APPLICATION OF SPINOR ANALYSIS TO THE MAXWELL AND DIRAC EQUATIONS

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Abstract

With the spinor analysis developed by B. van der Waerden which comprises all representations of the Lorentz group, even those not contained in ordinary tensor calculus, one is able to write all derivations and equations in an *automatically covariant form.* For the convenient translation into spinor language of the Maxwell equations, it becomes important to introduce three self-dual tensors, one representing the electromagnetic field, one corresponding to the Hertzian vector, and one representing a kind of current potential. These correspond to symmetric spinors of the 2nd rank. Many spinor equations thus become simpler than the corresponding tensorial equations, especially the expression for the stress energy tensor. From the 1st order Dirac equations in spinor form, as given by v.d. Waerden, we derived the 2nd order equation, which agrees with the Gordon-Klein form but for a correction term which again contains the self-dual field tensor. Further the expression for the current was derived, and its decomposition into conduction and polarization currents, and both Maxwell and Dirac equations were derived from a spinorial variation principle, analogous to the results of Gordon and Darwin. In addition to the divergence condition for the current three new invariant relations between the wave functions which are independent of the potentials were obtained (Chapter III, Eqs. (11), (12) and (13)).

Introduction

The Dirac equations of the electron have for the first time furnished an example of a system of equations, which show an invariance of form when subjected to a Lorentz transformation, but which only very artificially could be written with tensors. This difficulty was felt especially by Darwin[1] when he wrote: "The relativity theory is based on nothing but the idea of invariance, and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist, which it would be, to say the least, very artificial and inconvenient to express as tensors."

The Dirac equation is

$$[(\Gamma^k p_k) + mc]\psi = 0 \tag{1}$$

where p_k represents the 4-vector[2]

$$p_k = \frac{h}{i}\frac{\partial}{\partial x^k} + \phi_k \tag{2}$$

[1] C. G. Darwin, Proc. Roy. Soc. **118**, 657 (1928).

[2] We use the Dirac h, which is $1/2\pi$ times the Planck constant, and write here; and in Ch. III, ϕ_k for e/c times the ordinary four potential.

and Γ^k are four four-row matrices, given by:[3]

$$\Gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$
$$\Gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \Gamma^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3}$$

ψ is a function of two kinds of variables, of x^i and an inner variable, which can only assume four discrete values; the p_k act upon the x^k while the Γ^k operate on the inner variable. This way of describing Eq. (1) is of course equivalent to regarding Eq. (1) as *four* equations containing 4 ψ's, each of them a function of x^k only.

There are *two* points of view possible, with regard to the transformation properties of the Dirac equation: The *first* of these regards the Γ^k as a matrix four vector, and the ψ as constant. According to the *second* point of view, the Γ^k are constants and the ψ are being transformed.

One can easily show that these two methods are equivalent. For if we subject the coordinates x^k to a Lorentz transformation,

$$x = x'L$$

then the p_k will be transformed like

$$p = L^{-1}p'.$$

If we consider Γ^k as a four vector, it will transform like x^k, and the scalar product

$$(\Gamma p) = (\Gamma' L L' p') = (\Gamma' p')$$

will remain invariant. On the other hand, we can always find a matrix S such that

$$p = L^{-1}p' = S^{-1}p'S. \tag{4}$$

Now the Γ^k are kept constant, and since they act on the inner variable only, they are exchangeable with S, so that

$$(S^{-1}\Gamma p'S)\psi + mc\psi = 0$$

or

$$(\Gamma p')(S\psi) + mc(S\psi) = 0.$$

Putting

$$\psi' = S\psi \tag{5}$$

[3] When written out, Eq. (1) agrees with the form given by Weyl, Gruppentheorie and Quantenmechanik, Leipzig 1928, page 171, Eq. (45'). If the above given matrices for Γ^1, Γ^2, Γ^3 are divided by i, they become identical to Weyl's Γ_1, Γ_2, Γ_3; our Γ^4 is equal to his Γ_0.

we regain (1), now with transformed ψ. This point of view regards the Γ^k simply as determining the coefficients 0, ± 1, $\pm i$ in the four wave equations.

For special two dimensional Lorentz transformations, Darwin has written out the transformation (5). They have the peculiar character that the coefficients contain the angle $\theta/2$, when the coordinates are rotated by the angle θ. We can see already from (4) that S is something like a square root of L.[4] This shows that ψ cannot be a tensor, and that there are equations, which defy translation into tensor language and yet fulfill the relativity principle. Now it has been known to the mathematicians that the ordinary tensor language does not comprise all possible representations of the Lorentz group as had always been assumed tacitly by the physicists.[5] The necessary extension of the tensor calculus, the *spinor calculus*, was given by B. van der Waerden,[6] upon instigation of Ehrenfest, and indeed gives all possible representations.

These two points of view correspond in a certain way to the particle and wave description of the electron. The first may give additional information concerning the particle velocity,[7] the second, however, is necessary for the consideration of the Dirac equations as field equations. In this case the ψ must naturally be transformed to a new coordinate system just as E and H in Maxwell's equations. We shall restrict ourselves to this point of view.

Chapter I. The Mathematical Apparatus of the Spinor Analysis

§1. Since van der Waerden's article is not very easily accessible, and in order to make the spinor analysis more popular, we shall briefly develop the few necessary theorems and formulae here, following van der Waerden closely.

Consider the following binary transformation

$$\begin{aligned} \xi_1' &= \alpha_{11}\xi_1 + \alpha_{12}\xi_2 \\ \xi_2' &= \alpha_{21}\xi_1 + \alpha_{22}\xi_2 \end{aligned} \tag{1}$$

and its complex conjugate:

$$\begin{aligned} \bar{\xi}_1' &= \bar{\alpha}_{11}\bar{\xi}_1 + \bar{\alpha}_{12}\bar{\xi}_2 \\ \bar{\xi}_2' &= \bar{\alpha}_{21}\bar{\xi}_1 + \bar{\alpha}_{22}\bar{\xi}_2 \end{aligned} \tag{2}$$

with the determinant

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = 1. \tag{3}$$

All these transformations form a group of $8-2=6$ parameters. Any two numbers transforming like the ξ_1, ξ_2 in (1), we shall call a spinor of the 1st rank, and denote by

[4] Landau therefore calls ψ a half vector.

