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FERMI-DIRAC STATISTICS APPLIED TO THE PROBLEM OF SPACE CHARGE IN THERMIONIC EMISSION

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Abstract

This paper develops mathematically the state of an electron gas in equilibrium with a plane electrode when the electron gas obeys the Fermi-Dirac rather than the classical distribution law. For a part of the range of integration graphical methods were found necessary, but fortunately a change of variable leads to a solution, shown graphically, which is independent of the temperature and of the nature of the emitting electrode. Thus a single graphical integration can be applied to any emitting surface at any temperature, giving the density, electric intensity, and potential at any desired distance from the surface. A simple extension of the theory makes possible the calculation of the thermionic current between plane electrodes. Numerical examples are given, and the validity of the asumptions is discussed briefly.

When Schottky[1] first proposed that the thermionic work function could be explained by the attraction of an escaping electron to its electric image in the surface, he discussed at some length the possible effects of space charge and "structure effects," and finally concluded that other forces were small compared to the image force. Recently Waterman and the author[2] have questioned this conclusion, and have suggested that, at least under certain conditions, possibly always, space charge and structure effects are more important than the image force.

Schottky, assuming the image force to be correct, and assuming a Maxwell distribution of electron velocities, was able from measured electron emission currents to calculate the concentrations of the electron atmosphere at all points outside the metal and found it so rarefied that space charge could be neglected. This extension of the image force from a region out from the surface where it can be confirmed experimentally down towards the surface through a region where it could not be expected to be valid appears to be unjustifiable. The alternative treatment by space charge analysis, which is to be developed in this paper, is certainly not free from criticism. In particular, it is necessary to apply Poisson's equation to an electron gas, thus assuming a continuous distribution of electricity which certainly does not exist. On the other hand it would appear, as is pointed out in a previous paper by Waterman and the author, that the choice of suitable statistics may in part avoid that difficulty. In any case the space charge method of attack seems justified close to the surface where the image force certainly breaks down. In order to

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[1] Schottky, Phys. Zeits. **15**, 872 (1914).

[2] Bartlett and Waterman, Phys. Rev. [2] **37**, 279 (1931).

shed more light on this question certain calculations have been carried out, assuming that the electron concentration is controlled by space charge alone.

EQUILIBRIUM. CLASSICAL DISTRIBUTION LAW

After the manner of Fry,[3] Langmuir,[4] and others, consider a stream of electrons emerging from an infinite plane electrode with initial velocities distributed according to some law. The electrons outside the surface of the electrode will induce a charge on the electrode tending to draw the electrons back to it. There will be a potential distribution outside the metal surface through which the electrons move, those with higher initial velocities normal to the surface getting further out before they are turned back.

With no external field and no neighboring electrode, we may write an expression for the charge density at any point in potential space.

$$\rho = 2\epsilon \int_{(2V\epsilon/m)^{1/2}}^{\infty} \frac{f(v_0)}{v} dv_0 \tag{1}$$

where $f(v_0)dv_0$ gives the number of electrons emerging from the surface per square cm with initial velocities normal to the surface lying between v_0 and v_0+dv_0, while v represents the normal velocities of these electrons at the point in question. Clearly

$$v_0^2 - v^2 = \frac{2V\epsilon}{m} \cdot \tag{2}$$

Using Poisson's equation we get

$$\frac{d^2V}{dx^2} = -4\pi\rho = -8\pi\epsilon \int_{(2V\epsilon/m)^{1/2}}^{\infty} \frac{f(v_0)}{v} dv_0. \tag{3}$$

Multiplying both sides by 2 dV/dx and integrating in the usual manner gives

$$\left(\frac{dV}{dx}\right)^2 = 16\pi m \int_{(2V\epsilon/m)^{1/2}}^{\infty} v f(v_0) dv_0. \tag{4}$$

