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THE HIGH FREQUENCY BEHAVIOR OF A PLASMA

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Abstract

An analogue for a uniformly ionized gas on the basis of the known formula $K_p = 1 - \omega_0^2/\omega^2$ for its specific inductive capacity is found in a shunt-tuned circuit. In that formula $\omega_0^2 = 4\pi Ne^2/m$ and ω is the impressed angular frequency. This formula is extended to two simple cases of non-uniform ionization, with the conclusion that non-uniformity may be indistinguishable from large energy dissipation, insofar as reactive effects are concerned. Formulas have been derived for calculating the *conductivity and specific inductive capacity of a cylindrical plasma* (positive column of an arc) between parallel condenser plates from the measured impedance of this condenser by using a modification of Mossotti's Theory. The natural period of such a condenser occurs very nearly when $K_p = -1$. For a fixed frequency the reactance of such a composite condenser was followed for varying arc current by observing the length of Lecher System connected to the condenser plates which was required to cause circuit resonance. Two and sometimes three resonances were found. They occurred in the neighborhood of the calculated plasma-electron resonance as determined from electron density measurements, and varied with impressed frequency according to theory.

The theoretical variation of K_p was checked in the range $\omega_0^2 > 2\omega^2$ by the hyperbolic relation between change in condenser reactance and arc current there. Resistance measurements on the Lecher system, used in connection with a more detailed analysis, allowed *calculation of relative electron density* in the range from one-half to five times that giving resonance, checking the actual values. In the same range, the *resistance to electron motion* varied considerably. The maximum value occurred near resonance and checked, roughly, the value calculated from free path considerations. The alternating fields in the plasma varied from 0.17 to 1.2 $\text{v}\cdot\text{cm}^{-1}$ and the maximum amplitude of electron motion was 2.1×10^{-3} cm.

A transverse magnetic field was found to double the resonance as H. Gutton has found. The separation of resonances accords with theory.

I. Introduction

THE reaction of an ionized gas to very high frequency electric waves has been the subject of several investigations. Spontaneous oscillations in the neighborhood of 10^9 herz originally found by Penning[1] have been further investigated by Tonks and Langmuir[2] using the low pressure mercury arc. They found that the observed high frequency oscillations corresponded to the theoretical value for plasma-electron oscillations[3]

$$\nu = (Ne^2/\pi m)^{1/2} = 8980 \times N^{1/2} \tag{1}$$

[1] Penning, Nature **118**, 301 (1926) and Physica **6**, 241 (1926).

[2] L. Tonks and I. Langmuir, Phys. Rev. **33**, 195 and 990 (1929).

[3] This frequency appears so often in investigations of the high frequency behavior of ionized gases that it will be given a definite name in this paper, namely, "plasma-electron frequency."

where ν is the oscillation frequency, m is the electronic mass, and N is the ionization density. This frequency, looked upon as the limiting frequency at which radio waves can be propagated in an ionized medium, has been discussed at some length by T. L. Eckersley.[4] Since an ionized gas exhibits a resonant frequency, it is of interest to know the behavior of such a gas toward forced oscillations. W. H. Eccles[5] has derived formulae which have been interpreted by Bergmann and Düring[6] as attributing a dielectric constant

$$K = K_0 - 4\pi N e^2 m/(m^2\omega^2 + f^2) \tag{2}$$

and conductivity

$$\rho^{-1} = fNe^2/(m^2\omega^2 + f^2) \tag{3}$$

to the medium, where f is a dissipation factor appearing in the equation of motion of the electron.

H. Gutton and J. Clement[7] have made a rather thorough investigation of the dielectric properties of ionized hydrogen extending through the plasma resonance.[7.5] It was found that the natural period varied with the intensity of ionization roughly in the way that was theoretically expected, and they were able to obtain a fair numerical check between the ionization density calculated from Eq. (1) and that calculated rather indirectly from their results. More recently, C. Gutton[8] has confirmed the existence of a natural resonance period in an ionized gas by measuring the change in conductivity of ionized mercury vapor when a 2.76 meter wave was impressed on it. In his theoretical treatment of the gas reaction, however, H. Gutton introduces an elastic restoring force of unknown origin which seems to be not only unnecessary but somewhat in contradiction with the actual experimental results, as has been pointed out by J. Rybner.[9] Still more recently, Bergmann and Düring[6] have measured the change in dielectric constant of a small condenser in high vacuum arising from the introduction of electrons between the condenser plates. They have been able to check the theoretical value as given by Eq. (2) roughly, but the electron densities used were so small (in the neighborhood of 10^7 cm^{-3}) that K was but little different from K_0, that is, unity. Their measurements give an apparent conductivity which is of the order of magnitude given by Salpeter's[10] interpretation of f as arising from the momentum lost in electron collisions with gas molecules.

[4] T. L. Eckersley, Phil. Mag. **4**, 147 (1927).

[5] W. H. Eccles, Proc. Roy. Soc. **A87**, 79 (1912).

[6] L. Bergmann and W. Düring, Ann. d. Physik **1**, 1041 (1929).

[7] H. Gutton and J. Clement, L'Onde d'Electrique **6**, 137 (1927); H. Gutton, preliminary reports, Comptes Rendus **184**, 441 (1927); complete report, Ann. d. Physique **13**, 62 (1930).

[7.5] The term "plasma resonance" denotes the actual resonance of the system comprising the plasma and the condenser of which it forms part of the dielectric. This resonance occurs, in general, at a different ionization density from plasma-electron resonance. See the paragraph following Eq. (27) and the theoretical treatment in Section IX.

[8] C. Gutton, Ann. d. Physique **14**, 5 (1930).

[9] J. Rybner, L'Onde d'Electrique **7**, 428 (1928).

[10] J. Salpeter, Jahrb. der drahtl. Telegr. **8**, 252 (1914). But see the criticism of S. Benner, Annalen der Physik **3**, 993 (1929) regarding the application of Salpeter's results to this experiment. Salpeter treats the case in which the free time of an electron is variable, whereas in the Bergmann-Düring experiment, the effective free time of every electron is the same.

The present investigation extends the work of H. Gutton with the aim of making direct measurements of electron density so as to be able to check theoretical formulas more directly.

II. The Specific Inductive Capacity of a Plasma

The use of an ionized gas is necessary if high densities of electrons are to be obtained, but in a plasma the density of ionization is not uniform throughout the whole cross section. It therefore becomes of interest to know how non-uniformity in electron density affects plasma-resonance behavior.

The complex specific inductive capacity of a plasma can be found by direct substitution in the partial differential equation

$$\partial^2 Z/\partial x^2 = \mu K \partial^2 Z/\partial t^2 + [4\pi\mu N e^2/(f + jm\omega)]\partial Z/\partial t \tag{4}$$

as given by Eccles.[11] Substituting

$$Z = \exp\{j\omega[t - (\mu K_p)^{1/2}x]\}$$

where K_p denotes the complex specific inductive capacity and writing

$$4\pi\mu N e^2/m = \omega_x^2 \tag{5}$$

where ω_x is the local plasma-electron angular frequency and writing also

$$S = f/m \tag{6}$$

we find that

$$K_p = \frac{\omega_x^2 - \omega^2 + j\omega S}{-\omega^2 + j\omega S}. \tag{7}$$

This is the specific inductive capacity at each point of the ionized medium looked upon as a function of the electron density, since that enters ω_x and possibly S.

The response of the medium as a whole may be extremely complicated. The simplest case is the one in which the electron density is a function of one coordinate only and the impressed electric field lies in that same direction. In that case we may write

$$dV = (4\pi Q_0/K_p)dx$$

where V is the impressed voltage and Q_0 is the polarization of the medium. This may be integrated to the form

$$V = 4\pi Q_0 \overline{K_p^{-1}} x_p \tag{8}$$

for a plasma of thickness x_p where

$$\overline{K_p^{-1}} = (1/x_p)(-\omega^2 + j\omega S)\int_{\text{over } x_p} (\omega_x^2 - \omega^2 + j\omega S)^{-1} dx. \tag{9}$$

[11] Reference 5, page 87.

It is instructive to apply this formula to a hypothetical case. Neglecting the dissipation factor S, let us assume a simple linear law of variation of N in the plasma, as for instance,

$$N = N_0(1 - x/2a) \tag{10}$$

from $x=0$ to $x=a$. This leads to

$$\overline{K_p^{-1}} = -\frac{\omega^2}{\omega_0^2} \ln \left[\frac{\omega_0^2 - \omega^2}{\omega_0^2/2 - \omega^2}\right]^2 \tag{11}$$

where $\omega_0 = (4\pi N_0 e^2/m)^{1/2}$ is 2π times the plasma-electron frequency where the the ionization is densest.

