THE ACOUSTIC RESONATOR INTERFEROMETER: I. THE ACOUSTIC SYSTEM AND ITS EQUIVALENT ELECTRIC NETWORK

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Abstract

The steady state of motion of a fluid between two infinite plane parallel boundaries is found for the case in which one of the boundaries is given a prescribed periodic motion normal to its surface, the other boundary being infinitely rigid or being assigned a coefficient of reflection. The excess pressure at any point in the fluid is found, being of particular interest at the boundary of the source where it has a term in phase with the velocity of the source and one in phase with its displacement. These terms pass through cyclical values as the distance between the source and reflector is increased, the first passing through sharp maxima, the second changing rapidly from negative to positive values at reflector distances of an integral number of half wavelengths in the fluid. Application is made to the case where the source is the surface of a piezoelectric plate maintained in forced vibration. The equivalent electric network of the plate and coupled fluid column is found to be the same as that for the plate alone, with modified resistance and capacity coefficients, making possible consideration of the theory of the acoustic resonator interferometer in conjunction with driving and measuring circuits.

I. Introduction

The present paper considers the theory of an electro-mechanical system composed of a vibrator such as a piezoelectric plate which is electrically driven by an independent source and to which is coupled a column of fluid set into longitudinal vibration by the plate. This study was undertaken with reference to its application in acoustic interferometry, particularly the measurement of the velocity of high frequency compressional waves in gases and liquids, and their coefficients of absorption and reflection. Mr. A. L. Loomis and the writer[1] in collaboration have applied the piezoelectric resonator to the systematic study of the velocity of high frequency compressional waves in liquids, and the methods have been extended by the writer[2] to the study of gases. In these experiments a piezoelectric plate is maintained in forced vibration by an independent vacuum tube oscillator, while one of the vibrating faces of the plate produces plane compressional waves in the medium in contact with it. If a reflecting plate be set opposite and parallel to the vibrating face of the crystal) multiple reflection takes place and as the reflecting plate is made to approach or recede from the vibrator cyclical changes of phase and amplitude of the forced vibrations occur, the reflector passing through successive positions of resonance in the fluid medium. These changes of phase

[1] J. C. Hubbard and A. L. Loomis, Nature **120**, 189 (1927); Phil. Mag. **5**, 1178–1190 (1928); A. L. Loomis and J. C. Hubbard, J.O.S.A., and R.S.I. **17**, 295–307 (1928). See also E. B. Freyer with J. C. Hubbard and D. H. Andrews, J. Am. Chem. Soc. **51**, 759–770 (1929).

[2] J. C. Hubbard, Phys. Rev. **35**, 1442 (1930); **36**, 1668–1669 (1930).

and amplitude of the vibrator give rise to effects which can be measured electrically and the object of this study is to arrive at expressions for these effects in terms of mechanical and electrical constants of the system.

The theory of the piezoelectric resonator and methods for its study were first treated in the comprehensive paper of W. G. Cady[3] and further extensive analysis has been presented by D. W. Dye.[4] Part I of the present paper consists in a development of the theory of the vibrations of a fluid column and an extension of the methods of Cady and Dye so as to include the reaction of the fluid medium upon the resonator.

G. W. Pierce[5] was the first to develop methods of high frequency acoustic interferometry. He and his collaborators have made use of piezoelectric and magnetostriction oscillators in which the vibrating element serves not only as generator and indicator of acoustic vibrations in a gas, but also maintains electric oscillations in a vacuum tube circuit. The theory of the reaction of the acoustic system upon the circuit developed in the present paper is also

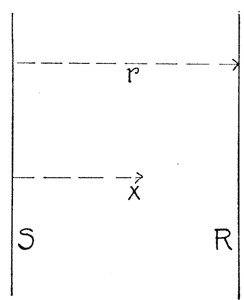

Fig. 1.

applicable in this case, but the problem is complicated by the necessity of considering the characteristics of the vacuum tube and its circuit and calls for separate treatment.