We cannot proceed further without knowing the distribution law of velocities expressed in $f(v_0)$. For the classical case

$$f(v_0) = n_0 \frac{mv_0}{kT} e^{-mv_0^2/2kT} \text{ or } f(v_0) = N_0 \left(\frac{m}{2\pi kT}\right)^{1/2} v_0 e^{-mv_0^2/2kT}, \tag{5}$$

where n_0 is the number of electrons passing through a square centimeter of the surface per second, and N_0 is the number of electrons per unit volume at the surface. Then

$$\left(\frac{dV}{dx}\right)^2 = 16\pi m N_0 \left(\frac{m}{2\pi kT}\right)^{1/2} \int_{(2V\epsilon/m)^{1/2}}^{\infty} v v_0 e^{-mv_0^2/2kT} dv_0. \tag{6}$$

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[3] Fry, Phys. Rev. [2] **17,** 441 (1921).

[4] Langmuir, Phys. Rev. [2] **21,** 419 (1923).

Changing the variable to v gives

$$\left(\frac{dV}{dx}\right)^2 = 16\pi m N_0\left(\frac{m}{2\pi kT}\right)^{1/2}\int_0^\infty v^2 e^{-mv^2/2kT - V\epsilon/kT}dv \qquad (7)$$

$$= 8\pi N_0 kT e^{-\epsilon V/kT}$$

since the integral is a well known form having the value $\frac{1}{4}(8\pi k^3T^3/m^3)^{1/2}$.

Taking the square root and performing the next step in the integration gives

$$x = \left(\frac{kT}{\pi N_0}\right)^{1/2}\frac{1}{2\epsilon}e^{\epsilon V/2kT}. \qquad (8)$$

The first boundary condition requires that V and x shall approach infinity together as dV/dx approaches zero, and this is satisfied by the above equations. The second condition specifies that at the surface the number of free electrons per cubic centimeter has a fixed value, in this case N_0. That requires that the surface of the metal shall be at x_0 given by

$$x_0 = \left(\frac{kT}{\pi N_0}\right)^{1/2}\frac{1}{2\epsilon}\cdot \qquad (9)$$

Combining Eqs. (1) and (5) and integrating at once gives Boltzmann's equation

$$N = N_0 e^{-\epsilon V/kT}. \qquad (10)$$

We could, in fact, have arrived at Eq. (8) by combining (10) with Poisson's equation. But (10) is of interest now in that, combined with (8) it gives us the concentration at various distances from the surface. The quantities V, dV/dx, x, and N are so related that as soon as we have fixed one all the others may be determined from various combinations of Eqs. (7), (8), and (10). It is also seen that a different metal, with a different electron concentration at the surface, could be handled by the same equations with an appropriate value of x_0 at the surface.

EQUILIBRIUM. FERMI-DIRAC DISTRIBUTION LAW

If in Eqs. (1) and (4) we replace the classical expression for $f(v_0)$ by that obtained from Fermi-Dirac statistics, the problem becomes much more complicated.

Now

$$f(v_0) = \frac{2\pi kTGm^2}{h^3}v_0 \log\left(A_0 e^{-mv_0^2/2kT} + 1\right) \qquad (11)$$

where A is a measure of the degeneracy of the electron gas. This gives us instead of (1) and (4) the following.

$$N = \frac{4\pi kTGm^2}{h^3}\int_{(2V\epsilon/m)^{1/2}}^{\infty}\frac{v_0}{v}\log\left(A_0 e^{-mv_0^2/2kT} + 1\right)dv_0 \qquad (12)$$

and

$$\left(\frac{dV}{dx}\right)^2 = 32\pi^2 kTG\left(\frac{m}{h}\right)^3 \int_{(2V\epsilon/m)^{1/2}}^{\infty} vv_0 \log\,(A_0 e^{-mv_0^2/2kT} + 1)dv_0. \tag{13}$$

Making the change of variable from v_0 to v gives

$$N = \frac{4\pi\, kTGm^2}{h^3} \int_0^{\infty} \log\,(Ae^{-mv^2/2kT} + 1)dv \tag{14}$$

where

$$A = A_0 e^{-\epsilon V/kT}$$

and

$$\left(\frac{dV}{dx}\right)^2 = 32\pi^2 kTG\left(\frac{m}{h}\right)^3 \int_0^{\infty} v^2 \log\,(Ae^{-mv^2/2kT} + 1)dv. \tag{15}$$

Since the evaluation of these integrals is rather complicated, it will be convenient to make certain substitutions.