The value of $\overline{K_p^{-1}}$ as a function of N_0 is shown as Curve 1 in Fig. 1.

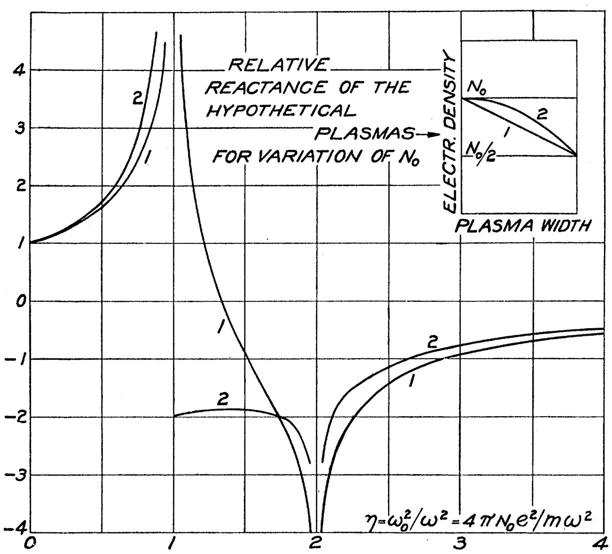

Fig. 1. Theoretical reactance of non-uniform plasmas.

Throughout the range $1 < \omega_0^2/\omega^2 < 2$ there is always a place in the plasma where local resonance occurs, giving rise to the middle section of the characteristic. The presence of resistance eliminates the infinities and leaves a curve resembling a resonance broadened by high dissipation, with the result that the width of the resonance region may only depend slightly upon the dissipative forces and may rather be a measure of the nonhomogeneity of the plasma. It was thought possible that a parabolic distribution of electron density

$$V = N_0(1 - x^2/2a^2) \tag{12}$$

might more nearly reproduce the assymmetry found in the actual resonance characteristics. Curve 2 was calculated with this in mind and although failing of its purpose, it may be useful in the appropriate case. Its equation is

$$
\begin{aligned}
\overline{K_p^{-1}} &= 2\eta^{-1}[2(\eta^{-1} - 1)]^{-1/2} \tan^{-1} [2(\eta^{-1} - 1)]^{-1/2} \\
&= \eta^{-1}[8(1 - \eta^{-1})]^{-1/2} \ln \left\{ \frac{[2(1 \;\; \eta^{-1})]^{1/2} + 1}{[2(1 \;\; \eta^{-1})]^{1/2} - 1} \right\}^2
\end{aligned}
\tag{13}
$$

according as $\eta < 1$ or $\eta > 1$.

With the dissipative factor omitted, Eq. (7) takes the form

$$K_p = 1 - \omega_x^2/\omega^2 \tag{14}$$

which shows that each element of the plasma has a simple circuit analogue in a fixed condenser C shunted by a variable inductor L. The shunt admittance of such a circuit is

$$Z^{-1} = j\omega KC = j\omega C + (j\omega L)^{-1}$$

where K is the apparent specific inductive capacity of the condenser. Putting $LC = \omega_0^{-2}$ we obtain

$$K = 1 - \omega_0^2/\omega^2.$$

Each elementary volume ($dx\ dy\ dz$, x being parallel to the electric field) in even a non-uniform plasma is thus equivalent to a vacuum parallel-plate condenser of capacity

$$C = dydz/4\pi dx$$

shunted by an inductor of inductance

$$L = mdx/Ne^2dydz.$$

If, for example, a plasma contains two layers, each of which is uniform in electron density, it can be represented by two shunt circuits in series. Such a system shows double resonance provided that the separation of the two natural periods relative to the damping of the individual circuits is great enough. This double resonance could be observed either by increasing the electron densities proportionally (decreasing inductances in the circuit analogue) while the impressed frequency was maintained constant, or by varying the impressed frequency the keeping ionization densities unchanged. Extension of this reasoning makes it possible to understand how even a continuous variation in electron density, such as found in an actual plasma, may give rise to multiple resonance.

Digressing slightly, it may be remarked that H. Guttonn arrives at a relation essentially different from Eq. (14), namely,

$$K = 1 + (4\pi Ne^2/m)/(\omega_0^2 - \omega^2)$$

In this equation, $w_0/2\pi$ is the resonant frequency of the electrons and ω_0 is related to N by the empirical equation

$$\omega_0^2 = AN^{3/4}.$$

These equations make K decrease initially from unity as N is increased from zero, which is qualitatively the same as Eq. (14). When N is infinite, however, these equations give K a weak positive infinity, whereas the strong negative infinity of Eq. (14) is qualitatively checked by H. Gutton's curves (Fig. 9 of Annales de Physique article) and by the measurements to be described below.

Turning now to the case of uniform ionization, it is evident that an analysis of the response of the medium reduces to an analysis of the electrostatic field between condenser plates in the presence of a dielectric of specific inductive capacity K_p. The simplest case is, of course, that of the plane parallel condenser, but that form is not suited to the positive column of an arc. The shape that has been adopted is that of a cylindrical plasma parallel to and midway between two plane condenser plates. This form has the experimental advantage that it is a natural form for a positive column, and that the electric lines of force in a uniform cylindrical plasma are straight and parallel so that there are no strong local fields at any point.

III. The Effective Specific Inductive Capacity of a Plane Parallel Condenser Containing a Cylindrical Plasma

Theoretically, a modification of Mossotti's theory of dielectrics makes the arrangement just outlined quantitative.[12] Mossotti's theory in its usual form contemplates a vacuum containing suspended conducting spheres. The response of such a configuration to an electric field can be treated quantitatively.[13] For present purposes the conducting spheres must be replaced by dielectric cylinders.

We consider then a large volume of dielectric (specific inductive capacity $=K$) made up of matter-free space in which are suspended infinitely long cylinders of specific inductive capacity K_p with their axes perpendicular to the uniform electric field E_e. The electric field tending to polarize any cylinder is

$$E_i = E_e + (4\pi/2)P \tag{15}$$

where P is the polarization of the composite dielectric, and the coefficient $4\pi/2$ applies to cylinders as $4\pi/3$ does to spheres. Within the polarized cylinder the field is

$$E_p = 2E_i/(K_p + 1) \tag{16}$$

analogous to $E_p = 3E_1/(K+2)$ in the case of spheres. The polarization of the cylinders is then

$$P_p = (K_p - 1)E_p/4\pi = (K_p - 1)E_i/2\pi(K_p + 1). \tag{17}$$

If θ denotes the fraction of the composite dielectric volume occupied by the cylinders,

$$P = \theta P_p = \theta(K_p - 1)E_i/2\pi(K_p + 1). \tag{18}$$

[12] H. Gutton's analysis based on Fig. 12 of his Ann. de Physique article cannot be justified theoretically. See J. Rybner, reference 9.

[13] Livens, Theory of Electricity, p. 228; Jeans, Electricity and Magnetism, Fifth Edition, p. 131 arrives at a formula which is not valid for sufficiently high values of dielectric constant.

Also

$$P = (K - 1)E_e/4\pi = (K - 1)E_i/2\pi(K + 1) \tag{19}$$

by using Eq. (15). Combining these two equations we find

$$(K - 1)/(K + 1) = \theta(K_p - 1)/(K_p + 1) \tag{20}$$

relating the dielectric constant, K_p, of the cylinders to that of the composite parallel-plate condenser, K. When K is not very different from unity, we can write

$$\delta K = 2\theta(K_p - 1)/(K_p + 1). \tag{21}$$

This formula was checked by the method H. Gutton used to determine the capacity values in the "equivalent" system shown in his Fig. 12. The arc tube, of 3.0 cm internal diameter, projecting well beyond and centered between two 7×7 cm condenser plates 7 cm apart was replaced by an open glass tube of the same size. With this tube empty, the Lecher System and condenser resonated with the thermocouple bridge at 17.9 cm, filled with mercury ($K_{Hg} = \infty$) at 16.3 cm, and filled with benzol ($K_B = 2.26$) at 17.35 cm. From Eq. (21)

$$\delta K_\infty = 2\theta$$

$$\delta K_B = 2\theta(2.26 - 1)/(2.26 + 1) = 0.386 \times 2\theta$$

whence

$$\delta K_B/\delta K_\infty = 0.386.$$

Since the changes in Lecher System length were small, they were proportional to capacity changes, whence

$$\delta K_B/\delta K_\infty = \delta L_B/\delta L_\infty = (17.9 - 17.35)/(17.9 - 16.3) = 0.34.$$

This is a satisfactory check considering that the experimental arrangement was only a rough approximation to that assumed mathematically. Since θ has been eliminated in the course of this calculation, the question naturally arises as to whether its value might not constitute an independent verification. The geometric capacity, C_0, terminating a resonant length, s, of Lecher system is given by

$$1/\omega C_0 K = Z_0 \tan (2\pi s/\lambda) \tag{22}$$

where Z_0 is the surge impedance of the system. It follows that the change in s required to retune after a slight change in K obeys the relation

$$\delta K/K = - 2\pi\delta s/\lambda \sin (2\pi s_0/\lambda) \cos (2\pi s_0/\lambda)$$

and using Eq. (21)

$$\theta = - 2\pi\delta s/\lambda \sin (4\pi s_0/\lambda) \tag{23}$$

since $K = 1$.