[5] Compare especially Hermann Weyl, Gruppentheorie und Quantenmechanik, Leipzig, 1928. Kap. III.

[6] B. van der Waerden, Göttinger Nachrichten 1929, page 100. The spinor formalism is already implicitly contained in the book of Weyl, and in a paper by V. Fock, Zeits. f. Physik **57,** 261 (1929).

[7] V. Fock, Zeits. f. Physik **55,** 127 (1929); G. Breit, Proc. Natl. Acad. **14,** 553 (1928); E. Schrödinger, Sitzungsber. Berliner Akad. **24,** 418 (1930).

$$a_k \qquad k = 1, 2$$

whereas any two quantities transforming like (2) will be written

$$b_{\dot{r}} \qquad \dot{r} = \dot{1}, \dot{2}.$$

Any four quantities transforming like the products $\xi_1\xi_1$, $\xi_1\xi_2$, $\xi_2\xi_1$, $\xi_2\xi_2$ we call a spinor of the 2nd rank and write

$$a_{kl} \qquad k, l = 1, 2.$$

Correspondingly 4 quantities transforming like the products of $\bar{\xi}_1$ and $\bar{\xi}_2$ we denote by

$$b_{\dot{r}\dot{s}} \qquad \dot{r}, \dot{s} = \dot{1}, \dot{2}.$$

There are also "mixed" spinors of the 2nd rank transforming like a product of a barred ξ and an unbarred ξ, denoted by

$$c_{\dot{r}k} \qquad \dot{r} = \dot{1}, \dot{2}; \; k = 1, 2.$$

Analogously we can define spinors of higher rank, like $a_{\dot{r}\dot{m}l}$.

One can show easily, that because of (3) the area of the parallelogram formed by two spinors ξ_k and η_l

$$\xi_1\eta_2 - \xi_2\eta_1$$

is invariant under transformation (1). This enables us to introduce *contravariant* spinors a^k or $b^{\dot{r}}$, according to

$$\boxed{\begin{array}{ll} a^1 = a_2 & b^{\dot{1}} = b_{\dot{2}} \\ a^2 = -a_1 & b^{\dot{2}} = -b_{\dot{1}} \end{array}} \tag{4}$$

because in this way the scalar products

$$a_1c^1 + a_2c^2 = a_\lambda c^\lambda$$
$$b_{\dot{1}}d^{\dot{1}} + b_{\dot{2}}d^{\dot{2}} = b_{\dot{\rho}}d^{\dot{\rho}}$$

are invariant.[8] When we establish the usual connection between covariant and contravariant indices by means of a spinor ϵ^{kl} according to

$$a^k = \epsilon^{k\lambda}a_\lambda \tag{5}$$

we find

$$\epsilon^{kl} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon_{kl} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

and

$$\epsilon^{k\lambda}\epsilon_{l\lambda} = -\delta_l{}^k \tag{6}$$

where

[8] Summation signs are as usual suppressed; dummy indices are always given Greek letters, free indices Latin letters.

$$\delta_l{}^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As in the usual tensor algebra, the only covariant operations are also here *multiplication* and *contraction*. For instance from the spinors $a_{\dot{r}\dot{s}}{}^l$ and $b^{\dot{m}}{}_{kt}$ we can form the spinor of the 6th rank

$$c_{\dot{r}\dot{s}}{}^{l\dot{m}}{}_{kt} = a_{\dot{r}\dot{s}}{}^l b^{\dot{m}}{}_{kt}$$

or the spinor of 4th rank

$$c_{\dot{s}}{}^l{}_{kt} = a_{\dot{\mu}\dot{s}}{}^l b^{\dot{\mu}}{}_{kt}$$

or the spinor of the 2nd rank

$$c_{\dot{s}t} = a_{\dot{\mu}\dot{s}}{}^{\lambda} b^{\dot{\mu}}{}_{\lambda t}.$$

The following two rules are essential in calculations. According to (6) we have

$$\boxed{a_\lambda b^\lambda = -\,a^\lambda b_\lambda} \tag{7}$$

An immediate consequence of this is that any spinor of odd rank has absolute value zero

$$a_\lambda a^\lambda = 0; \quad a^{\lambda\mu\nu} a_{\lambda\mu\nu} = 0. \tag{7a}$$

Similarly we have

$$\boxed{a^\lambda b_\lambda c_m + a_\lambda b_m c^\lambda + a_m b^\lambda c_\lambda = 0}\,. \tag{8}$$

Of course this rule may also be applied to the product $a^\lambda c_{\lambda m}$. It also holds for three dotted indices.

There are two more rules concerning the relations of dotted and undotted indices.

1° It is not necessary to fix the position of dotted and undotted indices belonging to the same spinor. Thus, for instance

$$a_{\dot{r}lt} = a_{l\dot{r}t} = a_{lt\dot{r}}.$$

On the other hand two dotted or two undotted indices are not necessarily interchangeable; if they are, the spinor in question has special symmetry properties.

2° The complex conjugate of any spinor equation is obtained by replacing all undotted indices by dotted ones and *vice versa*.

§2. We will now establish the connection between spinors and world tensors. We assert that the following combinations of components of a second rank spinor $a_{\dot{s}t}$ are to be associated with the components of a world vector A^1, A^2, A^3, A^4 as follows

$$\left.\begin{aligned} \tfrac{1}{2}(a_{\dot{2}1} + a_{\dot{1}2}) &= A^1 = A_1 \\ \frac{1}{2i}(a_{\dot{2}1} - a_{\dot{1}2}) &= A^2 = A_2 \\ \tfrac{1}{2}(a_{\dot{1}1} - a_{\dot{2}2}) &= A^3 = A_3 \\ \tfrac{1}{2}(a_{\dot{1}1} + a_{\dot{2}2}) &= A^4 = -A_4 \end{aligned}\right\} \tag{9}$$