Let

$$\frac{mv^2}{2kT} = u^2.$$

Let

$$f(A) = \int_0^{\infty} \log\,(Ae^{-u^2} + 1)du$$

and

$$g(A) = \int_0^{\infty} u^2 \log\,(Ae^{-u^2} + 1)du.$$

Then

$$N = \frac{2\pi G(2mkT)^{3/2}}{h^3} f(A) \tag{16}$$

and

$$\left(\frac{dV}{dx}\right)^2 = \frac{16\pi^2 Gm^{3/2}(2kT)^{5/2}}{h^3} g(A). \tag{17}$$

For the case where $A < 1$ the solution is fairly simple. Expanding the log $(Ae^{-u^2}+1)\,du$ we get

$$f(A) = \int_0^{\infty} [Ae^{-u^2} - \tfrac{1}{2}A^2e^{-2u^2} + \tfrac{1}{3}A^3e^{-3u^2} \cdots]du \tag{18}$$

and

$$g(A) = \int_0^{\infty} [Au^2e^{-u^2} - \tfrac{1}{2}A^2u^2e^{-2u^2} + \tfrac{1}{3}A^3u^2e^{-3u} \cdots]du. \tag{19}$$

The integrals involved are ordinary probability integrals. Hence

$$f(A) = \tfrac{1}{2}(\pi)^{1/2}[A - A^2/2^{3/2} + A^3/3^{3/2} \cdots] \tag{20}$$

$$g(A) = \tfrac{1}{4}(\pi)^{1/2}[A - A^2/2^{5/2} + A^3/3^{5/2} \cdots]. \tag{21}$$

Further, for the case where $A \ll 1$ we get the usual Boltzmann relation that $N = N_0 e^{-\epsilon V/kT}$. When the gas is not classical the Boltzmann relation becomes $A = A_0 e^{-\epsilon V/kT}$, as Waterman has found from a different point of view.

When $A > 1$ the solution becomes more difficult since the expansion used above is not valid over the total range of integration. It is necessary to divide the range of integration into two parts, for which Ae^{-u^2} is less than or greater than unity. Then

$$\begin{aligned} f(A) = \int_0^{(\log A)^{1/2}} \log\,[Ae^{-u^2}]du &+ \int_0^{(\log A)^{1/2}} \log\,[e^{u^2}/A + 1]du \\ &+ \int_{(\log A)^{1/2}}^{\infty} \log\,[Ae^{-u^2} + 1]du \end{aligned} \tag{22}$$

and

$$\begin{aligned} g(A) = \int_0^{(\log A)^{1/2}} u^2 \log\,[Ae^{-u^2}]du &+ \int_0^{(\log A)^{1/2}} u^2 \log\,[e^{u^2}/A + 1]du \\ &+ \int_{(\log A)^{1/2}}^{\infty} u^2 \log\,[Ae^{-u^2} + 1]du. \end{aligned} \tag{23}$$

If $A \gg 1$ (i.e., the electron gas completely degenerate) the first term is the only one of consequence.

