GROUP THEORY AND THE ELECTRIC CIRCUIT*

By Nathan Howitt
Naval Research Laboratory, Bellevue, Anacostia, D. C.

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Abstract

Electrical networks consisting of inductances, resistances, and capacitances form a group with the impedance function as an absolute invariant. That is, to a given impedance function there corresponds an infinite number of networks, any one of which can be obtained from any other by a special linear transformation of the instantaneous mesh currents and charges of the network. In this manner one may arrive at the complete infinite set of networks equivalent to a given network of any number of meshes. This is done by writing down the three fundamental quadratic forms of the network. Then a linear affine transformation of the instantaneous mesh currents and charges of the network results in the formation of new quadratic forms, the matrices of the coefficients of which represent a member of the group, i.e., an equivalent network. Instead of performing the substitutions, the three matrix multiplications $C'AC$ are used, one for each quadratic form, where A represents the original matrix, C the transformation matrix, and C' its conjugate. It may be possible to extend this theory to include continuous systems where the quadratic forms become integrals or infinite series and one deals with infinite matrices and infinite transformations.

In 1904, in an address before the Mathematics section of the International Congress of Arts and Science, Professor James Pierpont said, "The group concept, hardly noticeable at the beginning of the century, has at its close become one of the fundamental and most fruitful notions in the whole range of our science."[1] And now this abstract notion of groups finds application in an important branch of physics—electric circuit theory.

Considerable has been written on electrical networks and the impedance function,[2] but it has hardly been suspected that electrical networks formed a group with the impedance function as an absolute invariant and that it was possible to proceed in a continuous manner from one network to its equivalent network by a linear transformation of the instantaneous mesh currents and charges of the network.

Before proceeding with the general n-mesh network it will be instructive to construct the quadratic forms and the impedance function for the two-mesh network with all three network elements present, shown in Fig. 1.

The elements λ_{12}, ρ_{12} and σ_{12} are the elements common or mutual to meshes 1 and 2. λ_{11}, ρ_{11} and σ_{11} are the *total* parameters of mesh 1, that is, they are, respectively, the total inductance, resistance and elastance of mesh

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[1] J. Pierpont, Bulletin of the American Mathematical Society **2**, 144 (1904).

[2] O. Heaviside, Electromagnetic Theory, 1912, and Electrical Papers, 1925; J. R. Carson Electric Circuit Theory and the Operational Calculus, 1926; V. Bush, Operational Circuit Analysis, 1929.

1. Similarly, λ_{22}, ρ_{22} and σ_{22} are the total parameters of mesh 2. The quantities i_1 and i_2 are the *instantaneous mesh* currents, the arrows indicating their directions. Let q_1 and q_2 be the corresponding mesh charges, so that

$$i_1 = dq_1/dt \tag{1}$$

$$i_2 = dq_2/dt. \tag{2}$$

The total *instantaneous* magnetic energy in the complete network is given by

$$\begin{aligned} T &= \tfrac{1}{2}(\lambda_{11} - \lambda_{12})i_1^2 + \tfrac{1}{2}\lambda_{12}(i_1 + i_2)^2 + \tfrac{1}{2}(\lambda_{22} - \lambda_{12})i_2^2 \\ &= \tfrac{1}{2}(\lambda_{11}i_1^2 + 2\lambda_{12}i_1i_2 + \lambda_{22}i_2^2). \end{aligned} \tag{3}$$

Similarly, the total *instantaneous* electrostatic energy in the complete network is given by

$$\begin{aligned} V &= \tfrac{1}{2}(\sigma_{11} - \sigma_{12})q_1^2 + \tfrac{1}{2}\sigma_{12}(q_1 + q_2)^2 + \tfrac{1}{2}(\sigma_{22} - \sigma_{12})q_2^2 \\ &= \tfrac{1}{2}(\sigma_{11}q_1^2 + 2\sigma_{12}q_1q_2 + \sigma_{22}q_2^2). \end{aligned} \tag{4}$$

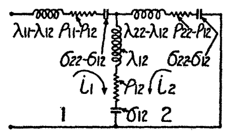


Fig. 1. General two-mesh network.