These combinations are real, and after a transformation (1) and (2) they will still be real; thus their transformation coefficients are real also. To prove that it is a Lorentz transformation we solve (9) for the $a_{\dot{s}t}$ and obtain using (4)

$$\left.\begin{aligned} a_{\dot{2}1} &= -a^{\dot{1}2} = A^1 + iA^2 = A_1 + iA_2 \\ a_{\dot{1}2} &= -a^{\dot{2}1} = A^1 - iA^2 = A_1 - iA_2 \\ a_{\dot{1}1} &= +a^{\dot{2}2} = A^3 + A^4 = A_3 - A_4 \\ -a_{\dot{2}2} &= -a^{\dot{1}1} = A^3 - A^4 = A_3 + A_4 \end{aligned}\right\}. \tag{10}$$

Now it is easily verified that

$$-\tfrac{1}{2}a_{\dot{\sigma}\tau}a^{\dot{\sigma}\tau} = A_\gamma A^\gamma. \tag{11}$$

To every transformation (1) there corresponds one Lorentz transformation; *vice versa*, since the connection formulae between the transformation coefficients of A and the α's of eq. (1) are quadratic, there are two transformations (1) differing in sign, corresponding to one Lorentz transformation.[9] Therefore (1) and (2) form a representation of the Lorentz group. It can be proved, and this is the fundamental theorem of the spinor analysis, that one obtains all representations of the Lorentz group by transforming all possible spinors according to (1) and (2). It follows that the true "quantities" belonging to the Lorentz group are spinors, of which tensors form only a special class. Analogous to (10) spinors of the fourth rank with two dotted and two un-

[9] For example using the transformation formulae (1) and (2) for a spinor $a_{\dot{r}l}$ we find that for a Lorentz transformation

$$\begin{aligned} ct' &= ct\cosh\theta + \sinh\theta \\ z' &= ct\sinh\theta + z\cosh\theta \end{aligned}$$

there correspond the two transformation matrices

$$\alpha_{kl} = \begin{pmatrix} \pm e^{\theta/2} & 0 \\ 0 & \pm e^{-\theta/2} \end{pmatrix}$$

and with a special rotation

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

there correspond the two matrices

$$\alpha_{kl} = \begin{pmatrix} \pm e^{-i\theta/2} & 0 \\ 0 & \pm e^{+i\theta/2} \end{pmatrix}$$

For general spacial rotations we see that because of the invariance of t, that is to say of $\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2$ the corresponding binary transformation (1) will be unitarian. For more information compare especially H. Weyl, Gruppentheorie und Quantentheorie, Leipzig 1928, p. 106–114.

dotted indices correspond to world tensors of the second rank. The formulae connecting them are obtained from (10) by multiplication. For instance

$$\left.\begin{aligned} a_{\dot{2}\dot{2}11} &= a^{\dot{1}\dot{1}22} = A^{11} - A^{22} + i(A^{12} + A^{21}) = A_{11} - A_{22} + i(A_{12} + A_{21}) \\ -\, a_{\dot{2}\dot{2}21} &= a^{\dot{1}\dot{1}12} = A^{31} - A^{41} + i(A^{32} - A^{42}) = A_{31} + A_{41} + i(A_{32} + A_{42}) \end{aligned}\right\} \quad (12)$$

In the following table all possible kinds of spinors of the 5 lowest ranks are written down and those corresponding to world tensors are underscored.

$$\begin{array}{ccccccccc} & & & & \underline{a} & & & & \\ & & & a_k & & a_{\dot{m}} & & & \\ & & a_{kl} & & \underline{a_{\dot{m}l}} & & a_{\dot{m}\dot{n}} & & \\ & a_{klt} & & a_{\dot{m}lt} & & a_{\dot{m}\dot{n}t} & & a_{\dot{m}\dot{n}\dot{r}} & \\ a_{kltv} & & a_{\dot{m}ltv} & & \underline{a_{\dot{m}\dot{n}tv}} & & a_{\dot{m}\dot{n}\dot{r}v} & & a_{\dot{m}\dot{n}\dot{r}\dot{s}} \end{array}$$

§3. Although the underscored spinors correspond *directly* to tensors of half their rank, the *spinors of even rank* can be related to tensors of higher rank, which possess certain symmetry properties. Let us consider the simplest spinors of this kind a_{kl} and its complex conjugate $a_{\dot{m}\dot{n}}$. We decompose a_{kl} into a symmetric and antisymmetric spinor, according to

$$a_{kl} = \tfrac{1}{2}(a_{kl} + a_{lk}) + \tfrac{1}{2}(a_{kl} - a_{lk}) = \sigma_{kl} + \alpha_{kl}. \quad (13)$$

The antisymmetric part α_{kl} has only one "Kennzahl"

$$\begin{aligned} \alpha_{12} &= -\,\alpha_{21} = \tfrac{1}{2}(a_{12} - a_{21}) = \tfrac{1}{2}(a_1{}^1 + a_2{}^2) \\ &= \tfrac{1}{2}a_\rho{}^\rho \end{aligned}$$

which is an invariant. The symmetric part σ_{kl} has three "Kennzahlen" and can be shown to correspond to an *antisymmetric self-dual tensor.* Consider a real antisymmetric world tensor F^{kl}. The dual tensor to this is obtained by means of

$$\overset{*}{F}_{kl} = \pm \frac{i}{2}\,\delta_{kl\alpha\beta}F^{\alpha\beta} \quad (14)$$

where $\delta_{klmn} = 0$ when any two indices are equal and $= \pm 1$ according to whether the indices form an even or odd permutation of the numbers 1234. Thus two dual tensors F^{*kl} are possible to an originally real tensor F^{kl}; they differ only by the sign, so that one is the complex conjugate of the other. Obviously the dual to F^{*kl} is again the original F^{kl} without asterisk. The self-dual tensor G^{kl} is the sum of the two[10]

$$\begin{aligned} G^{kl} &= F^{kl} + F^{*kl} \\ \overline{G}^{kl} &= F^{kl} - F^{*kl}. \end{aligned} \quad (15)$$