In the present case $s_0 = 27.2$ cm (for the bridge at 17.9 cm) and $\lambda = 187$ cm whence $\theta = 0.056$. The tube cross-section is $(\pi/4)\times 3.0^2 = 7.07$ cm^2, while the condenser cross-section is $7\times 7 = 49$ cm^2 giving $\theta = 0.14$. This large discrepancy is easily accounted for both by the large edge effect in a condenser of the proportions used and also by the individual capacity of each plate to ground. Under these circumstances it is permissable to treat θ as an empirical constant in Eqs. (20) and (21).

It is fairly obvious that Eq. (21) may not be applicable to plasma resonance since δK becomes very large when $K_p K_c$ nears -1. In addition, a dissipative term must be included if the analysis is to be complete. From Eq. (7) (treating the plasma as if uniform by using ω_0 instead of ω_x)

$$K_p = \frac{\omega_0^2 - \omega^2 + j\omega S}{-\omega^2 + j\omega S} = \frac{\eta - 1 + jS/\omega}{-1 + jS/\omega} \tag{24}$$

if we put $\omega_0^2/\omega^2 = N/N_\omega = \eta$, N_ω being the electron density for which the plasma-electron angular frequency is ω, that is, $N_\omega = [\omega/(2\pi\times 8980)]^2$. From Eq. (20),

$$K^{-1} = 1 - 2\theta(K_p - 1)/[K_p + 1 + \theta(K_p - 1)] \tag{25}$$

Combining, we have

$$K^{-1} = 1 + \theta\eta/[1 - (1+\theta)\eta/2 - 2jS/\omega] \tag{26}$$

for calculating the complex dielectric "constant" of the condenser when dissipation is included.

IV. Characteristics of a Tuned Circuit Made up of a Condenser Containing a Plasma and a Variable Inductor

If we suppose the condenser containing the plasma to be connected in series with an inductance, L, and denote its capacity when $N=0$ by C_0, the series impedance of the combination is

$$Z = j[\omega L - (\omega C_0 K)^{-1}]$$

Putting

$$L = L_0 + \delta L$$

where L_0 is the inductance which tunes C_0, that is, $L_0 = (\omega^2 C_0)^{-1}$, this equation becomes

$$Z = j[\omega\delta L - \omega L_0(K^{-1} - 1)].$$

Substituting from Eq. (26) and separating into real and imaginary parts we find

$$Z = j\omega\left[\delta L - L_0\frac{(\theta/\phi)\eta'(1-\eta'/2)}{(1-\eta'/2)^2 + S^2/\omega^2}\right] + 4\,\frac{\omega L_0\theta\eta S/\omega}{(1-\eta'/2)^2 + S^2/\omega^2}$$

where $\phi = 1+\theta$ and $\eta' = \phi\eta$. When L is adjusted so that the circuit is in resonance, the reactance is zero and δL_r, the value of δL at resonance, is given by

$$\delta L_r = L_0 \frac{\theta\eta(1 - \eta'/2)}{(1 - \eta'/2)^2 + S^2/\omega^2} \tag{27}$$

As N is increased from zero, η and η' increase proportionally and δL_r increases from zero, at first proportionally to N but later more rapidly. If S were zero, δL_r would approach $+\infty$ as $\eta' = 2$, to reappear at $-\infty$, but the finite value of S causes δL_r to vary rapidly from a positive maximum to a negative minimum as η' passes through 2. As N becomes still larger, δL_r increases once more approaching $\delta L_\infty = -2\theta L_0/\phi$ as $\eta \to \infty$. This is just the type of behavior approximated to by the curve in H. Gutton's Fig. 9.

Obviously, the plasma resonance occurs at $\eta' = 2$. The qualitative circuit equivalent of the plasma between its condenser plates, as shown in Fig. 2, makes it evident that the shunt resonance does not occur when the plasma

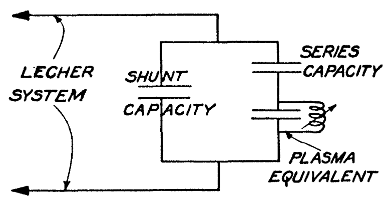

Fig. 2. Qualitative circuit equivalent of a plasma between condenser plates.

itself is tuned to the impressed frequency. This would be true only when the shunt capacity is zero. For example, the infinite plane parallel plasma is a (rather ideal) case in which plasma resonance would lie at $\eta = 1$. For a spherical plasma between parallel plates, resonance would occur near $\eta = 3$.

In the range where η' is different enough from 2 for S/ω to be neglected, Eq. (27) can be written

$$(1 - \eta'/2)\delta L_r = -(\eta'/2)\delta L_\infty(\theta\phi_\infty/\theta_\infty\phi)$$

the ratio in the right member being introduced to allow for changes in θ with N. It follows that

$$(\theta\phi_\infty/\theta_\infty\phi)\delta L_r^{-1} + \delta L_\infty^{-1}(2/\eta' - 1) = 0$$

and noting that $\eta' = \phi\eta = \phi N/N_\omega$ we have

$$(\theta\phi_\infty/\theta_\infty\phi)\delta L_r^{-1} = (2N_\omega/\delta L_\infty)(\phi N)^{-1} - \delta L_\infty^{-1}. \tag{28}$$

Now, changes in ϕ can be neglected since the actual value of θ is so small. It is thus justifiable to put $\phi/\phi_\infty = 1$, whence Eq. (28) becomes

$$(\theta/\theta_\infty)\delta L_r^{-1} = (2N_\omega/\delta L_\infty)(\phi_\infty N)^{-1} - \delta L_\infty^{-1}. \tag{29}$$

In the range of larger values of N (experimentally when $\eta' > 2$), the variation of θ with N was appreciable, but small, and the values of θ/θ_∞ could be calculated from the thickness of the wall sheath in the arc tube. For the smaller values of N, however, the variation of θ was not calculable and Eq. (29) could not be quantitatively tested in that range.

V. Preliminary Experiments

The plasma used in this investigation was the positive column of a mercury arc because the electrical methods for measuring ionization intensities have been applied most extensively to these arcs.[14] At first a tube containing an oxide-coated cathode was tried, but was found unsuitable on account of a conducting film which forms on the tube walls. This film appeared as a slight blackening, and its conductivity was made evident by the decrease in resonance current in the tuned circuit when the unexcited tube was placed between the condenser plates. That this film comes from the oxide cathode was proved by its absence in the next tube which had a hot tungsten filament. This tube, illustrated in Fig. 3, was a long cylinder of uniform internal diameter, 3.0 cm. The lower portion, containing a small quantity of mercury, was immersed in a water vath for pressure control. A short distance above the cathode was a ring electrode intended to be used as an auxiliary anode but soon abandoned. Thirty cm above this were two probes. One was a tungsten wire of 0.0127 cm diameter and 1.87 cm long placed in the axis of the tube. The other

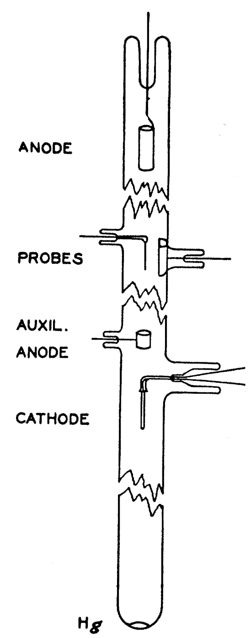

Fig. 3. Experimental arc tube.