2. The Forced Vibrations of a Fluid Column

Let the space between two infinite, parallel, plane surfaces at normal variable distance r be filled with an elastic fluid of density ρ and isothermal compressibility β. Let the boundary surface S, Fig. 1, be given a normal vibratory motion of infinitely small amplitude by an external agent such that when a steady state of motion is reached throughout the system the motion of S is given by

$$\dot{\xi}_{x=0} = \dot{\xi}_0 e^{i\omega t} \tag{1}$$

and the motion of R is given by

$$\dot{\xi}_r = \dot{\xi}_{r,0} e^{i(\omega t+\delta)}. \tag{2}$$

[3] W. G. Cady, Proc. I. R. E. **10,** 83–114 (1922).
[4] D. W. Dye, Proc. Phys. Soc. Lond. **38,** 399–458 (1926).
[5] G. W. Pierce, Proc. Am. Acad. **60,** 269–302 (1925).

If R is infinitely rigid, $\dot{\xi}_{r,0}=0$ and the surface R is a perfect reflector. Such is the case to a very high degree of precision when the fluid is a gas at ordinary pressures and the reflector is of metal. In the more general case the surface R will transmit motion from the fluid and the reflection coefficient will be less than unity.

It is assumed that the particles of fluid in contact with both S and R have the same motion as those surfaces. Particle velocity in the fluid is propagated with the velocity $v=(K/\beta\rho)^{1/2}$, K being the ratio of specific heats. We shall assume that the velocity is propagated in the fluid with an attenuation α due to absorption, such that

$$d\dot{\xi}/\dot{\xi} = -\alpha dx \tag{3}$$

or

$$\dot{\xi} = \dot{\xi}_{x=0}e^{-\alpha x} = \dot{\xi}_0 e^{-\alpha x}e^{i\omega t}. \tag{4}$$

If we define the coefficient of absorption μ by $I=I_0e^{-\mu x}$, then,[6] since $I=\frac{1}{2}(\delta p_{\max})^2\rho v$, where $\delta p_{\max}$ is the maximum pressure due to particle velocity, and is $\delta p_{\max}=v\rho\dot{\xi}_0e^{-\alpha x}$, we have $\mu=2\alpha$.

At any point x between R and S when the steady state has been reached we shall have at a given time t a total disturbance made up of a term due to particle velocity propagated directly from S, that once reflected from R, that once reflected from R and once from S, that twice reflected from R and once from S, and so on, and in addition a similar series of terms due to the motion, if any, of R. It comes to the same thing if we assume that at each reflection from R the particle velocity term is multiplied by a coefficient γ, the two modes of calculation yielding equations from which the phase and amplitude of the motion of R may be evaluated in terms of the reflection coefficient, or vice versa. The value of the particle velocity at x is thus

$$\begin{aligned}\dot{\xi}_x &= \dot{\xi}_0e^{i\omega t}\{(e^{-(\alpha+i\omega/v)x}+\gamma e^{-(\alpha+i\omega/v)(2r+x)}+\cdots)\\ &\quad-(\gamma e^{-(\alpha+i\omega/v)(2r-x)}+\gamma^2e^{-(\alpha+i\omega/v)(4r-x)}+\cdots)\}\\ &= \dot{\xi}_0e^{i\omega t}\sum(\gamma^m e^{-(\alpha+i\omega/v)(2mr+x)}-\gamma^{m+1}e^{-(\alpha+i\omega/v)[2(m+1)r-x]}).\end{aligned}$$

The total forward particle velocity at x is

$$\dot{\xi}_x(+) = \dot{\xi}_0e^{i\omega t}e^{-(\alpha+i\omega/v)x}/(1-\gamma e^{-(\alpha+i\omega/v)2r}) \tag{5}$$

and the total backward particle velocity at x is

$$\dot{\xi}_x(-) = -\dot{\xi}_0e^{i\omega t}\gamma e^{-(\alpha+i\omega/v)(2r-x)}/(1-\gamma e^{-(\alpha+i\omega/v)2r}). \tag{6}$$

Boundary conditions: When $x=0$, the total particle velocity is $\dot{\xi}_{x=0}=\dot{\xi}\,e^{i\omega t}$, which satisfies Eq. (1), and when $x=r$

$$\dot{\xi}_{x=r} = \dot{\xi}_0e^{i\omega t}(1-\gamma)e^{-(\alpha+i\omega/v)r}/(1-\gamma e^{-(\alpha+i\omega/v)2r}) = \dot{\xi}_re^{i(\omega t+\delta)} \tag{7}$$

from which the coefficients $\dot{\xi}_r$ and δ may be evaluated in terms of $\dot{\xi}_0$, α, γ and r. Thus