Finally, the total *instantaneous* power lost in the resistances of the complete network is given by

$$\begin{aligned} R &= (\rho_{11} - \rho_{12})i_1^2 + \rho_{12}(i_1 + i_2)^2 + (\rho_{22} - \rho_{12})i_2^2 \\ &= \rho_{11}i_1^2 + 2\rho_{12}i_1i_2 + \rho_{22}i_2^2. \end{aligned} \tag{5}$$

In more compact notation, T, V and R may, respectively, be written

$$T = \tfrac{1}{2}\Sigma_{jk=1}^{2}\lambda_{jk}i_ji_k \tag{6}$$

$$V = \tfrac{1}{2}\Sigma_{jk=1}^{2}\sigma_{jk}q_jq_k \tag{7}$$

$$R = \Sigma_{jk=1}^{2}\rho_{jk}i_ji_k \tag{8}$$

Since $\lambda_{jk} = \lambda_{kj}$, $\sigma_{jk} = \sigma_{kj}$, $\rho_{jk} = \rho_{kj}$ it is readily seen that by giving j and k all possible values from 1 to 2 in any manner, Eqs. (6), (7) and (8) reduce to Eqs. (3), (4), and (5).

It might be well at this point to generalize Eqs. (6), (7) and (8) for n meshes. This is done simply by changing the upper limit of the summation from 2 to n. For n meshes, then, these equations become

$$T = \tfrac{1}{2}\Sigma_1^n\lambda_{jk}i_ji_k \tag{9}$$

$$V = \tfrac{1}{2}\Sigma_1^n\sigma_{jk}q_jq_k \tag{10}$$

$$F = \tfrac{1}{2}\Sigma_1^n\rho_{jk}i_ji_k \tag{11}$$

where j and k take on all possible values from 1 to n, in any manner.

The quantities T, V and F are the so-called quadratic forms[3] which are positive and definite. That is, they are positive for all values of the variable i or q, and they are zero when and only when all the variables are zero or when the corresponding parameters are all zero. The positiveness of these forms follows at once from physical considerations since the magnetic energy, the electrostatic energy and the power lost in the resistances of the network are positive quantities, and are zero when and only when all the currents or charges are respectively zero, or when the corresponding parameters are zero. These quadratic forms play an important role in dynamics, and significant results are obtained from their positive and definite character.

It is instructive to point out here that the coefficients of the quadratic forms (6), (7) and (8) may be obtained directly from certain matrices. Thus, the coefficients of the quadratic form (6) are contained in the matrix

$$\begin{Vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{Vmatrix} \tag{12}$$

and the form is obtained at once by writing

$$\lambda_{11}i_1^2 + 2\lambda_{12}i_1i_2 + \lambda_{22}i_2^2$$

which is, of course, $2T$. Similarly, the coefficients of the forms (7) and (8), respectively, are contained in the matrices

$$\begin{Vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{Vmatrix} \quad \begin{Vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{Vmatrix} \tag{13}$$

Also, in the n-mesh case, the coefficients of the quadratic forms T, V and F are contained, respectively, in the matrices

$$\begin{Vmatrix} \lambda_{11} & \lambda_{12} \cdots & \lambda_{1n} \\ \lambda_{12} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \lambda_{1n} & \cdots & \lambda_{nn} \end{Vmatrix}, \quad \begin{Vmatrix} \sigma_{11} & \sigma_{12} \cdots & \sigma_{1n} \\ \sigma_{12} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \sigma_{1n} & \cdots & \sigma_{nn} \end{Vmatrix}, \quad \begin{Vmatrix} \rho_{11} & \rho_{12} \cdots & \rho_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \rho_{1n} & \cdots & \rho_{nn} \end{Vmatrix} \tag{14}$$

From these matrices, the respective quadratic forms of the n-mesh network as well as the respective networks are readily constructed.

The impedance function is obtained from the determinant of the network.[4] Thus the determinant of the network of Fig. 1 is

$$D_1 = \begin{Vmatrix} \lambda_{11}p + \rho_{11} + \sigma_{11}/p & \lambda_{12}p + \rho_{12} + \sigma_{12}/p \\ \lambda_{12}p + \rho_{12} + \sigma_{12}/p & \lambda_{22}p + \rho_{22} + \sigma_{22}/p \end{Vmatrix} \tag{15}$$

and the impedance function is obtained by dividing this determinant by the minor of the element in the first row and first column. Thus

$$Z(p) = \frac{D_1}{\lambda_{22}p + \rho_{22} + \sigma_{22}/p} \tag{16}$$

[3] See M. Bôcher, Introduction to Higher Algebra, 1927, p. 150.