[14] I. Langmuir and H. Mott-Smith, Jr., Gen. Elec. Rev., **27**, 449, 538, 616, 762 (1924); H. Mott-Smith, Jr. and I. Langmuir, Phys. Rev. **28**, 727 (1926); L. Tonks and I. Langmuir, Phys. Rev. **34**, 876 (1930).

was a square molybdenum plate, 1.95 cm on a side, on the tube wall opposite the probe and bent to fit the wall. These probes were intended to measure the intensity of ionization in the arc by well-known methods.[14] The main anode, a hollow cylinder, was 8 cm above these probes. The high frequency field was applied to the uniform column midway between the probes and the auxiliary anode by external electrodes of various shapes and sizes.

Initially a parallel wire system was connected with the external electrodes, later a small single loop inductance with variable condenser in series was employed, but finally a Lecher system consisting of two brass rods each 0.422 cm in diameter, spaced 3.90 cm on centers, and 180 cm long, was used for most of the measurements. The complete arrangement is shown schematically in Fig. 4. The Lecher system was tuned with a moveable bridge which consisted essentially of a low-resistance vacuum thermocouple of 2.6 ohms d.c. resistance. To prevent reflection effects from the free end of the Lecher sys-

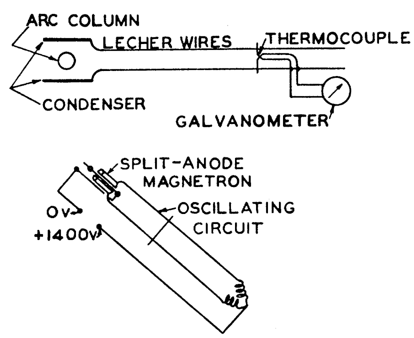

Fig. 4. Apparatus for observing ionized-mercury resonance.

tem, a short-circuiting bridge was attached to the thermocouple bridge and some 16 cm back of it, so that the two moved together. This 16 cm of conductor acts as a high impedance shunt to the thermocouple at all wave lengths used and effectively prevents any free-end oscillations without affecting the thermocouple calibration.[15] Stray effects arising from bringing the hand near to the free end of the Lecher system were minimized by by-passing the capacity currents along the thermocouple-to-galvanometer leads direct to the shorting bridge through small condensers.

A split-anode magnetron of Type FH-11 which can generate powerful oscillations up to 4×10^8 herz[16] was used as oscillation source. It was found more convenient to vary the arc current than to vary the frequency of the oscillator, since the e.m.f. in the Lecher system circuit then does not vary.

The earliest experiments showed two resonance frequencies in the plasma. In view of H. Gutton's single resonance, it was thought that this might arise in some way from the distributed excitation of the Lecher system together with its distributed inductance and capacity. With this in mind, a circuit with

[15] Paper to be submitted to I.R.E. Proc.

[16] W. C. White, Electronics **1**, 34 (1930).

lumped impedances was tried and as the same effect was observed, it was concluded that the double resonance was a property of the plasma itself.

Too high an excitation of the plasma circuit-resulted, as pointed out by C. Gutton, in additional ionization in the vicinity of plasma resonance which was often evident visibly by increased light from the region of the arc between the external electrodes. Such effects made measurements erratic and difficult of interpretation so that is was found best to work with very low excitation.

For the higher arc currents the tungsten wire cathode had to be run very hot. When it happened occasionally that it was too cool, the arc changed its appearance and the oscillation readings were erratic and quite different from the other condition. Too cool a cathode gives temperature-limited instead of space-charge-limited emission, the cathode drop increases, and the primary electrons, instead of ionizing the gas and causing glow on the anode side of the cathode only, penetrate down the tube and cause glow there also. With a plentiful supply of primary electrons, the cathode drop exceeds the ionizing potential only slightly so that although the electron at the cathode sheath edge has sufficient energy to ionize, the normal decrease in plasma potential as the electron moves away from the cathode is sufficient to bring its energy below the ionizing value. Of course, toward the anode, the gradient in the arc speeds up the electrons, but in the opposite direction ionization will be prevented. If, however, the electrons have several volts excess energy, as they have with temperature-limited currents, they are able to maintain a plasma even in the absence of an accelerating field in it. The change in oscillation readings undoubtedly arises from the change in arc conditions, and the fluctuations may be caused by line voltage variations which here appear as a variable cathode drop of about the same magnitude as the voltage variation itself, whereas with a hot enough cathode they cause only a small fractional change in arc current.

IV. Resonance Characteristic Measurements

In taking a resonance characteristic, the oscillation frequency of the magnetron oscillator was first fixed by adjusting the position of the bridge on the parallel conductors which, connected to the two anode sections of the magnetron, form the resonant circuit of the oscillator. The oscillation amplitude was fixed at a suitable value by adjusting the anode voltage, anode current (through filament current) and magnetic field of the magnetron. These three quantities were carefully maintained constant. The half wave-length was found from the separation of two successive nodes on a second Lecher system which could be coupled to the oscillator. This value agreed within 2 percent with the value found on the measuring system itself. The difference, which arises from the presence of the wood support in the measuring system, is small enough to be negligible in the present work.

The appendix of the arc tube was then immersed in a Dewar flask containing water at a temperature to give the desired mercury pressure.

With no arc in the tube, the resonance position of the bridge on the measuring system and the thermocouple galvanometer reading were recorded. The arc was then started and the same readings were made at arc currents increasing roughly in a geometric ratio.

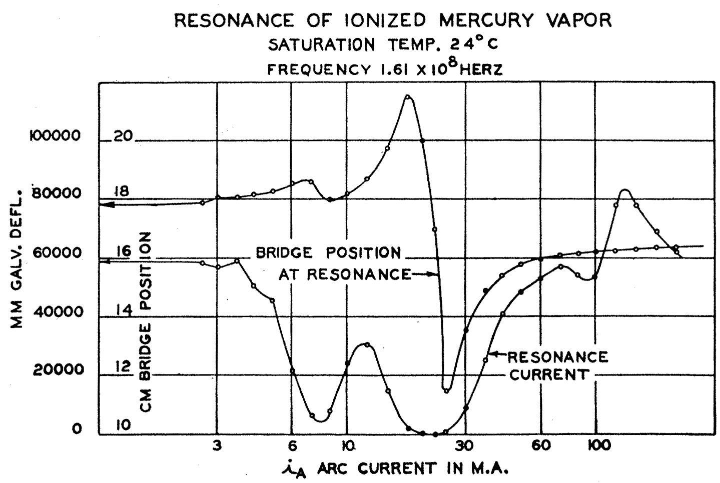

Fig. 5. Resonance characteristic of ionized mercury vapor.

The results of such a series of measurements are shown in Fig. 5. The two resonance points at 7.5 and 21.5 m.a., to be denoted as a- and b-resonances, respectively, are accompanied by the current minima expected at a shunt

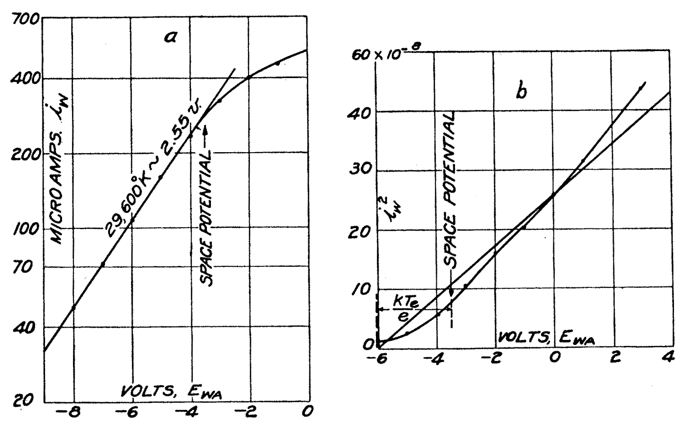

Fig. 6. Electrode volt-ampere characteristics for determining the electron density in the arc.

resonance. Excluding the immediate neighborhood of a resonance, the continual increase in s with increasing ionization intensity gives qualitative confirmation of the continual decrease in K_p with increase in N as required by Eq. (14). The variation in resonance current above 80 m.a. is not understood.