[6] I. B. Crandall, Vibrating Systems and Sound, p. 92.

$$\dot{\xi}_0(1-\gamma)e^{-(\alpha+i\omega/v)r}/(1-\gamma e^{-(\alpha+i\omega/v)2r}) = \dot{\xi}_r e^{i\delta}$$

from which

$$\dot{\xi}_r = \dot{\xi}_0(1-\gamma)e^{-\alpha r}\left\{1 - 2\gamma e^{-2\alpha r}\cos 2\frac{\omega}{v}r + \gamma^2 e^{-4\alpha r}\right\}^{-\frac{1}{2}} \tag{8}$$

and

$$\tan\delta = \left(\tan\frac{\omega}{v}r\right)(1+\gamma e^{-2\alpha r})/(1-\gamma e^{-2\alpha r}). \tag{9}$$

If $\gamma = 1$, $\dot{\xi}_r = 0$, (gas with rigid reflector). If $r = n\lambda/2$, $\omega r/v = n\pi$ and $\tan\delta = 0$. In resonance positions the phase of the motion of the reflector is the same as that for the vibrator in the even numbered nodal positions, and differs by π in the odd numbered nodal positions.

The excess pressure δp, hereafter referred to as p, in the medium at the point x is given by[7] $\pm v\rho\dot{\xi}_x$ the plus sign being used with the forward particle velocity and the minus sign with the backward particle velocity, thus:

$$p_x = v\rho\dot{\xi}_x(+) - v\rho\dot{\xi}_x(-). \tag{10}$$

Whence

$$p_x = \rho v\dot{\xi}P_x + \rho v\omega\xi Q_x \tag{11}$$

in which, from Eqs. (5) and (6)

$$P_x = \frac{[e^{-x\alpha} - \gamma^3 e^{-(4r-x)\alpha}]\cos(\omega x/v) + \gamma[e^{-(2r-x)\alpha} - \gamma e^{-(2r+x)\alpha}]\cos(\omega/v)(2r-x)}{1 - 2\gamma^2 e^{-2r\alpha}\cos(2r\omega/v) + \gamma^4 e^{-4r\alpha}} \tag{12}$$

and

$$Q_x = \frac{[e^{-x\alpha} + \gamma^3 e^{-(4r-x)\alpha}]\sin(\omega x/v) + \gamma[e^{-(2r-x)\alpha} + \gamma e^{-(2r+x)\alpha}]\sin(\omega/v)(2r-x)}{1 - 2\gamma^2 e^{-2r\alpha}\cos(2r\omega/v) + \gamma^4 e^{-4r\alpha}}. \tag{13}$$

The expressions (12) and (13) take various forms of interest depending upon the conditions to be imposed by experiment. For example if the fluid is a gas, we have at the surface of the source S, $x = 0$, and taking $\gamma = 1$,

$$P_{x=0} = \sinh 2r\alpha/(\cosh 2r\alpha - \cos(2r\omega/v)) \tag{14}$$

$$Q_{x=0} = \sin(2r\omega/v)/(\cosh 2r\alpha - \cos(2r\omega/v)) \tag{15}$$

giving as the excess pressure, $p_{x=0}$, at the surface of S

$$p_{x=0} = v\rho\dot{\xi}P_{x=0} + v\rho\omega\xi Q_{x=0}. \tag{16}$$

The first term on the right of Eq. (16) is in phase with the velocity of the vibrator and is responsible for the transmission of energy into the fluid. This term, as r is varied passes periodically through very sharp maxima, for values of $2r\omega/v = 2\pi n$, or $r = n\lambda/2$, that is, for positions of R, an integral number of half wave-lengths from S. The function $(\sinh 2r\alpha)/(\cosh 2r\alpha - \cos 2r\omega/v)$ is shown graphically in Figure 2 for values of α and ω chosen so as illustrate its general form. It will be seen that as a result of the damping as the path length is increased the maxima, given by $1/\tanh 2r\alpha$ decrease approximately

[7] Crandall, reference 6, p. 93.

along a hyperbola and not logarithmically as seems to have been assumed by several workers with the interferometer.

