solutions, it brings in no difficulties here. The independence of n is to be understood in a similar way.

By substitution of (19), (17) becomes:

$$\overline{I_n^2} = \frac{\omega_n^2 kT}{\rho\lambda_n} - \frac{kT}{2\rho\lambda_n}e^{-\beta t}\left\{\frac{4\lambda_n + \beta^2}{2\beta} + \omega_n \sin 2\omega_n t - \tfrac{1}{2}\beta\cos 2\omega_n t\right\} \tag{18}$$

and we obtain from (15):

$$\overline{s^2(x,t)}^0 = \frac{kT}{\rho}\sum_1^\infty \frac{X_n^2(x)}{\lambda_n}\left\{1 - \frac{\beta}{2\omega_n^2}e^{-\beta t}\left(\frac{4\lambda_n + \beta^2}{2\beta} + \omega_n \sin 2\omega_n t + \tfrac{1}{2}\beta\cos 2\omega_n t\right)\right\}. \tag{19}$$

Taking the limit of this expression for $t \to \infty$, we obtain for a canonical ensemble:

$$\overline{s^2} = \frac{kT}{\rho}F(x) \tag{20}$$

where:

$$F(x) = \sum_1^\infty \frac{X_n^2(x)}{\lambda_n}\cdot \tag{21}$$

[5] Quite analogous questions must be raised in the case of the harmonic oscillator.

4. We are now ready to calculate a quantity which is more easily measured than that given by (21), namely the mean-square deviation of a given point of the string, after having started at $t=0$ with a given initial deflection of that point. Let this given value be C, so that:

$$s(x,\,0) = C$$

for the particular value of x under consideration; this value will be taken to be the same throughout this discussion, so that in this section x is a constant. this means that:

$$\sum_1^\infty X_n(x)S_n(0) = C \tag{22}$$

and it is the value of this linear combination of the $S_n(0)$ which is to be given; otherwise we are to average over a canonical ensemble. That is to say, we consider a canonical ensemble of strings and, fixing on some particular numbers for x and C, pick out of it at $t=0$ a subensemble, all members of which have the deflections of the x-points equal to C. We then follow these strings in their motions, and it is the squared deflections of their x-points, averaged over this subensemble, which we wish to calculate.

To this end one may first calculate $\overline{s^2}^{s_0u_0}$, which is the mean-square deviation of a subensemble, each member of which has initially the same given shape and velocity distribution along it. This means that for each member, $S_n(0)$ and $S_n'(0)$ are the same; the difference in behavior of the different members arises only out of the fact that the fluctuating force, $F(x,t)$ varies from member to member. Next one calculates $\overline{\overline{s^2}}^{s_0}$, which is the mean-square deviation of a subensemble, each member of which has initially the same shape, and hence the same set of $S_n(0)$, no attention being paid to velocities. This is to be obtained by averaging $\overline{s^2}^{s_0u_0}$ over the $S_n'(0)$, using the facts that $\overline{S_n'(0)}=0$ (which follows from the lack of correlation between displacement and velocity in a canonical-ensemble) and that $\overline{S_n'^2(0)}$ has the value corresponding to equipartition of kinetic energy. The final result sought is to be obtained by averaging $\overline{\overline{s^2}}^{s_0}$ over the $S_n(0)$ with the restriction (22).

We find:

$$\overline{\overline{s^2}}^{s_0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_nX_m\left\{S_n(0)S_m(0)B_nB_m + \delta_{nm}\frac{kT}{\rho\lambda_n}(1 - B_n^2)\right\} \tag{23}$$

where

$$B_n(t) = e^{-\beta t/2}\left(\frac{\beta}{2\omega_n}\sin\omega_n t + \cos\omega_n t\right). \tag{24}$$