In order to compare the experimental results with the theoretical, it was necessary to determine the ionization intensity as a function of arc current. It developed that the wall electrode, Fig. 3, was unsuited to this purpose since the wall sheaths were too thick over the greater range of arc current. Accordingly, the cylindrical electrode w was employed for this purpose. Fig. 6 exhibits two plots of a typical volt-ampere characteristic. Fig. 6a is the semi-log plot which yields electron temperature and space potential. Fig. 6b is the i^2-V plot, the "slope" of which gives the electron density. Bearing in mind the ideal shape of this curve as shown by f^2 Fig. 7,[17] it is possible to make a

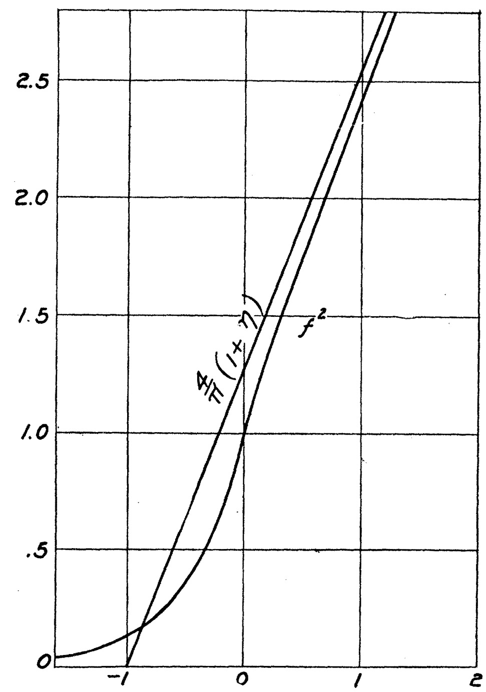

Fig. 7. Theoretical (f^2) volt-ampere characteristic and its approximation $[4(1+\eta)/\pi]$.

very good guess at the straight line whose slope is significant in the following way. This line intercepts the V-axis at the space potential less kT_e/e, the potential corresponding to the electron temperature. The space potential is -3.5 v, $kT/e=2.5$ v, whence the intersection is at -6 v. The upward bend above zero volts is obviously due to ionization in the sheath. Very little latitude is thus left for drawing the asymptote of the ideal characteristic. As drawn, the slope is 4.3×10^{-8} amps$^2\cdot$v^{-1} which gives[18]

[17] Reproduction of Fig. 7 from I. Langmuir and H. M. Mott-Smith, Jr., G. E. Review, p. 617, Sept. (1924).

[18] Eq. (45) I. Langmuir and H. M. Mott-Smith, Jr., G. E. Rev. **27,** 455 (1924) or L. Tonks and I. Langmuir, Phys. Rev. **34,** 913 (1929).

$$n_e = 3.32 \times 10^{11} S^{1/2}/A = 3.32 \times 10^{11} \times 2.07 \times 10^{-4}/0.080$$
$$= 8.6 \times 10^8 \text{ electrons} \cdot \text{cm}^{-3}$$

Like measurements at other arc currents and mercury pressures gave additional points on the curves of Fig. 8. Of course, at the higher pressures, special care was taken that the tube temperature at all points never fell below the appendix temperature.

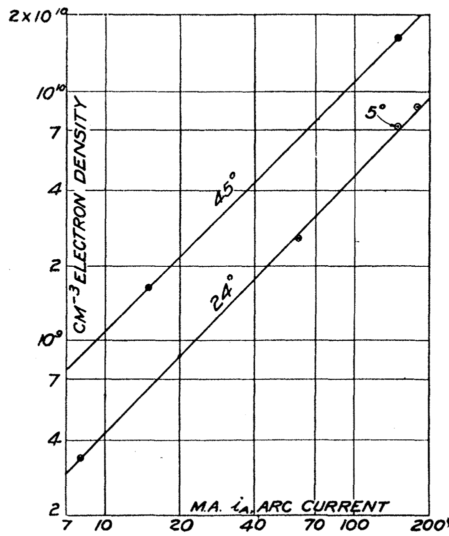

Fig. 8. Electron density in the arc as a function of arc current for different appendix temperatures.

With these data it became possible to correlate the frequency impressed on an ionized gas with the ionization density. The results of several resonance characteristics have been incorporated in Table I and Fig. 9. The "Theoretical Curve—Plane Plasma" is a plot of the equation $\nu = 8980\ N^{1/2}$, ($\omega_0 = \omega$ or

TABLE I. *Relation between plasma-resonance frequency and ionization density.*

Condenser plate size (cm²)	Temp. (°C)	Frequency ($\times 10^{-8}$)	a-resonance i_a (ma) $N_e \times 159$	b-resonance i_a (ma) $N_e \times 159$	aa-resonance i_a (ma) $N_e \times 159$	Plasma-resonance density, calculated $2\nu^2/(8980)^2$
2.5×2.5	13±	1.59	7.2 0.27	22 0.86		0.63×10^9
2.5×2.5	24	1.59	8.0 0.34	26± 1.12	4.1?	.63
1×1 approx.	24.5	3.66	60. 2.54	145± 6.7	30 1.28	3.30
1×1 "	24	1.60	7.7 0.32	24 1.03	4.3 0.18	.63
1×1 "	24	2.35	21. 0.91	54± 2.28	12. 0.51	1.36
7×7	24	1.60	7.5 0.31	22 0.95		.63
7×7	45	1.60	5.5 0.60	15.5 1.66		.63

$\eta = 1$ in other forms). It gives the theoretical relation between ionization and plasma-resonance frequency for a plane plasma. The "Theoretical Curve—Cylindrical Plasma" is a plot of $\eta = 2$ and gives the same relation for the cylindrical plasma. The a-resonance and b-resonance curves show the observed behavior of the a- and b-resonance conditions in a 2.2-bar arc. It is to be noted that the actual resonance frequency lies within 30 percent of the theoretical and that the slope of the straight lines is 0.42 and 0.45 compared to a theoretical 0.50. The *aa*-resonance is one which appeared at lower ionization densities in cases where the applied field was decidedly non-uniform. While it is possible to imagine that the *b*-resonance may arise primarily from the response of the outer portions of the plasma, and the *a*-resonance from the axial portion, it is rather difficult to account for a third resonance corresponding either to a still higher electron density or to a configuration approaching more nearly to the plane case.

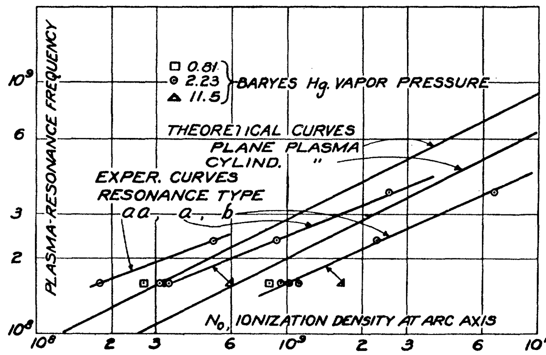

Fig. 9. Comparison of experimental and theoretical plasma-resonance frequencies.

The plasma resonance was found to be somewhat dependent on the gas pressure in the arc. Fig. 9 shows that a 14-fold pressure change caused a 2.2-fold change in resonant density. The resonance density thus varies roughly as the 0.3 power of the pressure. Compared to its variation with the square of the frequency, this is seen to be a small change which may arise from second order effects. It is probably significant that both *a*- and *b*-resonances were affected in the same ratio.

Although exact agreement of theory with experiment is lacking, it seems possible that the method used here may be useful in investigating electron concentration in cases where other methods fail. In recent unpublished work with Ne discharges, for instance, C. G. Found in this laboratory was found that the simple interpretation of probe characteristics fail, probably on account of the specific action of metastable atoms, whereas the present method appears to be directly applicable.

VII. Demonstration That $K_p \propto 1-\omega_0^2/\omega^2$ in the Range $\omega_0^2>2\omega^2$

It has been remarked that Eq. (29) can give a direct test of the specific inductive capacity formula. The δL's refer to inductance changes but, in connection with the Lecher system, they may just as well apply to the small changes in tuning length which have been found. Again, since tests have shown that N is very nearly proportional to i_A, we may interpret Eq. (29) as requiring that the reciprocal change in Lecher system tuning length caused by increasing the arc current from zero to i_A shall be a linear function of the reciprocal of i_A, insofar as θ/θ_∞ is constant. The solid dots of Fig. 10 are plotted in this way from the same data as Fig. 5. At 40 amp^{-1} the imaginary part of K_p is no longer negligible so that Eq. (29) no longer applies, and at the lower values of i_A^{-1}, the curvature of the plot indicates that a correction for variation of θ is necessary.