Calling the three Kennzahlen k_1, k_2, k_3, G^{kl} may be written

[10] We note the theorem, that for any two self-dual tensors G^{kl} and H^{kl} the product

$$G^{k\lambda}\overline{H}_{k\lambda} \equiv 0$$

$$G^{kl} = \begin{pmatrix} 0 & k_3 & -k_2 & -ik_1 \\ -k_3 & 0 & k_1 & -ik_2 \\ k_2 & -k_1 & 0 & -ik_3 \\ ik_1 & ik_2 & ik_3 & 0 \end{pmatrix} \tag{15a}$$

or taking the lower sign in (14)

$$\bar{G}^{kl} = \begin{pmatrix} 0 & \bar{k}_3 & -\bar{k}_2 & i\bar{k}_1 \\ -\bar{k}_3 & 0 & \bar{k}_1 & i\bar{k}_2 \\ \bar{k}_2 & -\bar{k}_1 & 0 & i\bar{k}_3 \\ -i\bar{k}_1 & -i\bar{k}_2 & -i\bar{k}_3 & 0 \end{pmatrix}. \tag{15b}$$

Then we form the spinor of the fourth rank $g_{\dot{r}\dot{s}kl}$ according to (12) which using G^{kl} can be written

$$g_{\dot{r}\dot{s}k}{}^{l} = g_{\dot{r}\dot{s}}\delta_k{}^l \tag{16a}$$

and which using $\bar{G}^{kl}$ can be written

$$g_{\dot{r}}{}^{\dot{s}}{}_{kl} = \delta_{\dot{r}}{}^{\dot{s}} g_{kl}. \tag{16b}$$

Here the spinors $g_{\dot{r}\dot{s}}$ and g_{kl} are *symmetric*. Solving (16) and (16a) we obtain

$$\left.\begin{aligned} g_{\dot{r}\dot{s}} &= \tfrac{1}{2} g_{\dot{r}\dot{s}\lambda}{}^{\lambda} \\ g_{kl} &= \tfrac{1}{2} g_{\dot{\sigma}}{}^{\dot{\sigma}}{}_{kl} \end{aligned}\right\}. \tag{17}$$

The formulae connecting the components of the symmetric spinors with those of the antisymmetric, self-dual world-tensor are

$$\left.\begin{aligned} g_{\dot{1}\dot{1}} &= 2(k_2 + ik_1) \\ g_{\dot{2}\dot{2}} &= 2(k_2 - ik_1) \\ g_{\dot{1}\dot{2}} = g_{\dot{2}\dot{1}} &= -2ik_3 \end{aligned}\right\} \tag{18}$$

and using (16a) we simply obtain the complex conjugate

$$\left.\begin{aligned} g_{11} &= 2(\bar{k}_2 - i\bar{k}_1) \\ g_{22} &= 2(\bar{k}_2 + i\bar{k}_1) \\ g_{12} = g_{21} &= 2i\bar{k}_3 \end{aligned}\right\}. \tag{18a}$$

§4. Corresponding to the introduction of a covariant gradient vector we now define a gradient spinor $\partial_{\dot{s}t}$ according to the connection formulae (10) as follows

$$\left.\begin{aligned} \partial^{\dot{1}}{}_1 = \partial_{\dot{2}1} &= \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} \\ -\partial^{\dot{2}}{}_2 = \partial_{\dot{1}2} &= \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ -\partial^{\dot{2}}{}_1 = \partial_{\dot{1}1} &= \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^4} \\ -\partial^{\dot{1}}{}_2 = -\partial_{\dot{2}2} &= \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4} \end{aligned}\right\} \tag{19}$$

where the contravariant vector (x^1, x^2, x^3, x^4) corresponds to (x, y, z, ct). Analogously to (11) we may now translate the familiar vector analytical operations like Div or $\Box$ into the spinor language. We have

$$\frac{\partial \phi^\alpha}{\partial x^\alpha} = -\frac{1}{2}\,\partial_{\dot{\sigma}\tau}\phi^{\dot{\sigma}\tau} \tag{20}$$

and

$$\frac{\partial^2 S}{\partial x_\alpha \partial x^\alpha} = -\frac{1}{2}\,\partial_{\dot{\sigma}\tau}\partial^{\dot{\sigma}\tau}s. \tag{21}$$

Chapter II. Maxwell Equations in Spinor Form

§5α. To facilitate the comparison, we shall briefly recapitulate the Maxwell equations in the ordinary tensor form. As usual we define the antisymmetric field tensor

$$F^{kl} = \begin{pmatrix} 0 & H_z & -H_y & -E_x \\ & 0 & H_x & -E_y \\ & & 0 & -E_z \\ & & & 0 \end{pmatrix} \tag{1}$$

then the dual tensor according to (14) is

$$F^{*kl} = -i\begin{pmatrix} 0 & E_z & -E_y & H_x \\ & 0 & E_x & H_y \\ & & 0 & H_z \\ & & & 0 \end{pmatrix} \tag{2}$$

The two Maxwell equations are

$$\frac{\partial F^{k\lambda}}{\partial x^\lambda} = S^k \tag{3}$$

$$\frac{\partial F^{*k\lambda}}{\partial x^\lambda} = 0$$

where S^k is the four-current $(\rho v_x/c,\ \rho v_y/c,\ \rho v_z/c,\ \rho)$ obeying the continuity equation

$$\frac{\partial S^\lambda}{\partial x^\lambda} = 0. \tag{5}$$

We can embody these two equations into one

$$\frac{\partial G^{k\lambda}}{\partial x^\lambda} = S^k \tag{6}$$

by using the self-dual tensor

$$G^{kl} = F^{kl} + F^{*kl} \tag{7}$$

which has the form (15), with[11]

$$\vec{k} = \vec{H} - i\vec{E}. \tag{8}$$

We can derive G^{kl} from the four-potential $\phi^k = (A_x, A_y, A_z, \phi)$ by means of

$$G^{kl} = \frac{\partial \phi^l}{\partial x_k} - \frac{\partial \phi^k}{\partial x_l} \cdot \tag{9}$$

The ϕ^k are subjected to the accessory condition

$$\frac{\partial \phi^\lambda}{\partial x^\lambda} = 0. \tag{10}$$

Introducing (9) into (6), using (10) we get the wave equation for ϕ^k

$$\frac{\partial^2 \phi^k}{\partial x_\lambda \partial x^\lambda} = S^k. \tag{11}$$