Now

$$\overline{\overline{\overline{s^2}}}^{C} = \frac{\displaystyle\int_C\cdots\int \overline{\overline{s^2}}^{s_0}\exp\,(-\,V/kT)dS_1(0)dS_2(0)\cdots}{\displaystyle\int_C\cdots\int \exp\,(-\,V/kT)dS_1(0)dS_2(0)\cdots}$$

where V is the potential energy given in Note 2, and the C written under the integral-signs (an infinite number of them are implied) means that the integrations are to be performed with the condition (22). Using (23) and interchanging summations and integrations we obtain a result which contains the quotient of two integrals. This quotient is evaluated in Note 3, and using this, we obtain as the final result:

$$\overline{\overline{s^2}}^{C} = \frac{kT}{\rho}F(x) + \left[C^2 - \frac{kT}{\rho}F(x)\right]e^{-\beta t}\left[\frac{1}{F(x)}\frac{\partial G}{\partial t} + \frac{\beta}{2F(x)}G\right]^2 \tag{25}$$

where

$$G(x,\,t) = \sum_{n=1}^{\infty} \frac{X_n{}^2 \sin \omega_n t}{\omega_n \lambda_n}\cdot \tag{26}$$

This is in complete correspondence with Eq. (1) for the harmonic oscillator, in which $(\sin \omega_1 t)/\omega_1$ plays the same part as $G(x,t)/F(x)$, and $1/\omega^2$ corresponds to $F(x)$. It is interesting to observe (as can be shown to follow at once from the form of (25)) that this mean-square deviation never crosses the equipartition value, and further, that the curves corresponding to a given value of β, but to different values of C, differ only in scale, when referred to the equipartition line. These statements are true also for the harmonic oscillator.

5. As an example we consider the string with fixed ends and constant tension. For this case, we find:

$$F(x) = \frac{2L}{a^2\pi^2}\sum_{1}^{\infty}\frac{\sin^2 n\pi x/L}{n^2} = \frac{L}{a^2}\left(\frac{x}{L} - \frac{x^2}{L^2}\right)$$

where

$$a = \left(\frac{\tau}{m}\right)^{1/2} = \text{wave velocity}.$$

This gives us:[6]

$$\overline{s^2(x)} = \frac{kTL}{\tau}\left(\frac{x}{L} - \frac{x^2}{L^2}\right). \tag{27}$$

The other result to be specialized is (25). In this there appears, in addition to $F(x)$, the function $G(x,t)$ and its time derivative. The corresponding series contain both x and t as variables, and in this general form we have not been able to sum them. However, if we restrict our attention to the midpoint of the string, we can further simplify matters, for we shall show in Note 4 that for the interval $0 \leqq t \leqq L/a$, we may write:

$$\begin{aligned} G(L/2,\,t) &= \frac{2L}{a^2\pi^2}\sum_{n\,\text{odd}}\frac{\sin\,(n^2a^2\pi^2/L^2 - \beta^2/4)^{1/2}t}{n^2(n^2a^2\pi^2/L^2 - \beta^2/4)^{1/2}} \\ &= \frac{L}{2a^2}\left\{\sinh\,(\beta t/2) - \frac{2at}{L}\,I_1(\beta t/2)\right\} \\ \frac{dG(L/2,\,t)}{dt} &= \frac{L}{4a^2}\left\{\cosh\,(\beta t/2) - \frac{2at}{L}\,I_0(\beta t/2)\right\} \end{aligned}$$

[6] For the mid-point this gives the value $kTL/4\tau$, which has been given by Ornstein, reference 2.

where

$$I_n(z) = i^{-n}J_n(iz)$$

and $J_n(x)$ is the Bessel function of order n. Using these expressions and reducing, (25) becomes:

$$\overline{\overline{S^2(L/2,t)}}^{C} = \frac{kTL}{4\tau} + \left[C^2 - \frac{kTL}{4\tau}\right]\left\{1 - \frac{2at}{L}e^{-\beta t/2}\left[I_0(\beta t/2) + I_1(\beta t/2)\right]\right\}^2 \quad (28)$$

for:

$$0 \leqq t \leqq L/a.$$

In Fig. 1 we have plotted some curves showing how $\overline{\overline{S^2}}^C$ depends on the time. For these we have taken $a=L=1$, $\beta=1,2,3,4$, and $C^2=0$ and $=3kT/4\tau$. For $0\leqq t\leqq 1$ (28) was used; for later times it was necessary to use the series themselves, although it was found possible to replace β by zero in these