This first term can be integrated directly to give

$$f(A) = 2/3(\log A)^{3/2} \tag{24}$$

$$g(A) = 2/15(\log A)^{5/2} \tag{25}$$

And finally by means of series expansion and integration by parts, involving steps of questionable rigor but justified by the result, one obtains

$$\begin{aligned} f(A) = \frac{2}{3}(\log A)^{3/2}\Big[1 &+ \frac{\pi^2}{8(\log A)^2} + \frac{7\pi^4}{640(\log A)^4} + \frac{31\pi^6}{6144(\log A)^6} \\ &+ \frac{4191\pi^8}{5 \times 2^{16}(\log A)^8} \cdots \Big] \end{aligned} \tag{26}$$

$$\begin{aligned} g(A) = \frac{2}{15}(\log A)^{5/2}\Big[1 &+ \frac{5\pi^2}{8(\log A)^2} - \frac{7\pi^4}{384(\log A)^4} - \frac{155\pi^6}{43008(\log A)^6} \\ &- \frac{381\pi^8}{2^{16}(\log A)^8} \cdots \Big] \end{aligned} \tag{27}$$

where these series, though not convergent, may be used for large values of A, and have been checked by graphical integration.

In the Sommerfeld-Fermi-Dirac statistics, the number of particles per unit volume and the energy per unit volume are expressed in terms of two functions $F(A)$ and $G(A)$. It turns out that $f(A)$ as used above is equal to $\pi^{1/2}/2 \cdot F(A)$, while $g(A) = \pi^{1/2}/4 \cdot G(A)$, thus confirming the results of the questionable steps mentioned above.

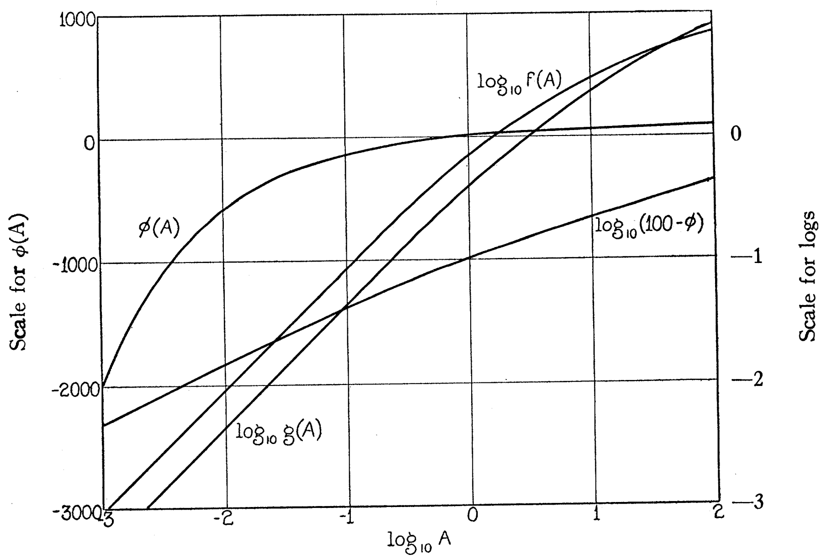

Fig. 1. $N = 5.4830 \times 10^{15}\ T^{3/2} f(A)/\text{cm}^3$ $E = 1843.9\ T^{5/4}[g(A)]^{1/2}$ volts/cm. $X = 2.1631 \times 10^{-9}$ $\phi(A)/T^{1/4}$ cm. $V = 1.9851 \times 10^{-4}\ T \log_{10} A$ volts.

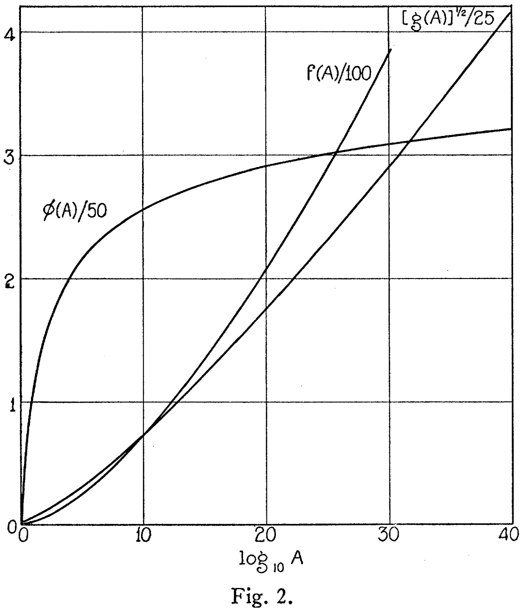

Fig. 2.