§5β. To free ourselves from the condition (10) imposed on the potentials, we express ϕ^k in terms of a self-dual antisymmetric tensor Z^{kl}, the analogue to the Hertzian vector in three dimensions by means of[12]

$$\phi^k = \frac{\partial Z^{k\lambda}}{\partial x^\lambda} \cdot \tag{12}$$

Introducing this in (11), we get for Z^{kl} a third order differential equation

$$\frac{\partial}{\partial x^\lambda} \frac{\partial^2 Z^{k\lambda}}{\partial x_\alpha \partial x^\alpha} = S^k.$$

In the three dimensional form one reduces this to a second order differential equation by an integration of the current $\rho v/c$ with respect to the time.[13] The analogue of this is the derivation of S^k from a stream potential Q^{kl}, which is again an antisymmetric self-dual tensor, according to

[11] Comp. Riemann-Weber, Die partiellen Differentialgleichungen der Physik, Braunschweig 1901, Vol. II, 348. L. Silberstein, Ann. d. Physik **22**, 579 (1907). See also F. Zerner, Handbuch der Physik vol. XII, p. 93.

[12] When we represent Z^{kl} by the scheme

$$Z^{kl} = i\begin{pmatrix} 0 & Z_z & -Z_y & iZ_x \\ & 0 & Z_x & iZ_y \\ & & 0 & iZ_z \\ & & & 0 \end{pmatrix}$$

then the three dimensional form of (12) is

$$A = i \operatorname{curl} Z - \frac{1}{c}\frac{\partial Z}{\partial t}$$

$$\phi = \operatorname{div} Z.$$

[13] See e.g. Madelung, Die mathematischen Hilfsmittel des Physikers, Berlin 1922, p. 196.

$$S^k = \frac{\partial Q^{k\lambda}}{\partial x^\lambda} \cdot \tag{13}$$

This causes the continuity equation (5) to be identically satisfied. For Z^{kl} we then get the wave equation

$$\frac{\partial^2 Z^{kl}}{\partial x_\alpha \partial x^a} = Q^{kl}. \tag{14}$$

§5γ. The stress-energy tensor $T_k{}^l$ whose divergence is equal to the components of the four force can be written

$$T_k{}^l = \tfrac{1}{2}(F_{k\alpha}F^{l\alpha} - F^*{}_{k\alpha}F^{*l\alpha}).$$

This can be written in terms of our tensor G_{kl}

$$T_k{}^l = \tfrac{1}{4}(G_{k\alpha}\overline{G}^{l\alpha} + \overline{G}_{k\alpha}G^{l\alpha}) \tag{15}$$

where the conjugate tensor is to be formed according to Eq. (15) of Chapter I. As is well known this tensor has a diagonal sum which is zero

$$T_\alpha{}^\alpha = 0 \tag{16}$$

as one sees using the theorem of footnote 10. Consequently $T_k{}^l$ has only nine linearly independent "Kennzahlen."

§5δ. It is important for the sequel briefly to discuss the phenomenological form of the Maxwell equations in matter. Since the dielectric displacment D and the magnetic induction B are connected with E and H through

$$\begin{aligned} D &= E + P \\ B &= H + I \end{aligned} \tag{17}$$

where P and I are the electric and magnetic polarization respectively, we can write the Maxwell equations

$$\left.\begin{aligned} \operatorname{div} E &= \rho' & \qquad \operatorname{div} B &= 0 \\ \operatorname{curl} B - \frac{1}{c}\frac{\partial E}{\partial t} &= \frac{J}{c} & \qquad \operatorname{curl} E + \frac{1}{c}\frac{\partial B}{\partial t} &= 0 \end{aligned}\right\} \tag{18}$$

where

$$\begin{aligned} \rho' &= \rho - \operatorname{div} P \\ \frac{J}{c} &= \frac{\rho v}{c} + \operatorname{curl} I + \frac{1}{c}\frac{\partial P}{\partial t} \cdot \end{aligned} \tag{19}$$

If ρ, $(\rho v/c)$, P and I are given, the above system agrees formally with the equations in vacuo. To write (18) and (19) in four dimensional form we introduce the two self-dual tensors G'^{kl} and M^{kl}. They have the form (15a) where

$$\begin{aligned} \vec{k}_{G'} &= \vec{B} - i\vec{E} \\ \vec{k}_M &= \vec{I} + i\vec{P}. \end{aligned} \tag{20}$$

We then write (18) and (19)

$$\left.\begin{aligned} &\frac{\partial G'^{k\lambda}}{\partial x^\lambda} = J^k \\ \text{where} \qquad &J^k = S^k + \frac{1}{2}\left(\frac{\partial M^{k\lambda}}{\partial x^\lambda} + \frac{\partial \overline{M}^{k\lambda}}{\partial x^\lambda}\right) \end{aligned}\right\} \tag{21}$$

§6α. Having been able to write the entire formalism of Maxwell equations with the help of the self-dual world tensor G^{kl} and its complex conjugate, we can now introduce by means of formulae (18) and (18a) two symmetric spinors $g_{\dot{m}\dot{n}}$ and g_{kl} of the second rank, and thus avoid the introduction of fourth-rank spinors altogether. It is obvious that the spinor g_{kl}, just like the tensor $\overline{G}^{kl}$, will only be needed in the formulation of the stress-energy tensor.

We know from Chapter II, that the four-current S^k becomes a spinor $s_{\dot{m}l}$, the gradient $\partial/\partial x^k$ according to (19) a spinor operator $\partial_{\dot{m}l}$; we see now that the simplest way of connecting $\partial_{\dot{m}l}$ and $s_{\dot{m}l}$ in a manner analogous to (6), is by letting $\partial_{\dot{m}l}$ act on a spinor with two dotted or two undotted indices. We choose the former and write Maxwell equations[14]

$$\partial^{\dot{\rho}}{}_l g_{\dot{\rho}\dot{m}} = 2s_{\dot{m}l}. \tag{6a}$$

The continuity equation for the four current reads

$$\partial^{\dot{\mu}\lambda} s_{\dot{\mu}\lambda} = 0. \tag{5a}$$

The identity of this with (5) was already noted in Chapter I formula (20).