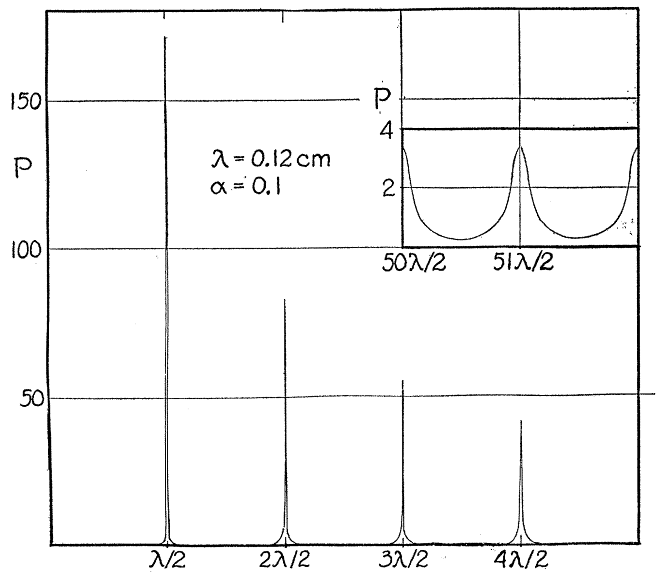

Fig. 2. $P = (\sinh 2r\alpha)/(\cosh 2r\alpha - \cos 2r\omega/v)$ as a function of r. $\alpha = 0.1$, $\lambda = 0.12$ cm, and $2r\omega/v = 4\pi r/\lambda$.

The second term on the right of Eq. (16) is in phase with the displacement and hence is proportional to the reaction of the fluid on the vibrator. The function $(\sin 2r\omega/v)/(\cosh 2r\alpha - \cos 2r\omega/v)$ is shown in Figure 3 as a function of r.

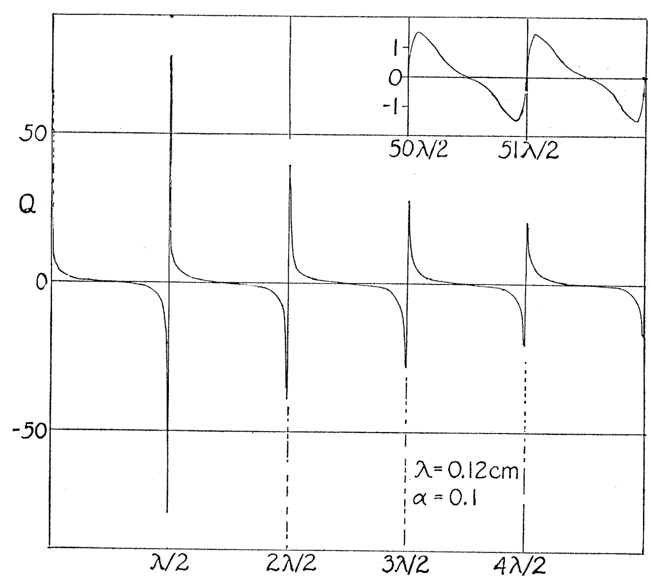

Fig. 3. $Q = (\sin 2r\omega/v)/(\cosh 2r\alpha - \cos 2r\omega/v)$ as a function of r. $\alpha = 0.1$, $\lambda = 0.12$ cm, and $2r\omega/v = 4\pi r/\lambda$.

If the reflector is itself a piezoelectric plate it may be used as a detector of excess pressure variations, and the expression for the latter at its boundary becomes of interest. In this case for a gas, $\gamma = 1$ and $x = r$, and we have

$$P_{x=r} = 2(\cos \omega r/v)(\sinh r\alpha)/(\cosh 2r\alpha - \cos 2r\omega/v) \quad (17)$$

$$Q_{x=r} = 2(\sin r\omega/v)(\cosh 2r\alpha)/(\cosh 2r\alpha - \cos 2r\omega/v). \quad (18)$$

In the case of liquids, except certain solutions and suspensions, α is so small that the decrement is difficult of observation. We may accordingly put $\alpha = 0$. In the case of liquids, however the reflector transmits a measurable part of the wave-motion out of the medium, particularly if the latter is of a density and compressibility of the same order of magnitude as for the substance of the