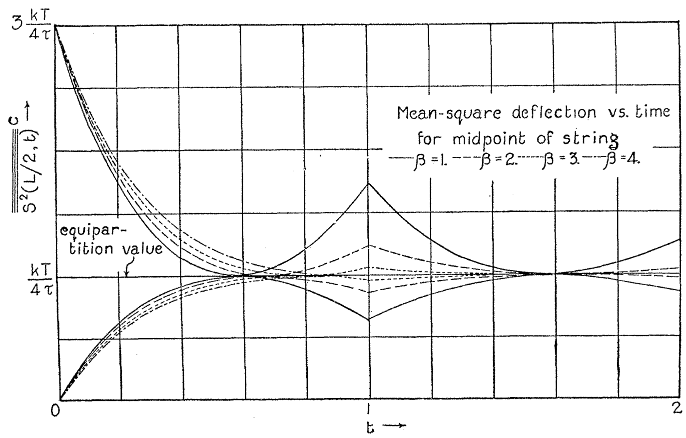

Fig. 1.

series without introducing too great an error. The apparent cusps in the curves at $t=1$ are curious; they are not in fact quite cusps if β is different from zero, but the difference is not apparent to the eye with the scale used.

II. The Brownian Motion of an Elastic Rod

6. We shall first treat, in the same manner as for the string, the problem of the Brownian motion of an elastic rod which has one end clamped and the other free, and later consider the effect of gravity. We take these end conditions because they are the ones most easily realized in experiments; the problem could as readily be carried through for any other combination of the usual conditions.

The equation of motion of an elastic rod under no forces is given by Rayleigh[7] as:

[7] Lord Rayleigh, Theory of Sound (London, Macmillan, 1894), vol. 1, p. 258.

$$\frac{\partial^2 S}{\partial t^2}+\frac{r^2E}{4d}\frac{\partial^4 s}{\partial x^4}-\frac{r^2}{4}\frac{\partial^4 s}{\partial x^2\partial t^2}=0$$

where E = Young's modulus, r = radius of cross-section (assumed circular and constant), d = volume-density; other symbols are as for the string. The third term of the first member arises out of the kinetic energy of rotation of the cross-sectional laminae; it is neglected by Rayleigh in his further treatment, and we shall do likewise. Adding the damping and accidental-force terms, we obtain:

$$\frac{\partial^2 s}{\partial t^2}+\beta\frac{\partial s}{\partial t}+\frac{r^2E}{4d}\frac{\partial^4 s}{\partial x^4}=A(x,t). \tag{29}$$

Taking the origin at the clamped end, and the end conditions (also given by Rayleigh) are:

$$\begin{aligned} s=0,\quad \frac{\partial s}{\partial x}&=0\quad\text{for } x=0\\ \frac{\partial^2 s}{\partial x^2}=0,\quad \frac{\partial^3 s}{\partial x^3}&=0\quad\text{for } x=L \end{aligned} \tag{30}$$

We make the same assumptions about $A(x, t)$ as for the string.