[4] V. Bush, reference 2, Chapter III.

It will be helpful first to consider the simple two-mesh network containing only two kinds of network elements shown in a, Fig. 2.

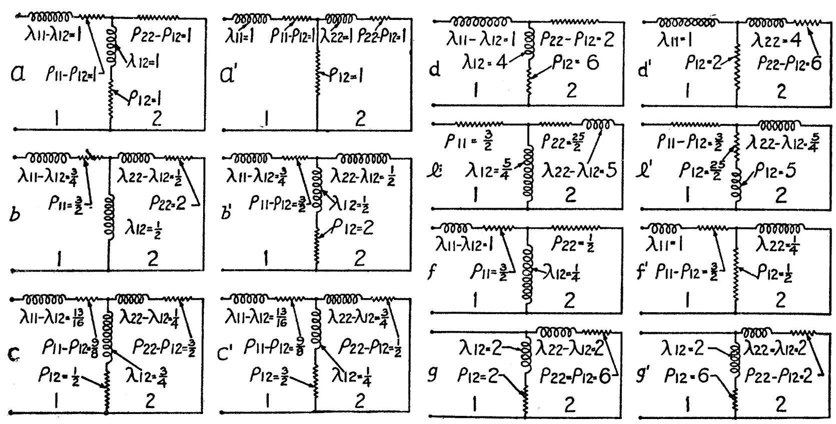
The parameters of the network are $\lambda_{11}=2$, $\lambda_{22}=1$, $\lambda_{12}=1$; $\rho_{11}=2$, $\rho_{22}=2$, $\rho_{12}=1$, and thus the quadratic forms are

$$T = \tfrac{1}{2}(2i_1^2 + 2i_1i_2 + i_2^2) \tag{17a}$$

$$F = \tfrac{1}{2}(2i_1^2 + 2i_1i_2 + 2i_2^2) \tag{17b}$$

and the matrices of the coefficients of these forms are

$$\begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix}. \tag{18}$$



Some members of the group of networks having $Z(p) = \dfrac{p^2+4p+3}{p+2}$

Fig. 2.

The minimal networks of the impedance function $Z(p) = \dfrac{p^2+4p+3}{p+2}$

Now perform the following linear transformations of the instantaneous mesh currents in the network

$$i_1 = i_1' \tag{19a}$$

$$i_2 = a_{21}i_1' + a_{22}i_2' \tag{19b}$$

where the a's are any real numbers, positive or negative. Substituting these values for i_1 and i_2 in the quadratic forms (17a) and (17b), we have

$$T = \tfrac{1}{2}[(2 + 2a_{21} + a_{21}^2)i_1'^2 + (2a_{22} + 2a_{21}a_{22})i_1'i_2' + (a_{22}^2)i_2'^2] \tag{20a}$$

$$F = \tfrac{1}{2}[(2 + 2a_{21} + 2a_{21}^2)i_1'^2 + (2a_{22} + 4a_{21}a_{22})i_1'i_2' + (2a_{22}^2)i_2'^2] \tag{20b}$$

Thus, the transformations (19) give the new quadratic forms (20). The two matrices containing the coefficients of these new forms are then

$$\begin{Vmatrix} 2 + 2a_{21} + a_{21}^2 & a_{22} + a_{21}a_{22} \\ a_{22} + a_{21}a_{22} & a_{22}^2 \end{Vmatrix}, \quad \begin{Vmatrix} 2 + 2a_{21} + 2a_{21}^2 & a_{22} + 2a_{21}a_{22} \\ a_{22} + 2a_{21}a_{22} & 2a_{22}^2 \end{Vmatrix} \tag{21}$$

These two matrices determine an infinite group of networks equivalent to the network shown in *a*, Fig. 2. The different networks are obtained by assigning different real values to a_{21} and a_{22}. Thus, for example, by giving the values of $+1$ and -1 respectively to a_{21} and a_{22}, the matrices (21) become under these substitutions

$$\left\| \begin{matrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{matrix} \right\|, \quad \left\| \begin{matrix} \frac{3}{2} & 0 \\ 0 & 2 \end{matrix} \right\|. \tag{22}$$