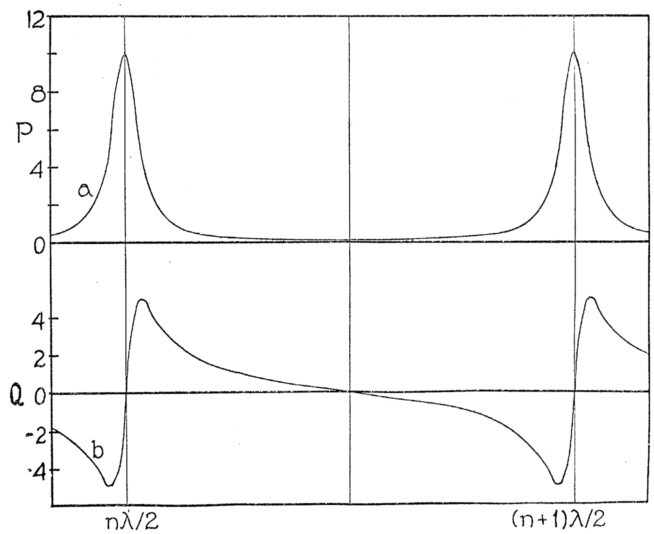

Fig. 4. (a) P and (b) Q as functions of r. $\alpha = 0$ and $\gamma = 0.9$, the fluid being a liquid.

reflector, R. Such a case is that where the liquid is mercury and the reflector is of steel, or if the liquid is water and the reflector glass. In this case

$$P_{x=0} = [1 + \gamma(1 - \gamma) \cos 2r\omega/v - \gamma^3]/(1 - 2\gamma^2 \cos 2r\omega/v + \gamma^4) \quad (19)$$

and

$$Q_{x=0} = [\gamma(1 + \gamma) \sin 2r\omega/v]/(1 - 2\gamma^2 \cos 2r\omega/v + \gamma^4). \quad (20)$$

These functions are shown graphically as functions of r in Fig. 4, at a and b respectively.

3. The Acoustic System and its Electrical Equivalent

If, in the foregoing discussion we limit the planes S and R to the finite area A, then we may ignore the effects of diffraction provided that, compared with the linear dimensions of A, the length of the compressional wave in the fluid is small, and, second, that the path r is not large. Neither of these conditions can conveniently be met in working with audible sound, but are not difficult of fulfillment in ultrasonic interferometry.

Let the vibrator S be a piezoelectric plate provided with electrodes and excited by an independent oscillator. Under these conditions the piezoelectric plate acts as a resonator, the theory of which has been worked out by W. G. Cady.[3] We will now extend this theory to include the combined action of the resonator and the fluid column coupled to it.

Let us assume that we have a plate of piezoelectric quartz with electrode faces perpendicular to an X or electric axis of the crystal. Following Cady's notation, let l, b and e be respectively the length, breadth and thickness of the plate, and let them be parallel respectively to the Y, Z and X axes of the crystal. The equation of motion along the X axis in the neighborhood of resonance is shown by Cady to be represented by

$$M\ddot{\xi} + N\dot{\xi} + G\xi = F \qquad (21)$$

where M, N and G are respectively the equivalent mass, resistance, and stiffness of the plate for the mode of motion considered, and F is the equivalent mechanical force exerted by the electric field applied between the electrodes, This field has the value V/e, where V is the instantaneous potential difference of the electrodes, and the mechanical stress in the crystal in the X direction is $X=\epsilon_{11}V/e$, where ϵ_{11} is the piezoelectric constant of quartz for the X axis. The equivalent mechanical force is shown by Cady to be $F=2beX=2\epsilon_{11}bV$. Let $V=V_0e^{i\omega t}$, then, in the steady state

$$\xi = \xi_0 e^{i(\omega t-\theta)} \qquad (22)$$

where

$$\xi_0 = 2\epsilon_{11}V_0/\omega[N^2 + (M\omega - G/\omega)^2]^{1/2} \qquad (23)$$