This correction has been made in the following manner. If the plasma occupies a radius $a-x$ in the tube of radius a, then $\theta/\theta_\infty=(a-x)^2/a^2$ since x, the sheath thickness, approaches zero as i_A approaches infinity. Thus, to a first approximation,

$$\Delta\theta/\theta_\infty = (\theta - \theta_\infty)/\theta_\infty = -2x/a.$$

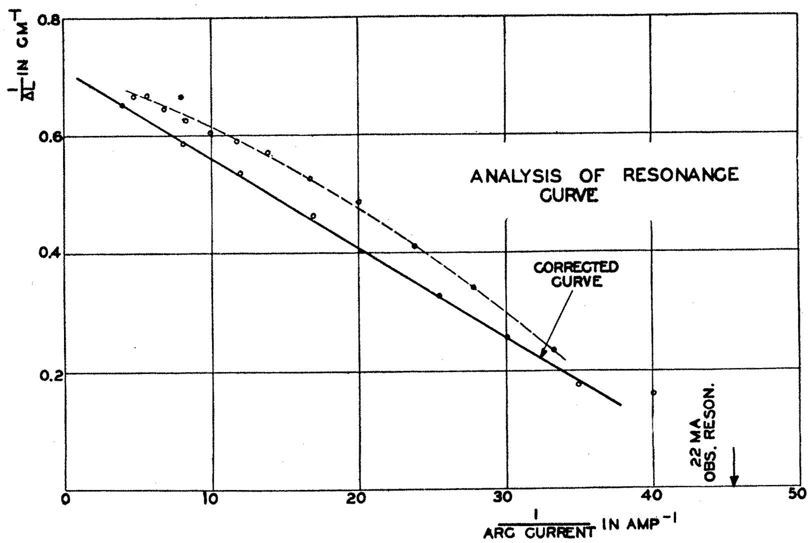

Fig. 10. Analysis of the resonance characteristic in Fig. 5.

Now x is related to I_p the positive ion current density to the wall through the space charge equation

$$I_p = 2.34 \times 10^{-6} V^{3/2}/600x^2$$

or

$$x = 6.25 \times 10^{-5} V^{3/4} I_p^{-1/2}$$

where V is the potential drop in the wall sheath and 600 is $(m_p/m_e)^{1/2}$. Now I. Langmuir and H. M. Mott-Smith, Jr., have found[19] that $I_p \propto i_A^{1.2 \text{ to } 1.3}$ and in the present tube I have found the relation to approximate

$$I_p = 1.08 \times 10^{-3} i_A^{1.4}$$

With $i_A=0.060$ amp, it was found that $T_e=28{,}600°$ whence[20]

[19] I. Langmuir and H. M. Mott-Smith, Jr., G. E. Rev. **27,** 764 (1924).
[20] L. Tonks and I. Langmuir, Phys. Rev. **34,** 898 (1929).

$$V = 6kT_e/e = 6 \times 28{,}600/11{,}600 = 14.8v$$

Assuming that V changes but little with $i_A(T_e$ changes but slowly) these equations lead to

$$\Delta\theta/\theta_\infty = -0.0191 i_A^{-0.7}.$$

Substituting $1+\Delta\theta/\theta_\infty$ for θ/θ_∞ in Eq. (29) we have

$$(1 - 0.0191 i_A^{-0.7})\delta L_r^{-1} = (2N_\omega/\delta L_\infty)(\phi N)^{-1} - \delta L_\infty^{-1}.$$

Thus the δL_r^{-1} coordinate of each point should be decreased by an amount directly calculable from i_A. The fractional corrections were tabulated for a series of values of i_A^{-1} and these were applied in several cases to the smooth curve through the experimental points. As in Fig. 10, the resulting curve is always a straight line, the continuation of which intercepts the δL_r^{-1} axis very near to the actual b-resonance, thus confirming Eq. (14). Later Fig. 11, showing θ/θ_∞ as a function of i_A, was plotted and was found useful in subsequent calculations.

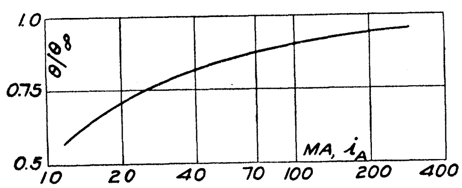

Fig. 11. Fraction of tube cross-section occupied by plasma.

The method here employed, by not requiring that the plasma fill the whole dielectric space, avoids the difficulties of interpretation encountered by Appleton and Childs[21] because of the formation of positive ion sheaths.

VIII. Detailed Analysis of Hg Resonance Characteristics

The resonant Lecher system length as a function of arc current and the thermocouple current as a function of arc current are the two characteristics so far mentioned. Another characteristic which can be obtained is the equivalent series resistance of the composite condenser.

It can be shown[15] that the resonance curve of a Lecher system readily lends itself to determinations of resistance. Let I_0 be the maximum (resonance) current in the thermocouple on the movable bridge and let I be the current at any nearby position s. The values of $[I^{-2}-I_0^{-2}]^{1/2}$ plotted against s give a straight line whose slope S_l in amp.$^{-1}\cdot$cm^{-1} is a measure of the resistance in the circuit. The relation is

$$R \cos^2 (2\pi s_r/\lambda) = 2\pi Z_0/S_l I_0 \lambda - R_t \tag{30}$$

where s_r is the effective length of Lecher system at resonance, R is the equivalent series resistance of the composite condenser, and R_t is the thermocouple resistance, the distributed resistance of the system being neglected.

[21] Appleton and Childs, Phil. Mag. 10, 969 (1930).

R_t was determined with no current in the arc. The resonance curve at the first resonance, together with $(I^{-2}-I_0^{-2})^{1/2}$ are shown in Fig. 12. Resonance occurred at 11.77 cm. The next resonance occurred at 89.39 cm, giving a wave-length, λ, of 155.2 cm. This compares with 158 cm measured on a sys-

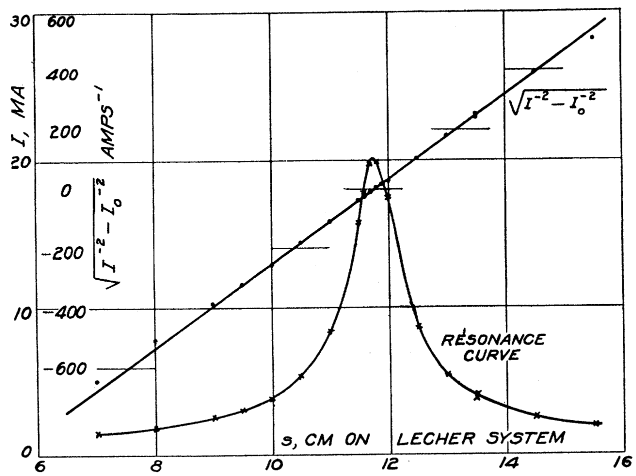

Fig. 12. Lecher system resonance curve and its analysis.

tem almost completely free of excess distributed capacity. The difference is too small to warrant detailed corrections, but in each case where λ appears the value which seems to be the more appropriate will be used.

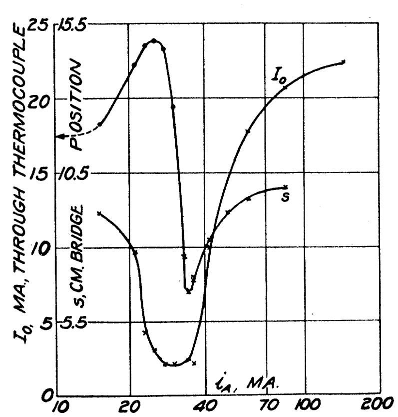

Fig. 13. Resonance characteristic for $\lambda=158$ cm.

From the figure $S_t=143$, $I_0=0.020$ amp., using $Z_0=353$, and $\lambda=155.3$ we have $R_t=4.98\ \omega$. At 89.39 cm the resonance curve yields $R_t=6.81\ \omega$. The differences arises from the distributed resistance of the system, but as long as measurements are confined to the neighborhood of the first resonance, it will doubtless be accurate enough to include this resistance in R_t.

Fig. 13 shows the resonance characteristic at this same frequency. For each value of arc current sufficient points (not less than 5) on the resonance curve were taken to permit the slope S_l to be determined. The bridge shortening had been determined by the double-hump resonance method[15] so that s_r could be found from the bridge position. Zero correction and bridge shortening together amounted to $+9.2$ cm. R was then calculated from Eq. (30), and the values obtained appear in Fig. 14.