We now have expressions for $f(A)$ and $g(A)$, hence for N and $(dV/dx)^2$ over four regions, completely degenerate, less degenerate, approaching classical, and completely classical. The two extremes are handled by exact expressions, the intermediate regions by series, and there is a region between for which neither series is valid. $f(A)$ and $g(A)$ have been evaluated graphically for this intermediate region. Numerical results of this graphical integration and of the series expansions are given in Figs. 1, 2 thus covering the regions between the exact expressions for completely degenerate and completely classical states.

For the degenerate state $N=4\pi G(2mkT)^{3/2}/3h^3\cdot(\log A)^{3/2}$ and for the classical state $N=G(2\pi mkT)^{3/2}/h^3\cdot A$. For intermediate states $N=2\pi G(2mkT)^{3/2}/h^3\cdot f(A)$ where $f(A)$ for any value of A may be found from the figures.

Further for the degenerate state

$$\left(\frac{dV}{dx}\right)^2 = 32\pi^2 Gm^{3/2}(2kT)^{5/2}/15h^3\cdot\left[\log\left(A_0e^{-\epsilon V/kT}\right)\right]^{5/2}.$$

Taking the square root of both sides and integrating directly gives

$$x = \frac{15^{1/2}h^{3/2}}{2^{7/4}\pi G^{1/2}m^{3/4}(kT)^{1/4}\epsilon}\log A^{-1/4}. \tag{28}$$

For the classical state

$$\left(\frac{dV}{dx}\right)^2 = \frac{4Gm^{3/2}(2\pi kT)^{5/2}}{h^3}A_0e^{-\epsilon V/kT}.$$

This also may be integrated directly to give

$$x = \frac{h^{3/2}}{2^{5/4}G^{1/2}m^{3/4}\pi^{5/4}(kT)^{1/4}\epsilon}\cdot A^{-1/2}. \tag{29}$$

For the intermediate region graphical integration is required, since there is no exact solution. It is seen that

$$\frac{dV}{dx} = \text{const }(g(A))^{1/2} = \text{const }\left(g\left(A_0e^{-\epsilon V/kT}\right)\right)^{1/2}.$$

Since values for the expression under the radical have been tabulated as a function of A, easily convertible to a function of V, it is possible to perform a graphical integration. To simplify the arithmetical work this has been done to give a new function $\phi(A)$ proportional to x, values for which are given in Figs. 1 and 2.

$f(A)$, $g(A)$, and $\phi(A)$ are perfectly general functions, independent of temperature and of the nature of the electron emitter. The coefficients of these functions in the various equations take care of the effect of temperature. The curves in Fig. 1 show these functions plotted against A. The origin at $\log A=0$ is purely arbitrary. In effect there is no origin, or better, perhaps, each particular problem provides its own origin, fixed by the conditions of the surface. In practice it is convenient to consider distances and potential differences fromx_1, V_1 corresponding to A_1, to x_2, V_2, corresponding to A_2. The formulae below together with the tables cover the entire range of an electron gas.

$$V(\text{volts}) = 300kT/\epsilon\cdot\log A = 690.8kT/\epsilon\cdot\log_{10} A \\ = 1.9851 \times 10^{-4}T\log_{10} A \tag{30}$$

$$N = 2\pi G(2mkT)^{3/2}/h^3\cdot f(A) = 5.4830\times 10^{15}T^{3/2}f(A) \tag{31}$$

$$A \gg 1\quad N = 4\pi G(2mkT)^{3/2}/3h^3\cdot(\log A)^{3/2} = 1.2772\times 10^{16}T^{3/2}(\log_{10} A)^{3/2} \tag{32}$$