If we consider the homogeneous equation obtained by replacing the second member of (29) by zero and separate the variables as usual, the resulting "space" equation is:

$$\frac{d^4X}{dx^4}=\frac{4\lambda d}{r^2E}X$$

where λ is the separation parameter. The situation is the same as that obtaining for the Sturm-Liouville equation for, as Rayleigh shows, this equation, together with the boundary conditions arising out of (30), gives rise to a complete set of orthogonal eigenfunctions:

$$X_n=b_n[a_n(\cos m_nx/L-\cosh m_nx/L)+(\sin m_nx/L-\sinh m_nx/L)] \tag{31}$$

where m_n is the nth positive root of the equation:

$$\cos m\cdot\cosh m+1=0 \tag{32}$$

$$a_n=\frac{\cos m_n+\cosh m_n}{\sin m_n-\sinh m_n} \tag{33}$$

$$\lambda_n=\frac{r^2Em_n{}^4}{4dL^4} \tag{34}$$

and the coefficients b_n are chosen for normalization. Rayleigh discusses the roots of (32) and calculates the first few, giving formulas from which the rest can be found. In addition he shows that:

$$\sum_{n=1}^{\infty} \frac{1}{m_n^4} = \frac{1}{12} \tag{35}$$

and

$$X_n^2(L) = \frac{4}{L} \cdot \tag{36}$$

We shall have occasion to use these results.

Returning to Eq. (29), we proceed just as for the string and expand $s(x, t)$ and $A(x, t)$ each in a series of eigenfunctions with coefficients which depend on the time; in the same way as before, we obtain Eq. (10) for $S_n(t)$, so that Eqs. (8)—(12) of article 2 apply as well as to this case as to the string.

7. The calculation of the mean-square deviation of any point, supposing that at $t=o$ the rod is at rest in its equilibrium position, follows here exactly as for the string; we may take all the equations of that article to apply to the present case. To obtain Eq. (17), however, we must use for the potential energy:

$$V = \frac{\pi r^4 E}{8} \int_0^L \frac{ds}{R^2}$$

where R is the radius of curvature of the rod at the point. In view of the approximate straightness we may replace this by:

$$V = \frac{\pi r^4 E}{8} \int_0^L \left(\frac{\partial^2 s}{\partial x^2}\right)^2 dx.$$

Using this, together with the differential equation and boundary conditions for the eigenfunctions, we again arrive at (B) of Note 2. The corresponding expression for the potential energy of the nth eigenvibration becomes, on using (36):

$$V_n = \frac{\pi r^4 E}{8L^4} m_n^4 S_n^2. \tag{37}$$

The further calculations are identical, and we again arrive at (17) where now $\rho = \pi r^2 d$. Hence results (18), (19) and (20) are valid for this case also; substituting for λ_n by (34) in (20) we obtain for the canonical-ensemble mean-square deviation of any point x:

$$\overline{s^2} = \frac{4L^4 kT}{\pi r^4 E} \sum_{n=1}^{\infty} \frac{X_n^2(x)}{m_n^4} \tag{38}$$

and for the free end, using (36) and (35) we obtain the result already given by Houdijk[3]:

$$\overline{s^2} = \frac{4L^3 kT}{3\pi r^4 E} \cdot \tag{39}$$

The calculations of article 4 for the string, in which we found an expression (given by Eq. (25)) for the mean-square deviation of a given point x, after having started at $t=o$ with a given initial deviation C of that point, apply just as well to the rod. The general result becomes especially simple for the free end because the series converge quite strongly on account of the rapid increase of m_n. Taking only the first terms, and using *12* for m_1^4 instead of *12.4* (since this makes the value at $t=o$ strictly correct), we obtain:

$$\overline{\overline{s^2(L,\,t)}}^{\,C} \cong \frac{4L^3kT}{3\pi r^4E} + \left(C^2 - \frac{4L^3kT}{3\pi r^4E}\right)e^{-\beta t}\left(\cos\omega_1 t + \frac{\beta}{2\omega_1}\sin\omega_1 t\right)^2 \tag{40}$$

where:

$$\omega_1^2 = \lambda_1 - \tfrac{1}{4}\beta^2 = \frac{3r^2E}{dL^4} - \tfrac{1}{4}\beta^2.$$

8. Houdijk[3] has observed the Brownian motion of the lower ends of thin filaments which were clamped vertically at their upper ends and were otherwise free, the surrounding medium being air. It is necessary in this case to take into account the effect of gravity, and we shall now modify the foregoing treatment in this regard. We shall assume that a rod under gravity executes the same eigenvibrations as without it and that their amplitudes will bear approximately the same ratios to each other, the effect of gravity being to increase the potential energy associated with each. We shall restrict our attention to the free (lower) end, and we may with good approximation assume that the form of the rod is given by the first eigenfunction.