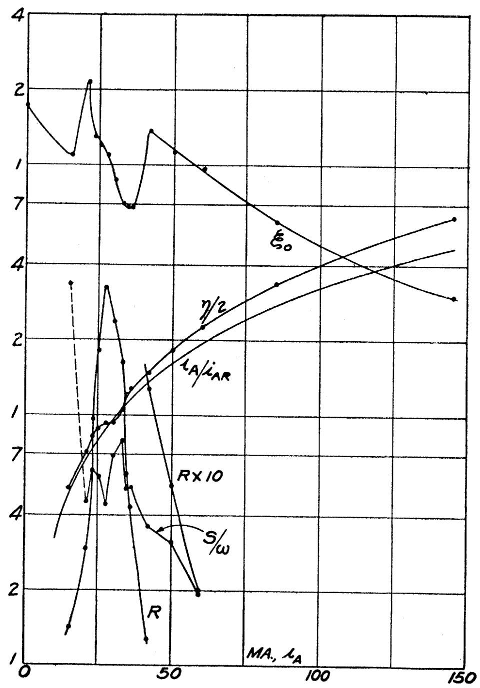

Fig. 14. Analysis of resonance and resistance characteristics.

Ranges: $\xi 0$ from 3.0×10^{-4} to 2.1×10^{-3} cm. $\eta/2$ from 0.51 to 6.2. i_A/i_{AR} is on the same scale. R from 14 to 320 ohms. S/w from 0.02 to 0.33.

It remains to interpret R and s_r in terms of the damping coefficient of the electron oscillations and the density of ionization. The specific inductive capacity K_0 of the composite condenser can be found from s_r readily since

$$\begin{aligned}(\omega K_0 C_0)^{-1} &= Z_0 \tan l \\ (\omega C_0)^{-1} &= Z_0 \tan l_0\end{aligned} \tag{31}$$

where $l = 2\pi s_r/\lambda$ and l_0 is the value of l for $N = 0(K_0 = 1)$. It follows that

$$K_0 = \tan l_0/\tan l \tag{32}$$

Thus the problem becomes that of a condenser of a complex capacity C_0K (Eq. (26)) equivalent to the combination, C_0K_0 in series with R;

$$(j\omega KC_0)^{-1} = R + (j\omega K_0C_0)^{-1}. \tag{33}$$

Substituting from Eq. (26) and separating into real and imaginary parts we have

$$(2S/\omega)(1 - K_0^{-1}) - (\eta/2)\omega C_0R(1 + \theta) = - \omega C_0R$$
$$(2S/\omega)\omega C_0R + (\eta/2)[1 - \theta - K_0^{-1}(1 + \theta)] = 1 - K_0^{-1}$$

whence

$$S/\omega = \theta\omega C_0R/D \tag{34}$$
$$\eta/2 = [(1 - K_0^{-1})^2 + \omega^2C_0^2R^2/D] \tag{35}$$

where

$$D = (1 - K_0^{-1})[1 - K_0^{-1} - \theta(1 + K_0^{-1})] \tag{36}$$

The necessary data are all available. The calculations of R and K_0 have been discussed, $\theta_\infty = 0.056$, and $\omega C_0 = 0.00250$ by Eq. (31). The values of S/ω and $\eta/2$ calculated in this way are shown in Fig. 14. Since theory requires that

$$\eta/2 = N/N_{\text{resonance}} = i_A/0.031$$

very nearly, $i_A/0.031$ has been plotted for comparison. The higher values of $\eta/2$ for the larger values of i_A may well be due to an error in θ_∞. This failed to appear in Fig. 10 since there only relative values of θ were involved. A change to $\theta_\infty = 0.053$ would make $\eta/2$ coincide with $i_A/0.031$ at $i_A = 145$ m.a. and would not spoil the present agreement near resonance. It thus appears that the experimental results confirm the oscillation theory, not only to one side of the resonance region, but even through this region. The check is surprisingly good in view of the non-uniformity of the plasma.

It may be that this factor has more influence on the value of S/ω. Salpeter[10] has shown that collisions of electrons with gas atoms introduce a damping factor in the electron equation of motion and S is then the number of collisions per electron per second. In the present experiment with an electron mean free path of about 3 cm[22] and a speed of 1.22×10^8 cm/sec ($T_e \sim 30{,}000°$) $S = 4.1\times10^7$. For a 158 cm wave, $\omega = 1.19\times10^9$, whence $S/\omega = 3.4\times10^{-2}$. On the other hand the electrons are reflected much more frequently from the sheath on the tube wall. Such reflections should, on the average, destroy the directed momentum just as collisions with gas atoms do. In the tube of 3.0 cm diameter, the mean free path for wall collisions may be estimated at 1.5 cm, and the effective speed at $[1.22/(1.5)^{1/2}]\times10^8$ cm sec, whence, due to this cause, $S/\omega = 0.056$. Adding the two dissipation factors we obtain, finally, $S/\omega = 0.090$, a value of the same order as found experimentally. Since the gas density and the electron temperatures change but little with a change in arc current, S/ω should, theoretically, be constant. Aside from experimental errors, which undoubtedly affect the values when R becomes comparable with R_t, that is, above $i_A = 50$ m.a., the non-homogeneity of the plasma may possibly account for a large part of the variation. The high 15-m.a. value may arise from the proximity to the a-type resonance.

[22] T. J. Killian, Phys. Rev. **35**, 1238 (1930), Table I.

Expressed as a decrement, δ,

$$\delta = \pi S/\omega$$

whence, theoretically, $\delta = 0.28$ and the amplitude of a free oscillation would decrease by 24 percent per cycle.

The analysis we have made makes it possible to calculate the electric field in the plasma and the amplitude of the electron motions. From Eqs. (16) and (19)

$$E_p/E_e = (K+1)/(K_p+1)$$

from which, using Eq. (20), we find

$$E_p/E_e = \frac{K - 1 - \theta(K+1)}{-2\theta} = \frac{1 - K^{-1} - \theta(1 + K^{-1})}{-2\theta K^{-1}}$$

for the ratio of the field strength in the plasma to that in the composite condenser. Substituting for K from Eq. (33) we have

$$E_p/E_e = \frac{1 - K_0^{-1} - \theta(1 + K_0^{-1}) - j\omega C_0 R(1+\theta)}{-2\theta(K_0^{-1} + j\omega C_0 R)} \tag{37}$$

or, in absolute magnitude

$$E_p/E_e = \frac{1}{2\theta}\left\{\frac{[1 - K_0^{-1} - \theta(1 + K_0^{-1})]^2 + [\omega C_0 R(1+\theta)]^2}{K_0^{-2} + (\omega C_0 R)^2}\right\}^{1/2} \tag{38}$$

Now

$$E_e = V/d$$

where V is the condenser voltage and $d(=7 \text{ cm})$ is the plate separation. Also

$$v = i_c Z_0 \tan l \tag{39}$$

where i_c is the condenser current, and

$$i_c = i_t \cos l \tag{40}$$

where i_t is the thermocouple current, whence it follows that,

$$E_e = (i_t Z_0/d) \sin l \tag{41}$$

The equation of motion of an electron on which Eq. (4) was based is

$$\ddot{\xi} + S\dot{\xi} = (eZ/m)\epsilon^{j\omega t}$$

whence the amplitude of motion is

$$\xi_0 = 2^{1/2} e E_p / m\omega^2 (1 + S^2/\omega^2)^{1/2}$$

on the understanding that E_p is the r.m.s. value. In practical units

$$\xi_0 = 2.50 \times 10^{15} E_p/\omega^2(1 + S^2/\omega^2)^{1/2}$$

and for the present case ($\omega = 1.194 \times 10^9$)

$$\xi_0 = 1.75 \times 10^{-3}(1 + S^2/\omega^2)^{-1/2} E_p.$$

Only at $i_A = 15$ m.a. does S/ω make an appreciable contribution to the radical so that throughout most of the range ξ_0 is proportional to E_p. ξ_0 is plotted in Fig. 14 and it is interesting to note that the greatest amplitude was 1/500 cm. For this amplitude the maximum velocity is $\omega/500 = 0.6 \times 10^6$ cm/sec corresponding to a voltage of $10^{-4}v$.

The valley in the ξ_0 curve between 21 and 42 m.a. certainly arises from the effect of the increased resistance near plasma resonance in decreasing the resonant current in the Lecher System.