From these matrices the parameters of the network are readily obtained. They are

$$\lambda_{11} = \frac{5}{4}, \quad \lambda_{22} = 1, \quad \lambda_{12} = \frac{1}{2}; \quad \rho_{11} = \frac{3}{2}, \quad \rho_{22} = 2, \quad \rho_{12} = 0$$

and the corresponding network is shown in *b*, Fig. 2. It is a simple matter to verify the fact that the networks *a* and *b* of Fig. 2 have the same impedance function, namely,

$$Z(p) = \frac{p^2 + 4p + 3}{p + 2}. \tag{23}$$

In the same way, by assigning different real values to a_{21} and a_{22}, one can obtain the complete infinite group of networks having (23) for an impedance function. Thus, for example, all the networks shown in Fig. 2 have the same impedance function, namely, (23). These networks are some of the members of the infinite group of networks contained in the tensors (21). It is not difficult to ascertain what values of a_{21} and a_{22} in the transformation matrix will give these networks.

Note that the networks a'-g' are, respectively, identical with the networks *a*-*g*, except that the branches in mesh 2 are interchanged. The former networks may thus be considered images of the latter. Mathematically, two networks with their branches in mesh 2 interchanged, are considered different networks, and to exhaust the complete infinite group of networks, both networks and their images must be included.

Note also that the networks *d*-*g* and their respective images d'-g' are minimal networks. That is, they are the networks of the group containing the least number of network elements. These can be easily obtained from the tensors (20). Finally, note that it is unnecessary to go through the work of substituting (19) in (17) to obtain the matrices (21) of the quadratic forms (20). We merely make use of an important theorem on matrices, namely, that if we subject the x's in a quadratic form with matrix A to a linear transformation with matrix C, we obtain a new quadratic form with the matrix $C'AC$, where C' is the conjugate of C.[5] In open form, the matrix of the new quadratic form is obtained by multiplying the three matrices

[5] M. Bôcher, reference 3, Theorem 1, p. 129.

$$\begin{Vmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & & & \vdots \\ \vdots & & & \vdots \\ c_{1n} & \cdots & \cdots & c_{nn} \end{Vmatrix} \times \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{Vmatrix} \times \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & & \vdots \\ \vdots & & & \vdots \\ c_{n1} & \cdots & \cdots & c_{nn} \end{Vmatrix}. \quad (24)$$

In our problem, the linear transformation is (19), the matrix of which is

$$\begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix} \quad (25)$$

which corresponds to the C matrix. Hence, using this matrix and the matrices (18), we obtain for the matrices of the transformed quadratic forms

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix} \quad (26a)$$

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}. \quad (26b)$$

Performing the multiplication of the matrices in (26a), we have

$$\begin{Vmatrix} 2 + 2a_{21} + a_{21}^2 & a_{22} + a_{21}a_{22} \\ a_{22} + a_{21}a_{22} & a_{22}^2 \end{Vmatrix}. \quad (27a)$$

Note that this is the left-hand matrix of (21), which was obtained from the transformed quadratic forms (20). In the same way, performing the matrix multiplication in (26b), we have

$$\begin{Vmatrix} 2 + 2a_{21} + 2a_{21}^2 & a_{22} + 2a_{21}a_{22} \\ a_{22} + 2a_{21}a_{22} & 2a_{22}^2 \end{Vmatrix} \quad (27b)$$

which is the right-hand matrix of (21).

Thus, however complicated a network may be, and however numerous its meshes, a transformation (24) will give the complete set of equivalent networks. Some of the networks of this infinite set may contain negative as well as positive elements. To obtain networks with only positive elements, it is necessary that the transformation matrix be such that the elements in the main diagonal of the transformed matrix are positive and greater than the corresponding non-diagonal elements.

The infinite group of networks with all three kinds of network elements, namely, inductance, resistance and capacity elements, which have the same impedance function, are obtained exactly in the same manner. Now, however, the transformations are