and

$$\theta = [\tan^{-1}(M\omega - G/\omega)]/N. \qquad (24)$$

The leads going to the electrodes of the crystal carry the current i which is made up in part of the current i_1 due to the capacity of the system as a condenser and in part of the current i_2 generated by the crystal itself as a result of the excess of its instantaneous strain due to its motion over that which would be the equilibrium value produced by V itself. The current i_1 will be in phase with V and i_2 will have a phase depending upon the phase of the velocity of extension of the crystal in the direction of its vibration and is shown by Cady to be $i_2=bl\dot{D}$, where $D=(2\xi-\delta_{11}lv/e)\epsilon_{11}/l$, 2ξ being the total elongation of the crystal, ξ for each face, and ϵ_{11} and δ_{11} being respectively the piezoelectric constant and modulus of the crystal for the axis considered. The equilibrium elongation of the quartz for the potential V is $\delta_{11}lV/e$. Near resonance the crystal reaches great amplitude of vibration and the static elongation may be neglected, giving for the piezoelectric current generated by the crystal $i_2=2b\epsilon_{11}\dot{\xi}$. Thus the total current to the electrodes is

$$i = \dot{V}K_1 + 2b\epsilon_{11}\dot{\xi} \qquad (25)$$

where K_1 is the capacity of the crystal and its electrodes considered as a condenser, and $\dot{\xi}$ is subject to Eq. (21).

An examination of Eqs. (21) and (25) shows that the crystal and its electrodes behave as a pure capacity shunted by an effective capacity, inductance and resistance in series, for, substituting $\dot{\xi} = i_2/2b\epsilon_{11}$ in equation (21) we have, since $F = 2\epsilon_{11}bV$, and putting $4\epsilon_{11}^2b^2 = 1/\mathrm{B}$,

$$Ld^2i_2/dt^2 + Rdi_2/dt + i_2/K = dV/dt \tag{26}$$

where

$$L = BM, \ \ R = BN, \ \ 1/K = BG. \tag{27}$$

K. S. Van Dyke[8] has shown that the equivalent inductance, resistance, and capacity of the series branch of the equivalent electric network of the crystal have in fact these respective values for the mode of vibration here considered. The crystal and its electrodes may therefore be replaced by the network shown in Fig. 5.

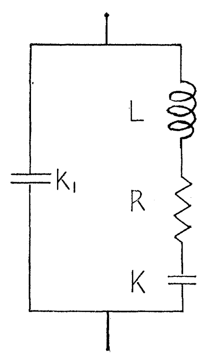

Fig. 5. Equivalent network of piezoelectric plate, and, with modified R and K, of plate coupled to fluid column.

Let us now assume that the electrodes are of negligibly small mass, stiffness, and resistance, and that they are in contact with the surface of the crystal, and that furthermore one of the electrodes serves as the boundary of the fluid medium. Consideration will be given later to the slight modifications introduced by variation of these restrictions. In addition to the force F, we now have the reaction of the excess pressure on the face of the crystal given by Eq. (16), where P and Q have the respective values given by Eqs. (14) and (15) if the fluid is a gas, and by Eqs. (19) and (20) in most cases if the fluid is a liquid. The equation of motion of the crystal is now

$$M\ddot{\xi} + N\dot{\xi} + G\xi = F - Ap \tag{28}$$

where A is the area of the vibrating surface in contact with the fluid. The excess pressure p as has been seen has a term in phase with $\dot{\xi}$ and one in phase with ξ. We may accordingly write

[8] K. S. Van Dyke, Proc. I. R. E. **16**, 742–764 (1928); See also D. W. Dye, reference 4, p. 401.

$$M\ddot{\xi} + N'\dot{\xi} + G'\xi = F \tag{29}$$

or

$$Ld^2i_2/dt^2 + R'di_2/dt + i_2/K' = dV/dt \tag{30}$$

where we now have

$$R' = R + AB\rho vP \tag{31}$$

and

$$1/K' = 1/K + AB\rho v\omega Q. \tag{32}$$

It is to be noted that the effective resistance R' and capacity K' of the acoustic system are constants involving the values of α, γ, r, and ω. The acoustic system may thus be represented by the same network as the piezoelectric plate alone (Fig. 5), the resistance R and capacity K of that network being replaced by R' and K' defined respectively by Eqs. (31) and (32). It is often convenient, especially in the interferometry of gases, to work with a transverse face of a crystal instead of with one of the electrode surfaces. In this case the piezoelectric constant and modulus are different, leading to a different value for B in equations (31) and (32) but the form of those equations is unchanged.

In Part II of this paper the acoustic system will be considered in combination with suitable driving and measuring circuits in accordance with the necessities of specific experimental objectives, such as the determination of compressional velocities in liquids and gases, and the absorption coefficients of gases.