The analogue of the four potential ϕ^k will be the spinor $\phi_{\dot{m}l}$. It is connected with the field spinor $g_{\dot{r}\dot{s}}$ by the following curl-like operation, which, however, in this case is symmetrical:

$$g_{\dot{r}\dot{s}} = \partial_{\dot{r}\lambda}\phi_{\dot{s}}{}^\lambda + \partial_{\dot{s}\lambda}\phi_{\dot{r}}{}^\lambda \tag{9a}$$

The $\phi_{\dot{m}l}$ are subjected to the divergence condition

$$\partial^{\dot{\mu}\lambda}\phi_{\dot{\mu}\lambda} = 0. \tag{10a}$$

Introducing (9a) into (6a) we have

$$\partial^{\dot{\rho}}{}_l(\partial_{\dot{\rho}\alpha}\phi_{\dot{m}}{}^\alpha + \partial_{\dot{m}\alpha}\phi_{\dot{\rho}}{}^\alpha) = 2s_{\dot{m}l}.$$

Using (8) Chapter I, we can transform each term according to

$$\begin{aligned} \partial^{\dot{\rho}}{}_l\partial_{\dot{\rho}\alpha}\phi_{\dot{m}}{}^\alpha &= -\,\partial^{\dot{\rho}}{}_\alpha\partial_{\dot{\rho}}{}^\alpha\phi_{\dot{m}l} - \partial^{\dot{\rho}\alpha}\partial_{\dot{\rho}\alpha}\phi_{\dot{m}l} \\ \partial^{\dot{\rho}}{}_l\partial_{\dot{m}\alpha}\phi_{\dot{\rho}}{}^\alpha &= -\,\partial_{\dot{m}l}\partial_{\dot{\rho}\alpha}\phi^{\dot{\rho}\alpha} - \partial_{\dot{\rho}l}\partial^{\dot{\rho}}{}_\alpha\phi_{\dot{m}}{}^\alpha. \end{aligned}$$

[14] For $l=1$, $\dot{m}=\dot{1}$ and for $l=2$, $\dot{m}=\dot{2}$ we have

$$\begin{aligned} \partial^{\dot{1}}{}_1 g_{\dot{1}\dot{1}} + \partial^{\dot{2}}{}_1 g_{\dot{2}\dot{1}} &= 2s_{\dot{1}1} \\ \partial^{\dot{1}}{}_2 g_{\dot{1}\dot{2}} + \partial^{\dot{2}}{}_2 g_{\dot{2}\dot{2}} &= 2s_{\dot{2}2} \end{aligned}$$

With (4), (10), (19), (18) of Ch. I, and (8) Chapter II, this becomes:

$$\begin{aligned} \left(\operatorname{curl}_z H - \frac{1}{c}\dot{E}_z + \operatorname{div} E\right) - i\left(\operatorname{curl}_z E + \frac{1}{c}\dot{H}_z - \operatorname{div} H\right) &= \frac{\rho v_z}{c} + \rho \\ \left(-\operatorname{curl}_z H + \frac{1}{c}\dot{E}_z + \operatorname{div} E\right) + i\left(\operatorname{curl}_z E + \frac{1}{c}\dot{H}_z + \operatorname{div} H\right) &= \frac{\rho v_z}{c} + \rho \end{aligned}$$

from which follow four of the eight Maxwell equations.

Adding these, the last two terms cancel because of (7) Chapter I, and due to (10a), we get

$$\partial_{\dot{\rho}\alpha}\partial^{\dot{\rho}\alpha}\phi_{\dot{m}l} = 2s_{\dot{m}l}. \tag{11a}$$

§6β. The analogue of the self-dual Hertzian tensor Z^{kl} is the symmetric Hertzian spinor $z_{\dot{r}\dot{s}}$. The potential spinor $\phi_{\dot{m}l}$ is derived from this according to

$$\phi_{\dot{m}l} = \partial^{\dot{\sigma}}{}_{l}z_{\dot{m}\dot{\sigma}}. \tag{12a}$$

By this operation the divergence condition (10a) is identically fulfilled, for we obtain, introducing (12a) into (10a)

$$\partial^{\dot{\mu}\lambda}\phi_{\dot{\mu}\lambda} = \partial^{\dot{\mu}\lambda}\partial^{\dot{\sigma}}{}_{\lambda}z_{\dot{\mu}\dot{\sigma}}$$

using the symmetry of $z_{\dot{\mu}\dot{\sigma}}$ we have

$$\partial^{\dot{\mu}\lambda}\phi_{\dot{\mu}\lambda} = \tfrac{1}{2}\left\{\partial^{\dot{\mu}\lambda}\partial^{\dot{\sigma}}{}_{\lambda} + \partial^{\dot{\sigma}\lambda}\partial^{\dot{\mu}}{}_{\lambda}\right\}z_{\dot{\mu}\dot{\sigma}} \equiv 0.$$

This vanishes applying (7) Chapter I to the index λ. Analogous to the stream potential Q_{kl}, a self-dual tensor we now derive the stream spinor $s_{\dot{m}l}$ from

$$s_{\dot{m}l} = \partial^{\dot{\sigma}}{}_{l}q_{\dot{m}\dot{\sigma}} \tag{13a}$$

where $q_{\dot{m}\dot{s}}$ is again a symmetric spinor of the second rank. Introducing (12a) and (13a) into (11a), and dropping a ∂ operator on both sides, we obtain the wave equation for $z_{\dot{m}\dot{s}}$

$$\partial_{\dot{\rho}\alpha}\partial^{\dot{\rho}\alpha}z_{\dot{m}\dot{s}} = 2q_{\dot{m}\dot{s}}. \tag{14a}$$

§6γ. The spinor analogue of the stress-energy tensor T_{kl} will be a spinor of the fourth rank with two dotted and two undotted indices. Its divergence will have to be equal to the spinor $f_{\dot{m}l}$ which corresponds to the four vector of the Lorentz force density and the action density. To derive this expression we write down the Maxwell equations and their conjugates according to (6a), using (7) Chapter I

$$\partial_{\dot{\rho}l}g^{\dot{\rho}\dot{m}} = -2s^{\dot{m}}{}_{l}$$
$$\partial_{\dot{r}\lambda}g^{\lambda k} = -2s_{\dot{r}}{}^{k}.$$