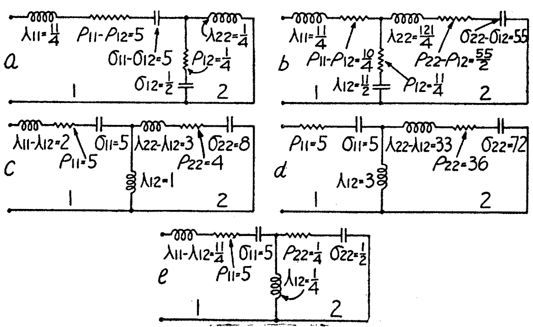
$$\begin{aligned} i_1 &= i_1' \\ i_2 &= a_{21}i_1' + a_{22}i_2' \end{aligned} \quad (28a)$$

$$\begin{aligned} q_1 &= q_1' \\ q_2 &= a_{21}q_1' + a_{22}q_2' \end{aligned} \quad (28b)$$

and we will have three matrices representing a network instead of two. Fig. 3 shows some of the members of the group of networks having the impedance function

$$Z(p) = \frac{11p^4 + 32p^3 + 64p^2 + 60p + 40}{p(4p^2 + 4p + 8)}\cdot \tag{29}$$

As before, the number of arbitrary constants, namely a_{21} and a_{22}, tells us the number of network elements which may be eliminated from the network without disturbing the invariance of the impedance function. Thus, for the two-mesh network containing three kinds of elements, the minimal forms will have, in general, seven elements.



Some members of the group of networks having

$$Z(p) = \frac{11p^4 + 32p^3 + 64p^2 + 60p + 40}{p(4p^2 + 4p + 8)}\cdot$$

Fig. 3.

For the general case, then, of networks of any number of meshes containing all three kinds of network elements, namely, inductance, resistance and elastance elements, we have the following three matrices which represent or definitely fix the network.

$$\left\|\begin{matrix} \lambda_{11} & \lambda_{12} \cdots & \lambda_{1n} \\ \lambda_{12} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \lambda_{1n} & \cdots\cdots & \lambda_{nn} \end{matrix}\right\|, \quad \left\|\begin{matrix} \rho_{11} & \rho_{12} \cdots & \rho_{1n} \\ \rho_{12} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \rho_{1n} & \cdots\cdots & \rho_{nn} \end{matrix}\right\|, \quad \left\|\begin{matrix} \sigma_{1n} & \sigma_{12} \cdots & \sigma_{1n} \\ \sigma_{12} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \sigma_{1n} & \cdots\cdots & \sigma_{nn} \end{matrix}\right\| \tag{30}$$

Making the following linear transformations of the instantaneous mesh currents or charges in the network, we have

$$\left.\begin{aligned} i_1 &= i_1' \\ i_2 &= a_{21}i_1' + a_{22}i_2' + \cdots + a_{2n}i_n' \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ i_n &= a_{n1}i_1' + a_{n2}i_2' + \cdots + a_{nn}i_n' \end{aligned}\right\} \tag{31a}$$

for the currents, and

$$
\begin{aligned}
q_1 &= q_1' \\
q_2 &= a_{21}q_1' + a_{22}q_2' + \cdots a_{2n}q_n' \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
q_n &= a_{n1}q_1' + a_{n2}q_2' + \cdots + a_{nn}q_n'
\end{aligned} \tag{31b}
$$

for the charges.

The three fundamental forms of the electric network of n meshes, whose coefficients are determined from the three matrices (31), are, respectively,

$$T = \tfrac{1}{2}\Sigma_1{}^n\lambda_{jk}i_ji_k \tag{32a}$$

$$F = \tfrac{1}{2}\Sigma_1{}^n\rho_{jk}i_ji_k \tag{32b}$$

$$V = \tfrac{1}{2}\Sigma_1{}^n\sigma_{jk}q_jq_k. \tag{32c}$$

The substitution of the transformations (31) in (32), results in three new quadratic forms, namely,

$$T' = \tfrac{1}{2}\Sigma_1{}^n\lambda_{jk}'i_j'i_k' \tag{33a}$$

$$F' = \tfrac{1}{2}\Sigma_1{}^n\rho_{jk}'i_j'i_k' \tag{33b}$$

$$V' = \tfrac{1}{2}\Sigma_1{}^n\sigma_{jk}'q_j'q_k'. \tag{33c}$$

The coefficients of these new quadratic forms, λ_{jk}', ρ_{jk}' and σ_{jk}' will of course be functions of the elements of the matrices (30) of the original quadratic forms (32) and of the a coefficients of the transformations (31). This has already been noted in the previous two-mesh example.