To obtain the contribution, ΔV_1, of gravitation to the potential energy, we treat the rod as a string under the tension $\rho g(L-x)$, where we again use ρ for the linear density; thus we have:

$$\Delta V_1 = \frac{\rho g}{2}\int_0^L (L-x)\left(\frac{\partial s}{\partial x}\right)^2 dx; \tag{41}$$

this is completely equivalent to multiplying the total mass by the distance through which its center of gravity is raised in bending it to a given shape. In accordance with the approximation mentioned above, we write:

$$s = S_1(t)X_1(x)$$

Substituting in (41), using the expression (31) for $X_1(x)$, and using the fact that to our present approximation, $s^2(L,\,t) = 4/L\cdot S_1^2\,(t)$ we obtain:

$$\Delta V_1 = \frac{0.98\rho g}{8}\,s^2(L,\,t)$$

To the same approximation,

$$V_1 = \frac{\pi r^4 E}{32L^3}\,m_1^4 s^2(L,\,t)$$

so that

$$V_1 + \Delta V_1 = \frac{\pi r^4 E m_1^4 + 3\cdot 96 \rho g L^3}{32L^3} s^2(L, t) = \frac{kT}{2}\,.$$

Subsituting $\pi r^2 d$ for ρ and 12 for m_1^4, we obtain:

$$\overline{s^2(L)} = kT\frac{4L^3}{3\pi r^4 E + 0.98\pi r^2 dgL^3}\,. \tag{42}$$

This formula is practically identical with the one given by Houdijk, the only difference being that he has the number *1* where we have 0.98, and this difference is much too small to be significant in the comparison with experimental results. We have given the above derivation because we believe it to be more consequent then Houdijk's. His observations agree very well with the values predicted by this formula. The closeness of agreement is indicated by the fact that he uses it to determine Avogadro's number, N, (substituting R/N for k) from his observations, finding as the average of ten determinations the value 6.36×10^{23}, which differs from the accepted value of 6.06×10^{23} by only *5* percent.

Concerning the dependence of the mean-square deviation on the initial conditions and the time, to adapt formula (40) to this case we need only replace the equipartition value given there by the second member of (42). This formula cannot yet be tested, for Houdijk gives only limiting values in his publication.

Notes

Note 1. We are to investigate the quantity $\overline{I_n I_m}$; it is clear that if we write it as a double integral, it will contain under the integral signs the quantity $\overline{A_n(v)A_m(w)}$, and by using (10) we obtain for this:

$$\overline{A_n(v)A_m(w)} = \int_0^L \int_0^L \overline{A(x, v)A(y, w)}X_n(x)X_m(y)dxdy.$$

Now let $x-y=u$ and $y=z$; using (5),

$$\overline{A_n(v)A_m(w)} = \int_0^L dz \int_{-\infty}^{+\infty} \phi(u, v-w)X_n(u+z)X_m(z)du.$$

But $\phi(u, v-w)$ differs from zero only for u very little different from zero, hence we replace $X_n(u+z)$ by $X_n(z)$ in the integrand, and by using the orthogonality property we obtain.

$$\overline{A_n(v)A_m(w)} = \delta_{nm}\psi_n(v-w)$$

where

$$\psi_n(v-w) = \int_{-\infty}^{+\infty} \phi(u, v-w)du$$

So that

$$\overline{I_n I_m} = 0 \quad \text{for} \quad n \neq m.$$

This procedure is of course not rigorous unless $\phi(u, v-w)$ is infinitely "sharp" in u, which we have not assumed.