$$A \ll 1\quad N = G(2\pi mkT)^{3/2}/h^3\cdot A = 4.8591\times 10^{15}T^{3/2}A \tag{33}$$

$$E = 2^{13/4}\pi G^{1/2}m^{3/4}(kT)^{5/4}/h^{3/2}\cdot[g(A)]^{1/2}\text{E.S.U/cm} \\ = 1843.9T^{5/4}[g(A)]^{1/2}\text{ volts/cm} \tag{34}$$

$$A \gg 1\quad E = 2^{15/4}\pi G^{1/2}m^{3/4}(kT)^{5/4}/15^{1/2}h^{3/2}\cdot(\log A)^{5/4} \\ = 1909.8T^{5/4}(\log_{10} A)^{5/4}\text{ volts/cm} \tag{35}$$

$$A \ll 1\quad E = 2^{9/4}\pi^{5/4}G^{1/2}m^{3/4}(kT)^{5/4}/h^{3/2}\cdot A^{1/2} = 1227.4T^{5/4}A^{1/2}\text{ volts/cm} \tag{36}$$

$$x = h^{3/2}\times 0.046052/2^{13/4}\pi G^{1/2}m^{3/4}(kT)^{1/4}\epsilon\cdot\phi(A)\text{ cm} \\ = 2.1631\times 10^{-9}\phi(A)/T^{1/4}\text{ cm} \tag{37}$$

$$A \gg 1\quad x = 15^{1/2}h^{3/2}/2^{7/4}\pi G^{1/2}m^{3/4}(kT)^{1/4}\epsilon\cdot(\log A)^{-1/4} \\ = 4.1770\times 10^{-7}/T^{1/4}\cdot(\log_{10} A)^{-1/4}\text{ cm} \tag{38}$$

$$A \ll 1\quad x = h^{3/2}/2^{5/4}G^{1/2}m^{3/4}\pi^{5/4}(kT)^{1/4}\epsilon\cdot A^{-1/2} \\ = 1.4112\times 10^{-7}/T^{1/4}\cdot A^{-1/2}\text{ cm.} \tag{39}$$

A small additive correction is necessary to join different regions. This may easily be calculated, but has been omitted here to avoid further complications.

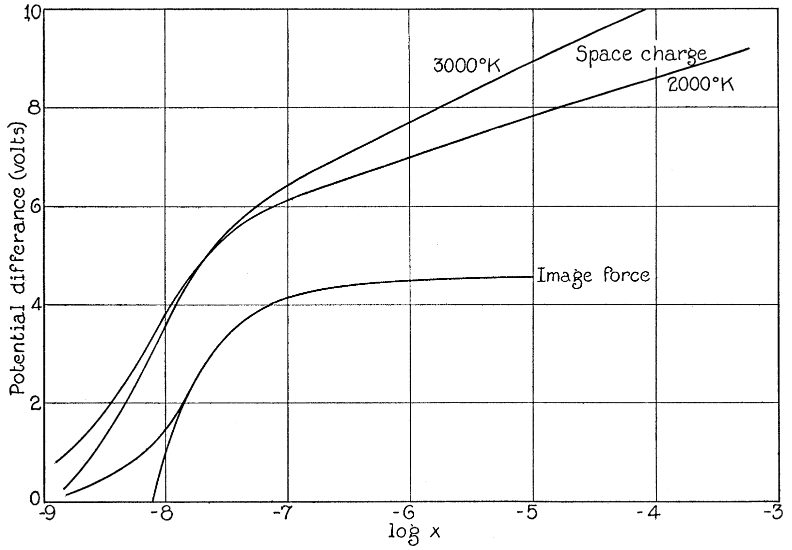

Fig. 3. Height of potential barrier at different distances from the surfaces.

If then N is known for any point, A, V, x, and E may be found from these formulae used in conjunction with the figures. And as soon as one of these is fixed for another point, all the others may be found. The separation of the points in space and in potential is found by subtracting the V's and x's for the two points. Figs. 3, 4 give some characteristic results for various conditions.