The transformation matrix, which contains the coefficients of the transformations (31) may be written

$$
C = \left\| \begin{matrix} 1 & 0 \cdots & 0 \\ a_{21} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{matrix} \right\| \tag{34}
$$

The matrices containing the coefficients of the new quadratic forms (33) are of course

$$
\left\| \begin{matrix} \lambda_{11}' \cdots \lambda_{1n}' \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \lambda_{1n}' \cdots \lambda_{nn}' \end{matrix} \right\|, \quad
\left\| \begin{matrix} \rho_{11}' \cdots \rho_{1n}' \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \rho_{1n}' \cdots \rho_{nn}' \end{matrix} \right\|, \quad
\left\| \begin{matrix} \sigma_{11}' \cdots \sigma_{1n}' \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \sigma_{1n}' \cdots \sigma_{nn}' \end{matrix} \right\|. \tag{35}
$$

These matrices contain the complete infinite group of networks having for an impedance function the impedance of the network of (30). The impedance function is thus an *absolute* invariant to a linear transformation of the instantaneous currents or charges of the networks in which the indicial current and corresponding charge are kept invariant.[6] The matrices (35) include within them the matrices (30), which are obtained by the *identity* transformation, namely,

[6] See also the Appendix.

$$\begin{aligned} i_1 &= i_1' \\ i_2 &= \quad i_2' \\ i_3 &= \qquad i_3' \\ &\cdots\cdots\cdots \\ i_n &= \qquad\qquad i_n' \end{aligned} \tag{36}$$

The C matrix corresponding to this transformation is the *identity* matrix

$$\begin{Vmatrix} 1 & \cdots & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & 1 \end{Vmatrix} \tag{37}$$

As in the two-mesh example, the actual substitution of the transformations (31) in the quadratic forms can be avoided by making use of the transformation theorem (24). Thus, the tensors (35) are obtained from the matrices (30), and the transformation matrix C (34) by the following matrix multiplications:

$$\begin{Vmatrix} 1 & a_{21} & \cdots & a_{n1} \\ 0 & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & a_{2n} & \cdots & a_{nn} \end{Vmatrix} \times \begin{Vmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{1n} & \cdots & \lambda_{nn} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix} \tag{38a}$$

$$\begin{Vmatrix} \text{“} \end{Vmatrix} \times \begin{Vmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & & \vdots \\ \rho_{1n} & \cdots & \rho_{nn} \end{Vmatrix} \times \begin{Vmatrix} \text{“} \end{Vmatrix} \tag{38b}$$

$$\begin{Vmatrix} \text{“} \end{Vmatrix} \times \begin{Vmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{1n} & \cdots & \sigma_{nn} \end{Vmatrix} \times \begin{Vmatrix} \text{“} \end{Vmatrix} \tag{38c}$$

The result of the matrix multiplications will be the three tensors (35) where the elements λ', ρ' and σ' are expressed in terms of the elements of the given network, λ, ρ and σ, and the elements a of the transformation matrix C (34).[7]

[7] The expressions for λ', ρ' and σ' in terms of λ, ρ and σ and the elements a of the transformation matrix C may be expressed by the following summations:

$$\lambda_{ik}' = \sum_{r,s=1}^{n} a_{ri}a_{sk}\lambda_{rs}$$

$$\rho_{ik}' = \sum_{r,s=1}^{n} a_{ri}a_{sk}\rho_{rs}$$

$$\sigma_{ik}' = \sum_{r,s=1}^{n} a_{ri}a_{sk}\sigma_{rs}$$

where the summation may be carried out in any order.

I am indebted to Professor E. A. Guillemin for these compact expressions which give the transformed parameters directly in terms of the given parameters and the elements of the transformation matrix.

The above network transformations have been made for the purpose of preserving the invariance of the *driving-point* impedance function. These transformations need not be so limited. Transformations may be made whereby the invariance of the transfer-impedance function is preserved. This gives rise to a new notion of equivalence, namely, equivalence with respect to a *definite mesh*.

It has been noted that the number of arbitrary constants in the transformation matrix determined the number of elements which could be eliminated from the network without disturbing the invariance of the impedance function. Thus, the least number of elements necessary in any network to realize a definite driving-point impedance function, or a definite transfer-impedance function, can be readily determined. This is important because one can tell whether a communication network of any number of meshes has superfluous elements.

Instead of imposing conditions on the a coefficients of the transformation to give minimal networks, it may be possible to obtain equivalence with respect to more than one mesh in a network; that is, to make the instantaneous currents in both the k-mesh and r-mesh, for example, invariant for the complete infinite group of networks. Finally, we may obtain equivalence with respect to, say, j-meshes, by using a still more general transformation.