For the string with fixed ends and constant tension, however, the eigenfunctions are sines, and in this case we may improve the argument. We have:

$$\overline{A_n(v)A_m(w)} = \frac{2}{L}\int_0^L dz \int_{-\infty}^{+\infty} \phi(u, v-w)\left[\sin\frac{n\pi u}{L}\cos\frac{n\pi z}{L} + \cos\frac{n\pi u}{L}\sin\frac{n\pi z}{L}\right]\sin\frac{m\pi z}{L}du.$$

When integrated, the first term of this gives zero because in the integration over u the integrand is an odd function, and the second term gives zero in the integration over z if $n \neq m$ because of the orthogonality of the sines. If $n=m$, however, we obtain:

$$\overline{A_n(v)A_m(w)} = \int_{-\infty}^{+\infty} \phi(u, v-w) \cos\frac{n\pi u}{L}\, du = \psi_n(v-w).$$

It is clear from this that for this case, ψ_n should be expected to depend on n, in that it will be diminished for very high values of n if ϕ is not infinitely sharp in its dependence on u, and therefore we have allowed for this also in the general case.

Note 2. We are to determine the constants α_n and γ_n. The potential energy of the system, which includes the potential energy of the binding at the ends is:

$$V = \tfrac{1}{2}\rho \left\{ \int_0^L p(x)\left(\frac{\partial s}{\partial x}\right)^2 dx + h_0 p(0)s^2(0, t) + h_L p(L)s^2(L, t) \right\}.$$

When we substitute (8) into this, interchange summations and integration, and integrate by parts (using (7) and the orthogonality property), we find:

$$V = \tfrac{1}{2}\rho \sum_{n=1}^{\infty} \lambda_n S_n^2 \tag{A}$$

or

$$V_n = \frac{\rho\lambda_n}{2} S_n^2$$

where V_n is the potential energy of the n-th eigenvibration. In a similar manner, although much more simply, we find for the kinetic energy:

$$K = \tfrac{1}{2}\rho \sum S_n'^2$$
$$K_n = \tfrac{1}{2}\rho S_n'^2.$$

For $t \rightarrow \infty$, the subensemble which started at $t=0$ from rest and equilibrium position will become a canonical ensemble; thus:

$$\lim_{t\to\infty} \overline{V_n}^0 = \frac{\rho\lambda_n}{2}\overline{S_n^2(\infty)}^0 = \frac{\rho\lambda_n}{2\omega_n^2}\overline{I_n^2(\infty)} = \frac{kT}{2}\cdot$$

Substituting (16) into this, we obtain the relation:

$$4\alpha_n\lambda_n - \gamma_n\beta^2 = \frac{8\omega_n^2\beta kT}{\rho}\cdot \tag{B}$$

If we similarly equate the kinetic energy of the n-th eigenvibration to $kT/2$ we obtain, not another relation, but the same one again, as might have been expected from the results for the harmonic oscillator. To confirm this statement, one must calculate $S_n'^2(\infty)$ in the same way that we have calculated $S_n^2(\infty)$.

To get another relation we make use of the fact that in a canonical ensemble of strings there will be no correlation between the displacement and the velocity for any point. Thus, writing $u = ds/dt$:

$$\overline{su}^0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\omega_n\omega_m}\overline{I_n J_m} X_n X_m$$

where J_m is an integral similar to I_m. Now it is clear that $\overline{I_n J_m}$ will contain under the integral signs the quantity $\overline{A_n(v)A_m(w)}$, and we have shown that this vanishes unless $m=n$. Therefore, the double sum reduces to a single sum. Our requirement that $\mathrm{Lim}_{t\to\infty}\, \overline{su}^0 = 0$ gives the result $\alpha_n = \gamma_n$ and putting this into (B),

$$\alpha_n = \gamma_n = \frac{2\beta kT}{\rho}\cdot$$

Note 3. The ratio of integrals to be calculated is.