Then we multiply the upper equation with g^{lk} and the lower with $g^{\dot{r}\dot{m}}$ and contract with respect to r and l. After adding we can write this

$$\partial_{\dot{\rho}\lambda}\left\{g^{\dot{\rho}m}g^{\lambda k}\right\} = -2\left\{g^{\dot{\rho}\dot{m}}s_{\dot{\rho}}{}^{k} + g^{\lambda k}s^{\dot{m}}{}_{\lambda}\right\}.$$

Introducing

$$t^{\dot{r}\dot{m}lk} = \tfrac{1}{4}g^{\dot{r}\dot{m}}g^{lk} \tag{15a}$$

we have

$$\partial_{\dot{\rho}\lambda}t^{\dot{\rho}\dot{m}\lambda k} = 2f^{\dot{m}k} \tag{15b}$$

where the force spinor

$$f^{\dot{m}k} = -\tfrac{1}{4}\left\{g^{\dot{\rho}\dot{m}}s_{\dot{\rho}}{}^{k} + g^{\lambda k}s^{\dot{m}}{}_{\lambda}\right\}. \tag{15c}$$

It follows from the symmetry of the spinors $g^{\dot{r}\dot{m}}$ and g^{lk} that $t^{\dot{r}\dot{m}lk}$ has only nine "Kennzahlen." Thus relation (16) is already embodied in the structure of the stress-energy spinor.

§6δ. Analogous to the developments of §5δ we introduce two symmetric spinors $g'_{\dot{r}\dot{s}}$ and $m_{\dot{r}\dot{s}}$ which correspond to the self-dual tensors G'^{kl} and M^{kl}. We then write the spinor analogue of (21)

$$\partial^{\dot{\sigma}}{}_l g'_{\dot{\sigma}\dot{r}} = 2j_{\dot{r}l}$$

where

$$j_{\dot{r}l} = s_{\dot{r}l} + \tfrac{1}{4}(\partial^{\dot{\sigma}}{}_l m_{\dot{\sigma}\dot{r}} + \partial_{\dot{r}}{}^{\alpha} m_{\dot{\sigma}r}) \tag{21a}$$

where the first term on the right side is due to conduction and the second term to electric and magnetic polarization.

Chapter III. The Dirac Equations in Spinor Form

§7. Van der Waerden has shown how to write the Dirac equations in spinor form. The four wave functions ψ of Dirac correspond to two spinors of the first rank $\psi_{\dot{m}}$ and χ_l, and his equations become

$$mc\chi_l - \left(\frac{h}{i}\,\partial^{\dot{\sigma}}{}_l + \phi^{\dot{\sigma}}{}_l\right)\psi_{\dot{\sigma}} = 0 \tag{1a}$$

$$mc\psi_{\dot{m}} + \left(\frac{h}{i}\,\partial_{\dot{m}}{}^{\lambda} + \phi_{\dot{m}}{}^{\lambda}\right)\chi_{\lambda} = 0 \tag{1b}$$

where $\phi^{\dot{s}}{}_l$ is the potential spinor as according to (9a) Chapter II. We shall also need the complex conjugate equations which read

$$mc\chi_{\dot{m}} - \left(-\frac{h}{i}\,\partial_{\dot{m}}{}^{\lambda} + \phi_{\dot{m}}{}^{\lambda}\right)\psi_{\lambda} = 0 \tag{2a}$$

$$mc\psi_l + \left(-\frac{h}{i}\,\partial^{\dot{\sigma}}{}_l + \phi^{\dot{\sigma}}{}_l\right)\chi_{\dot{\sigma}} = 0. \tag{2b}$$

We now wish to obtain the second order Dirac equations. Introducing the abbreviation

$$p^{\dot{s}}{}_l = \frac{h}{i}\,\partial^{\dot{s}}{}_l + \phi^{\dot{s}}{}_l \tag{3}$$

we eliminate χ_l from (1) and get

$$m^2c^2\psi_{\dot{m}} + p_{\dot{m}}{}^{\lambda}p^{\dot{\sigma}}{}_{\lambda}\psi_{\dot{\sigma}} = 0.$$

We apply the identity (8) Chapter I to the second term and have, using also identity (7) Chapter I,

$$p_{\dot{m}}{}^{\lambda}p^{\dot{\sigma}}{}_{\lambda}\psi_{\dot{\sigma}} = -\tfrac{1}{2}p^{\dot{\sigma}\lambda}p_{\dot{\sigma}\lambda}\psi_{\dot{m}} + \tfrac{1}{2}p_{\dot{\sigma}\lambda}p_{\dot{m}}{}^{\lambda}\psi^{\dot{\sigma}} - \tfrac{1}{2}p_{\dot{m}}{}^{\lambda}p_{\dot{\sigma}\lambda}\psi^{\dot{\sigma}}.$$

Taking (3) into account we see that

$$p_{\dot{\sigma}\lambda}p_{\dot{m}}{}^{\lambda}\psi^{\dot{\sigma}} - p_{\dot{m}}{}^{\lambda}p_{\dot{\sigma}\lambda}\psi^{\dot{\sigma}} = \frac{h}{i}\,\psi^{\dot{\sigma}}(\partial_{\dot{\sigma}\lambda}\phi_{\dot{m}}{}^{\lambda} + \partial_{\dot{m}\lambda}\phi_{\dot{\sigma}}{}^{\lambda})$$
$$= \frac{h}{i}\,\psi^{\dot{\sigma}}g_{\dot{m}\dot{\sigma}},$$

the latter because of (9a) Chapter II. Thus the second-order wave equation becomes

$$-\frac{1}{2}\left(\frac{h}{i}\,\partial^{\dot{\sigma}\lambda} + \phi^{\dot{\sigma}\lambda}\right)\left(\frac{h}{i}\,\partial_{\dot{\sigma}\lambda} + \phi_{\dot{\sigma}\lambda}\right)\psi_{\dot{m}} + m^2c^2\psi_{\dot{m}} = -\frac{h}{2i}\,g_{\dot{m}\dot{\sigma}}\psi^{\dot{\sigma}}. \tag{4}$$

Correspondingly one obtains

$$-\frac{1}{2}\left(\frac{h}{i}\,\partial^{\dot{\sigma}\lambda} + \phi^{\dot{\sigma}\lambda}\right)\left(\frac{h}{i}\,\partial_{\dot{\sigma}\lambda} + \phi_{\dot{\sigma}\lambda}\right)\chi_k + m^2c^2\chi_k = \frac{h}{2i}\,g_{k\lambda}\chi^{\lambda}. \tag{5}$$

The left side is identical with the Gordon wave equation written in spinor form, whereas the right side represents the spin correction. It is satisfactory, that the field only occurs in the form of our symmetric spinor $g_{\dot{m}\dot{n}}$ resp. g_{kl}.