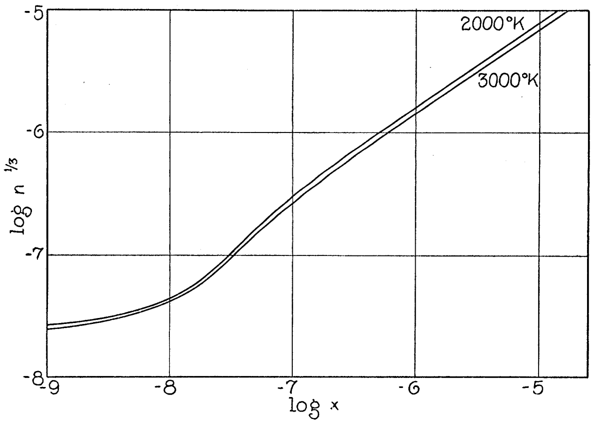

Fig. 4. Distances between electrons at different distances from the surface.

CURRENT TO THE ANODE

The analysis just given applies to the case of no current, effectively an equilibrium case, though it was considered from the point of view of emerging and returning streams. Fortunately it is possible to take advantage of earlier work by Fry and Langmuir in going over to the case where a current is flowing from the hot cathode. In Langmuir's paper two variables are defined

$$\xi = 4(\pi/2kT)^{3/4}m^{1/4}(\epsilon i)^{1/2}(x - x_m) \tag{40}$$

$$\eta = \epsilon(V - V_m)/kT. \tag{41}$$

It is easily seen that $\eta = \log A - \log A_m$ in our notation. Now the current passing a potential barrier V is given by

$$i = 2\pi\epsilon kTGm^2/h^3 \cdot \int_{(2V\epsilon/m)^{1/2}}^{\infty} u \log (A_0 e^{-mu\ /2kT} + 1) du \tag{42}$$

$$i = 2\pi Gm\epsilon(kT)^2/h^3 \cdot A_0 e^{-\epsilon V/kT} = 2\pi Gm\epsilon(kT)^2/h^3 \cdot A \tag{43}$$

provided $A_0 e^{-\epsilon V/kT} < 1$.

If this value of i is put into the expression for ξ above we get

$$\xi = 2^{9/4}\pi^{5/4}(kT)^{1/4}m^{3/4}\epsilon A_0^{1/2}e^{-\epsilon V/kT}(x - x_m)/h^{3/2} \tag{44}$$

from which it appears that $\xi = 0.046052\pi^{1/4}/2^{3/4} \cdot \phi(A) \cdot A^{1/2}$ with similar relations beyond the range of $\phi(A)$. Thus it is possible to apply the Langmuir Fry analysis to this case if a little care is exercised, using the relation $\xi = \xi_1 A_0^{1/2} e^{-\epsilon V/2kT} = \xi_1 A^{1/2}$.

The Langmuir-Fry analysis assumes electrons streaming out from the electrode with velocities according to Maxwell's distribution law. Of these the highest velocity electrons escape completely, so that the returning stream consists of only a part of the Maxwellian distribution. Now in thermionic emission the currents are generally so small that the loss of electrons to the returning stream is of no consequence close to the electrode where the distribution is no longer classical. In the region where the loss of electrons is of consequence, the distribution is classical or so nearly so that the Langmuir-Fry analysis can be joined to mine as indicated without serious error.

ILLUSTRATION OF THE METHOD OF CALCULATION

Let us take the case of an emitter at 3000°K.