$$L_{nm}=\frac{\int_C\cdots\int S_n(0)S_m(0)\exp\left\{-\rho\sum_1^\infty\lambda_iS_i^2(0)/2kT\right\}dS_1(0)dS_2(0)\cdots}{\int_C\cdots\int\exp\left[-\rho\sum_1^\infty\lambda_iS_i^2(0)/2kT\right]dS_1(0)dS_2(0)\cdots}$$

where the integration are to be carried out with the condition:

$$\sum_{n=1}^\infty X_nS_n(0)=C.$$

Let us first take the integrals to be only N-tuple, and the summations correspondingly to go only from 1 to N, afterward taking the limit for $N\rightarrow\infty$. Now introduce the new variables:

$$y_n=S_n(0)\left(\frac{\rho\lambda_n}{2kT}\right)^{1/2}.$$

The Jacobian of the transformation of course does not matter, being the same in the numerator and denominator; this gives us:

$$L_{nm},_N=\frac{2kT}{\rho(\lambda_n\lambda_m)^{1/2}}\frac{\int_{C_N}\cdots\int y_ny_m\exp\left(-\sum_1^N y_i^2\right)dy_1dy_2\cdots dy_N}{\int_{C_N}\cdots\int\exp\left(-\sum_1^N y_i^2\right)dy_1dy_2\cdots dy_N}$$

and the condition now becomes:

$$\sum_{n=1}^N a_ny_n=C_N$$

where

$$a_n=X_n\left(\frac{2kT}{\rho\lambda_n}\right)^{1/2}.$$

We now rewrite each integral so that the integrations over y_n and y_m are the last ones; these integrations are carried from $-\infty$ to $+\infty$, and the condition is satisfied by keeping constant

$$\sum_{j=1}^N{}''a_jy_j=C_{nm}=C_N-a_ny_n-a_my_m$$

in the integration over the remaining $N-2$ variables, where the double accent signifies that j does not take on the values n and m.

The integration over this hyperplane, whose distance from the origin is $C_{nm}D_{nm}^{-1/2}$, where $D_{nm}=\sum''a_j^2$, is carried out by rotating our coordinate system so that one of the axes coincides with the normal; the condition is now satisfied by keeping this variable constant and integrating over the others from $-\infty$ to $+\infty$. This transformation involves no calculation because the variables appear in the integrand in an invariant form, and we obtain:

$$L_{nm},_N=\frac{2kT}{\rho(\lambda_n\lambda_m)^{1/2}}\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}y_ny_m\exp\left(-y_n^2-y_m^2-C_{nm}^2/D_{nm}\right)dy_ndy_m}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left(-y_n^2-y_m^2-C_{nm}^2/D_{nm}\right)dy_ndy_m}.$$

We now take the limit for $N\rightarrow\infty$ by appropriately changing the definitions of C_{nm} and D_{nm}. The remaining integrations are carried out in a straight-forward manner by transforming the expression in the exponent onto its principle axes, etc. We obtain:

$$L_{nm}=\frac{X_nX_m}{F^2(x)\lambda_n\lambda_m}\left[C^2-\frac{kT}{\rho}F(x)\right].$$

In the above discussion we have assumed $n\neq m$; a similar calculation for L_{nn} shows that the general result may be written:

$$L_{nm} = \frac{X_n X_m}{F^2(x)\lambda_n\lambda_m}\left[C^2 - \frac{kT}{\rho}F(x)\right] + \delta_{nm}\frac{kT}{\rho\lambda_n}\cdot$$

Note 4. We have to consider:

$$\frac{\partial G(L/2,\,t)}{\partial t} = \frac{2L}{a^2\pi^2}\sum_{n_{\text{odd}}}\frac{1}{n^2}\cos\,(n^2a^2\pi^2/L^2 - \beta^2/4)^{1/2}t.$$