The values of S/ω given in Fig. 14 are based on simultaneous measurements of tuning length and resistance. But on the assumption of a uniform plasma, the Lecher System characteristic is, by itself, capable of giving a value of S/ω by using either the maximum variation in L_r (Lecher System tuning length) or the change in i_A accompanying this variation. By setting the derivative of Eq. (27) with respect to η (assuming that variations in θ and ϕ are negligible) equal to zero, we obtain the condition for the extreme values of L_r:

$$\eta' = 2\{1 + S^2/\omega^2 \pm [(1 + S^2/\omega^2)^2 - (1 + S^2/\omega^2)]^{1/2}\}. \tag{42}$$

The difference between the values of η' at the extremes of L_r is, therefore,

$$\Delta\eta' = \phi(N_2 - N_1)/N_\omega = 4[(1 + S^2/\omega^2)^2 - (1 + S^2/\omega^2)]^{1/2} \tag{43}$$

Solving for S/ω, noting that $\Delta\eta'^2/4 \ll 1$,

$$S/\omega = \Delta\eta'/4. \tag{44}$$

At plasma resonance, theory (Eq. (17)) requires that

$$\eta' = 2 = \phi N_r/N_\omega \tag{45}$$

where N_r is the electron density at plasma resonance [7,5]. Combining Eqs. (43), (44), and (45) we have

$$S/\omega = (N_2 - N_1)/2N_r \tag{46}$$

or, in terms of arc current

$$S/\omega = (i_{A2} - i_{A1})/2i_{AR} \tag{47}$$

where i_{AR} is the arc current at plasma resonance. The other relation mentioned is

$$S/\omega = [\delta L_\infty/(L_1 - L_2)]\theta\phi_\infty/\theta_\infty\phi \tag{48}$$

and is deduced by eliminating η' between Eqs. (27) and (42).

Applying these to the present case, Fig. 13 gives,

$$S/\omega = (34.5 - 25.)/2 \times 31 = 0.15$$

in the one case and

$$S/\omega = [1.3/(15 - 6.5)] \times 0.79 = 0.12,$$

in the other. These values are roughly twice as large as the maximum values shown in Fig. 14. This fact together with the asymmetry of the resonance characteristic indicates that other factors may be important here. The analysis of Section II shows, for instance, that if frictional factors were absent, the whole spread $\Delta\eta'$ could be due to a 30 percent variation in electron density.

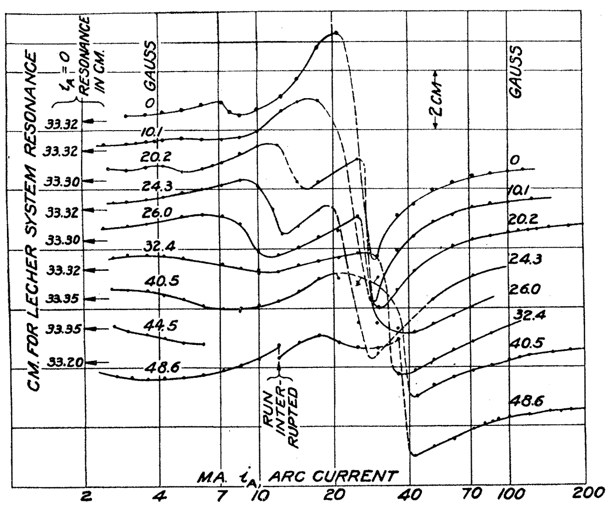

Fig. 15. Effect of transverse magnetic field on plasma-resonance.

IX. The Effect of a Magnetic Field

Appleton and Childs[21] have observed an absorption maximum arising from a constant magnetic field at right angles to the impressed electric field and Benner[23] has observed an accompanying change in dielectric constant. In neither of these cases was the electron density involved as a vital factor. Gutton,[7] however, observed a doubling of the resonance when he applied such a magnetic field to the ionized gas.

Resonance characteristics have been made of the mercury vapor plasma at a succession of magnetic fields. The field used was that from a pair of Helmholtz coils so placed that the oscillating portion of the positive column was in the uniform part of the field. The field was calculated from the coil dimensions as 8.1 gauss·amp.$^{-1}$. The characteristics obtained for a frequency of 1.605×10^9 herz are shown in Fig. 15. The dashed portions of the curves

[23] S. Benner, Die Naturwissenschaften, **17**, 120 (1929).

are those which are doubtful, either because experimental points were not obtained or because the points which were obtained were very uncertain on account of the flatness of the resonance curve. The splitting of the b-type resonance is clearly seen for magnetic field strengths from 20.2 gauss up, until at 48.6 gauss the curve begins to take on a more complicated form, due perhaps to the approach to the purely magnetic resonance of Appleton and Childs and Benner. For the frequency used this would occur with a magnetic field of $H = 2\pi mc\nu/e = 56.8$ gauss.

The expectation is that the two resonances will lie one to either side of the original. In the figure only the lower current resonance shows displacement for the smaller magnetic fields. This arises from the concentration of the arc by the magnetic field, which results in higher electron densities for the same arc current when the magnetic field is present.

In analyzing these curves we might proceed along the lines already developed, but more insight into the processes involved can be gained by examining the whole phenomena from a less formal point of view.[24]

Consider a uniform plasma bounded by the sheaths on two plane parallel non-conducting boundary walls. A uniform displacement of the electrons by an amount, ξ, throughout the plasma will develop a surface charge density of

$$\sigma_e = Ne\xi$$

at the sheath edge which exerts the restoring force

$$F_p = -4\pi Ne^2\xi$$

on each electron throughout the plasma. The electrons in the plasma are thus capable of free simple harmonic oscillations in accordance with the equation of motion

$$4\pi Ne^2\xi + m\ddot{\xi} = 0$$

whence the natural frequency is that found for plasma-electron oscillations, Eq. (1).

If the plasma is bounded by a cylinder, and the uniform parallel electron displacements are perpendicular to the axis, the restoring force is only

$$F_c = F_p/2 = -4\pi Ne^2\xi/2$$

and if the boundary is spherical

$$F_s = F_p/3.$$

The natural frequency is lowered in these cases, as has already been seen in the analysis of Section IV.

In proceeding to the case of superposed magnetic field, we may denote this shape factor by ϕ,

$$F = -4\pi Ne^2\phi.$$

[24] J. J. Thomson, Phil. Mag., **11**, 697 (1931) develops the same viewpoint along somewhat different lines.

This factor may be different in the ξ-direction from what it is at right angles to both this direction and the magnetic field. For instance, taking ξ perpendicular to infinite parallel plane boundaries, $\phi_\xi = 1$, while, introducing ζ for the electron displacement perpendicular to both ξ and H, $\phi_\zeta = 0$. In general, then, both shape factors must be included in the treatment. We have,

$$F_\xi = -4\pi Ne^2\phi_\xi\xi - (eH/c)\dot{\zeta} = m\ddot{\xi}$$

$$F_\zeta = -4\pi Ne^2\phi_\zeta\zeta + (eH/c)\dot{\xi} = m\ddot{\zeta}.$$

Putting $\xi = \xi_0\epsilon^{jt\omega}$, $\zeta = \zeta_0\epsilon^{j\omega t}$, $\omega_H = eH/mc = 1.767\times 10^7 H, \omega_0^2 = 4\pi Ne^2/m = 3.18 \times 10^9 N$ and eliminating ζ_0 and ξ_0, we have

$$(\omega^2 - \phi_\xi\omega_0^2)(\omega^2 - \phi_\zeta\omega_0^2) = \omega^2\omega_H^2,$$

as the condition for resonance. For a circular cylinder with axial magnetic field, $\phi_\xi = \phi_\zeta$ from symmetry, so that we may write

$$(\omega^2 - \phi\omega_0^2)^2 = \omega^2\omega_H^2.$$

Since N was varied in the experiment to obtain resonance, we solve for the two values of ω_0^2, say ω_1^2 and ω_2^2, which satisfy this equation. It is readily found that

$$(\omega_2^2 - \omega_1^2)/(\omega_2^2 + \omega_1^2) = \omega_H/\omega$$

so that, in terms of arc currents at resonance, i_{A1} and i_{A2},

$$(i_{A2} - i_{A1})/(i_{A1} + i_{A2}) = \omega_H/\omega.$$

Table II based on Fig. 15 shows the agreement between calculated and observed resonance behavior. Columns 4 and 5 should agree. The bad discrep-

TABLE II. *Analysis of magnetic effect.*

H, (gauss)	0	10.1	20.2	24.3	26.0	32.4	40.5	44.5	48.6
i_{A1}, (m.a.)	25	22?	13	11	9	7?	4.5?	3.5?	2.
i_{A2}, (m.a.)			27	22	27	32	37		37?
$\dfrac{i_{A2}-i_{A1}}{i_{A2}+i_{A1}}$			0.35	0.33	0.50	0.64	0.78		0.90
ω_H/ω			0.35	0.43	0.46	0.57	0.71		0.85

ancy at 24.3 gauss is doubtless due to an erroneous determination of the 25-m.a. resonance on that curve, since this curve fails to fall in line with thetrend of all the others here. Otherwise the agreement is reasonably good.