In the foregoing theory, we have limited ourselves to networks of a finite number of meshes, that is, to networks with n degrees of freedom. There is no reason physically why this theory cannot be applied to networks of an infinite number of meshes, that is, an infinite number of degrees of freedom. Here interesting problems arise which bear intimately on mathematical theory, acoustics, electromagnetic wave theory, elastic waves,—in short, all branches of physics involving oscillations. This is also true for the finite problem, since the theory explained above can be applied to any physical vibrational problem involving a finite number of degrees of freedom, not merely to electric circuit theory. The latter province, however, appears to offer the most fertile soil for further investigation, and to provide a physical picture of the phenomena which occur.

In the problem involving networks with an infinite number of degrees of freedom, we have to deal with matrices and tensors containing an infinite number of elements as well as with quadratic forms which are power series or integrals. The matrices containing the coefficients of the three fundamental quadratic forms will contain an infinite number of elements, as will the transformation matrix and the resulting tensors. But for a physical network of an infinite number of degrees of freedom, we know physically that the total instantaneous magnetic energy, the total instantaneous electrostatic energy and the total instantaneous power lost, are finite quantities. Hence the three fundamental quadratic forms, which are now power series or integrals, are properly convergent. Likewise, the linear transformations, which are linear forms of an infinite number of terms, have meaning, as have the infinite transformation matrices which contain the coefficients of the transformations. Finally, the resulting tensors, which contain an infinite number of elements have physical meaning. They represent the complete infinite group of networks of an infinite number of degrees of freedom, discrete or continuous systems, which are equivalent in one or all the ways defined above.

The network with an infinite number of degrees of freedom may be merely a continuous system such as the smooth transmission or communication line. Thus, not only may there be an infinite number of different terminal networks which may perform the same function in a communication or transmission system, but also an infinite number of communication or transmission lines, which likewise may perform the same function. That is, there exists an infinite group of lines all of which have the same impedance function.

It will be recalled that this investigation has considered essentially two-terminal networks. By the principle of superposition, it should be possible to extend the theory to networks of any number of terminals. This extension is important, since by means of it any section of a communication network can be removed and replaced by an equivalent section.

In conclusion, it may be useful to suggest problems for further investigation. First, it should be mentioned that the conception that networks form a group in which the impedance is an invariant may prove useful in simplifying many problems in network theory. Thus, for example, the solution for the instantaneous currents of a network of the group at once results in the solutions for the instantaneous currents of all of the infinite number of networks in the group, since these currents are obtained from the former by a simple linear transformation. Furthermore, the solution for the instantaneous currents in one network in the group may be much simpler than for another; and there may be one network in the group for which the computations are least complicated. Hence, if it is necessary to obtain currents and voltages in one network, it may be simpler first to transform the network to an equivalent one, for which the computations are much simpler. This is already recognized, for example, when we transform from Y to Δ and vice versa in three-phase alternating-current network problems. Thus it is probable that simplification may result in operational circuit analysis by the above method.

It should be noted that in the matrix multiplication which gives the tensor containing the complete infinite group of equivalent networks, the impedance function vanishes from the scene. This suggests the possibility that the notion of the impedance function, which is a special creation of the electrical engineer, may perhaps disappear in the future. What we have to deal with are networks, currents and energies; and while the impedance function may be helpful for visualization, it may not be necessary to obtain the final important results.

As has been indicated, the problem of currents and charges in an electrical network is identical with the problem of velocities and displacements in a dynamical system. Although this is generally recognized, there is much in classical dynamic theory that remains to be translated in appropriate language for electric circuit theory. Many questions suggest themselves. What in electric circuit theory corresponds to the principal or normal coordinates in dynamic theory? Is it possible to eliminate the cross product terms in the fundamental quadratic forms of the electric circuit, thereby giving expressions which are sums of squares of the currents or charges? If it is, can a physical network be built which realizes this?

Problems of networks with an infinite number of degrees of freedom, equivalence with respect to transfer-impedance, equivalence with respect to more than one mesh, networks with more than two terminals—these have been merely intimated. Furthermore, it appears that mathematics does not discriminate against negative network elements, which seems to indicate that perhaps they may be realized physically, though not, of course, by coils, resistors and condensers.