Let

$$H(t) = \sum_{n_{\text{odd}}}\frac{1}{n^2}\cos\,(k_n^2 - w^2)^{1/2}t$$

where

$$k_n = \frac{n a\pi}{L},\quad w = \tfrac{1}{2}\beta$$

thus,

$$\frac{\partial G}{\partial t} = \frac{2L}{a^2\pi^2}H(t)$$

$$G(L/2,\,t) = \frac{2L}{a^2\pi^2}\int H(t)dt + A.$$

We now make use of a method due to Versluys.[8] He shows by making a Taylor expansion that:

$$\begin{aligned}\cos\,(k_n^2 - w^2)^{1/2}t &= \cos k_n t - \frac{w^2t^2}{1!}\frac{d}{d(t^2)}\frac{\cos k_n t}{k_n^2} + \frac{w^4t^4}{2!}\frac{d^2}{d(t^2)^2}\frac{\cos k_n t}{k_n^4} + \cdots\\ &= \left\{1 + \frac{w^2t^2}{1!}\frac{d}{d(t^2)}\int_C^t\int_0^t dtdt + \frac{w^4t^4}{2!}\frac{d^2}{d(t^2)}\int_C^t\int_0^t\int_C^t\int_0^t (dt)^4 + \cdots\right\}\cos k_n t\\ &= V\cdot\cos k_n t\end{aligned}$$

where:

$$c = \frac{L}{2a}$$

and the operation V through which $\cos k_n t$ is transformed into $\cos\,(k_n^2 - w^2)^{1/2}t$ is independent of k_n; it may be written as follows:

$$V = \sum_{p=0}^{\infty}\frac{(w^2t^2)^p}{p!}\frac{d^p}{d(t^2)^p}\int_C^t\int_0^t\cdots\int_C^t\int_0^t (dt)^{2p}.$$

Thus we have:

$$H(t) = \sum_{n_{\text{odd}}}\frac{1}{n^2}V\cdot\cos\frac{n\pi a t}{L} = V\cdot\sum_{n_{\text{odd}}}\frac{1}{n^2}\cos\frac{n\pi a t}{L}$$

as it can be proved (according to Versluys) that here the operations V and Σ can be interchanged. Now:

$$\sum\frac{1}{n^2}\cos\frac{n\pi a t}{L} = \frac{\pi^2}{4}(\tfrac{1}{2} - at/L)\quad\text{for}\quad 0 \leqq t \leqq L/a$$

$$\therefore H(t) = \frac{\pi^2}{4}\cdot\left(\tfrac{1}{2}V\cdot 1 - \frac{a}{L}V\cdot t\right).$$

We find:

$$V\cdot 1 = \sum_{p=0}^{\infty}\frac{(w^2t^2)^p}{(2p)!} = \cosh wt = \cosh\tfrac{1}{2}\beta t$$

and

$$V\cdot t = t\sum_{p=0}^{\infty}\frac{(wt)^{2p}}{2^{2p}(p!)^2} = tI_0(wt) = tI_0(\tfrac{1}{2}\beta t).$$

[8] W. A. Versluys, Proc. Acad. Amst. **31,** 670 (1928).

Therefore

$$\frac{\partial G}{\partial t} = \frac{L}{4a^2}\left[\cosh \tfrac{1}{2}\beta t - \frac{2at}{L} I_0(\tfrac{1}{2}\beta t)\right]$$

$$G = \frac{L}{2a^2\beta}\left[\sinh \tfrac{1}{2}\beta t - \frac{2at}{L} I_1(\tfrac{1}{2}\beta t)\right];$$

here we have used the fact that

$$\int zI_0(z)dz = zI_1(z)$$

and evaluated A so as to make $G(L/2, 0) = 0$, which gives $A = 0$.

For other intervals than the one used above, the series

$$\sum \frac{1}{n^2} \cos (n\pi at/L)$$

represents a "saw tooth" function having discontinuities in its first derivative at the points nL/a, $n = 1, 3, 5 \cdots$ and the attempt to extend the calculation past the point $t = L/a$ was not successful.