$N_0=6.2\times10^{22}$ (tungsten)	Parallel plane electrodes 1 cm apart.
From the figures $\log A_s=9.534$	$A_s=A$ at surface
Arbitrarily $\log A_m=-3.0000$	$A_m=A$ at potential minimum

A simple calculation gives i corresponding to A_m as 3.612×10^{-4} amps/cm^2

From the figures $\phi(A_s)=126.59$ $\quad\phi(A_m)=2002.06$

Using equation (45) $\xi_{1m}=43.399$ $\quad\xi_{1s}=-2.744$ $\quad\xi_1=46.143$

Multiplying by $A^{1/2}$ gives $\xi=1.4592$

To this we add 1.1397 to fit classical to Fermi,

giving ξ_1' (from surface to potential minimum) $=2.5989$

From 40 or 44 $\xi'=43.041$

so that ξ_2' (from potential minimum to collecting electrode) $=40.442$

From tables of Langmuir, extended to cover some cases dealt with here, $\eta_2=87.49$ (from potential minimum to collecting electrode.)

$\eta(\text{total})=74.96$ $\quad V=19.387$ from equation 41

Repetition of this process for a series of values of A_m leads to the results given in Fig. 5.

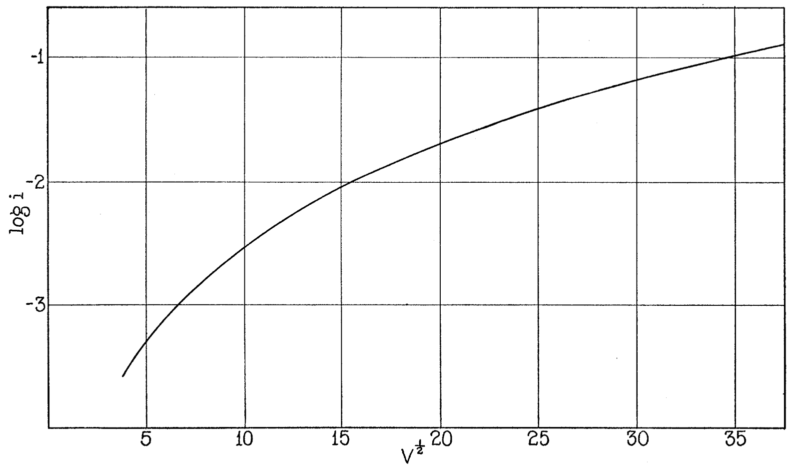

Fig. 5. Current from emitter at 3000° to parallel electrode 1 cm distant for different accelerating potentials.

DISCUSSION OF RESULTS

Fig. 3, showing the height of the potential barrier at various distances from the surface, for tungsten, $N_0 = 6.2 \times 10^{22}/\text{cm}^3$ is interesting in that it shows a value for the work function much larger than that conventionally accepted, this larger value being necessary to nullify the effect of the zero-point energy of a highly concentrated electron gas.

Fig. 4 shows in effect the average distance between electrons as a function of distance from the surface. The distance between electrons is comparable with the distance from the surface at a point so close that the image force is meaningless, and slightly further out it is seen that an electron is further from the surface than its nearest neighbor, showing that space charge is certainly important.

When considering the calculations shown in Fig. 5, it should be remembered that the calculations were carried out for plane parallel electrodes, whereas experiments have almost invariably used coaxial cylinders. The rapid change of space charge outward radially from a fine wire would naturally lead to different results. In order to test this question, an attempt is now being made to reproduce experimentally the conditions here treated theoretically.

There are two more questions of importance that must be recognized here. In the first place, the calculated electric field at the surface is so large that a great reduction of electron concentration within the surface should result from the very large surface charge. Just what effect this might have on the case of a current cannot be predicted. The calculations for the equilibrium case are still valid, if one chooses a proper electron concentration at the surface. A second question, of course, is that of the validity of Poisson's equation. It seems to the author that probably this calculation is reasonably close to the truth in regions where the concentrations of electrons is large. But at considerable distances from the surface, where the concentration is low, the approximation to the truth cannot be as good. It is for this reason that for the calculation of emission currents the electrodes were chosen so close together. Also for this reason results are not given for currents at lower temperatures. This particular problem will be investigated further.

In conclusion the author wishes to thank Professor A. T. Waterman for many helpful discussions of the problem.