Finally, in the study of an electrical network and its response to an impressed electromotive force, one continually encounters many seemingly unrelated branches of mathematics, such as (1) continued fractions, (2) Cauchy residue theory, (3) asymptotic series, (4) fractional and irrational derivatives and integrals, (5) group theory, (6) Fourier series and transforms, (7) integral equations, etc. It seems almost as if something were there, inarticulately trying to make itself understood. But perhaps it must await a modern Euler.

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Appendix

Consider the pair of quadratic forms representing, respectively, the inductance and resistance quadratic forms of an n-mesh network containing inductance and resistance elements:

$$\phi(i_1 \cdots i_n) = \Sigma_1{}^n \lambda_{jk} i_j i_k \tag{39a}$$

$$\psi(i_1 \cdots i_n) = \Sigma_1{}^n \rho_{jk} i_j i_k \tag{39b}$$

and form the pencil of quadratic forms

$$\phi p + \psi = \Sigma_1{}^n (\lambda_{jk} p + \rho_{jk}) i_j i_k. \tag{40}$$

The discriminant of this pencil is the determinant of the network:

$$D(p) = \begin{vmatrix} \lambda_{11} p + \rho_{11} \cdots \lambda_{11} p + \rho_{1n} \\ \lambda_{1n} p + \rho_{1n} \cdots \lambda_{nn} p + \rho_{nn} \end{vmatrix} \tag{41}$$

This is a polynomial which is in general of degree n and may be written

[8] Bell System Technical Journal, vol. 3, 1924,

[9] *Ibid.*

[10] Archiv. für Elektrotechnik, Heft 4, Band 17, 1926.

[11] Elektrischen Nachrichtentechnik, Heft 7, Band 6, 1929.

$$D(p) = \Delta(\lambda)p^n + \Delta(\lambda, \rho)p^{n-1} + \Delta_2(\lambda, \rho)p^{n-2} + \cdots + \Delta_2(\rho,\lambda)p^2 + \Delta_1(\rho,\lambda)p + \Delta(\rho) \quad (42)$$

where $\Delta(\lambda)$ and $\Delta(\rho)$ are the discriminants of ϕ and ψ respectively, while $\Delta_k(\lambda, \rho)$ is the sum of the different determinants which can be formed by replacing k columns of the discriminant of ϕ by the corresponding columns of the discriminant of ψ.

Likewise, form the first minor of $D(p)$, namely,

$$M_{11}(p) = \begin{vmatrix} \lambda_{22}p + \rho_{22} & \cdots & \lambda_{2n}p + \rho_{2n} \\ \vdots & & \vdots \\ \lambda_{2n}p + \rho_{2n} & \cdots & \lambda_{nn}p + \rho_{nn} \end{vmatrix}. \quad (43)$$

This may be written as a polynomial in general of degree n–1, namely,

$$M_{11}(p) = M_{11}(\lambda)p^{n-1} + M_{11}^{(1)}(\lambda,\rho)p^{n-2} + \cdots + M_{11}^{(1)}(\rho, \lambda)p + M_{11}(\rho) \quad (44)$$

Now it can be shown that the coefficients $\Delta(\lambda)\cdots\Delta(\lambda, \rho)\cdots\Delta(\rho)$ of $D(p)$ are integral rational invariants of weight two of the pair of quadratic forms ϕ and ψ.[12] Similarly, the coefficients $M_{11}(\lambda)\cdots M_{11}{}^k(\lambda, \rho)\cdots M_{11}(\rho)$ are integral rational invariants of weight two. Thus it is that the linear transformations of the variables of the quadratic forms make the impedance function $Z(p)$, which is the ratio of $D(p)$ and $M_{11}(p)$; that is, the ratio of two relative invariants of the same weight, an absolute invariant. The foregoing is true also of n-mesh networks containing all three kinds of elements, where we now have in addition to the inductance and resistance quadratic forms, the elastance quadratic form. Thus

$$Z(p) = D(p)/M_{11}(p) \quad (45)$$

becomes under a linear transformation with matrix

$$\begin{Vmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{Vmatrix} \quad (46)$$

$$Z(p) = \frac{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{vmatrix}^2}{\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}^2} \times \frac{D(p)}{M_{11}(p)} \quad (47)$$

$$= D(p)/M_{11}(p).$$

[12] See M. Bôcher, Introduction to Higher Algebra, 1927, p. 166.