§8. To derive the expression for the current we multiply (1a) with ψ^l, (1b) with $-\chi^{\dot{m}}$, (2a) with $-\psi^{\dot{m}}$, (2b) with χ^l, and add all four equations. Using repeatedly identity (7) Chapter I all terms containing the mass and the potentials cancel and we can write the result

$$\partial^{\dot{\mu}\lambda}j_{\dot{\mu}\lambda} = 0 \tag{6}$$

where

$$j_{\dot{m}l} = \psi_{\dot{m}}\psi_l + \chi_{\dot{m}}\chi_l. \tag{7}$$

§9. We shall split up the above expression after the fashion of Eq. (1a) Chapter II. We replace $\psi_{\dot{m}}$ and χ_l in (7) by their expressions following from (1a) and (1b) and have, using identity (8) Chapter I

$$mcj_{\dot{m}l} = -\psi_l\left(\frac{h}{i}\,\partial_{\dot{m}}{}^{\alpha} + \phi_{\dot{m}}{}^{\alpha}\right)\chi_{\alpha} + \chi_{\dot{m}}\left(\frac{h}{i}\,\partial^{\dot{\sigma}}{}_l + \phi^{\dot{\sigma}}{}_l\right)\psi_{\dot{\sigma}}$$
$$= +\psi^{\alpha}\left(\frac{h}{i}\,\partial_{\dot{m}\alpha} + \phi_{\dot{m}\alpha}\right)\chi_l - \chi^{\dot{\sigma}}\left(\frac{h}{i}\,\partial_{\dot{\sigma}l} + \phi_{\dot{\sigma}l}\right)\psi_{\dot{m}}$$
$$+ \psi_{\alpha}\left(\frac{h}{i}\,\partial_{\dot{m}l} + \phi_{\dot{m}l}\right)\chi^{\alpha} - \chi_{\dot{\sigma}}\left(\frac{h}{i}\,\partial_{\dot{m}l} + \phi_{\dot{m}l}\right)\psi^{\dot{\sigma}}.$$

We then replace in (7) ψ_l and $\chi_{\dot{m}}$ by the expressions following from (2a) and (2b) and transform the equations by means of (8) Chapter I in a completely analogous fashion. We thus obtain four expressions for $mcj_{\dot{m}l}$. Adding all of them we have after a few elementary transformations

$$j_{\dot{r}l} = s_{\dot{r}l} + \tfrac{1}{4}(\partial^{\dot{\sigma}}{}_l m_{\dot{\sigma}\dot{r}} + \partial_{\dot{r}}{}^{\alpha}m_{\alpha l}) \tag{8}$$

where

$$s_{\dot{r}l} = \frac{h}{4imc}\left(\psi_\alpha \partial_{\dot{r}l}\chi^\alpha + \chi_\alpha \partial_{\dot{r}l}\psi^\alpha + \psi^{\dot{\sigma}}\partial_{\dot{r}l}\chi_{\dot{\sigma}} + \chi^{\dot{\sigma}}\partial_{\dot{r}l}\psi_{\dot{\sigma}}\right) + \frac{1}{2mc}\phi_{\dot{r}l}\left(\psi_\alpha\chi^\alpha + \psi_{\dot{\sigma}}\chi^{\dot{\sigma}}\right) \tag{9}$$

$$\left.\begin{aligned} m_{\dot{r}\dot{s}} &= \frac{h}{imc}\left(\psi_{\dot{r}}\chi_{\dot{s}} + \psi_{\dot{s}}\chi_{\dot{r}}\right) \\ m_{kl} &= \frac{-h}{imc}\left(\psi_k\chi_l + \psi_l\chi_k\right) \end{aligned}\right\} \tag{10}$$

Equations (8) (9) and (10) are the spinor form for the decomposition of the current first given independently by Gordon[15] and Darwin.[16] Similarly to the correction terms on the right side of (4), the spinor (10) expresses the existence of a spin.

§10. The multiplication process described in §8 is not the only process by means of which the potentials $\phi_{\dot{m}l}$ may be eliminated from the Dirac equations. In fact, besides the one leading to the continuity Eq. (6), it is possible in three more ways to eliminate the $\phi_{\dot{m}l}$ each of which lead to an invariant relation between the ψ and χ.

For the sake of convenience we write down the four Dirac equations, but using only covariant ∂-operators. In four columns at the right side of the equations, the various factors, with which we multiply are given.

	a	b	c	d
$mc\chi_l + \frac{h}{i}\partial_{\dot{\sigma}l}\psi^{\dot{\sigma}} + \phi_{\dot{\sigma}l}\psi^{\dot{\sigma}} = 0$	$+\psi^l$	$+\psi^l$	$+\chi^l$	$+\chi^l$
$mc\psi_{\dot{m}} - \frac{h}{i}\partial_{\dot{m}\lambda}\chi^\lambda - \phi_{\dot{m}\lambda}\chi^\lambda = 0$	$-\chi^{\dot{m}}$	$+\chi^{\dot{m}}$	$+\psi^{\dot{m}}$	$+\psi^{\dot{m}}$
$mc\chi_{\dot{m}} - \frac{h}{i}\partial_{\dot{m}\lambda}\psi^\lambda + \phi_{\dot{m}\lambda}\psi^\lambda = 0$	$-\psi^{\dot{m}}$	$-\psi^{\dot{m}}$	$+\chi^{\dot{m}}$	$-\chi^{\dot{m}}$
$mc\psi_l + \frac{h}{i}\partial_{\dot{\sigma}l}\chi^{\dot{\sigma}} - \phi_{\dot{\sigma}l}\chi^{\dot{\sigma}} = 0$	$+\chi^l$	$-\chi^l$	$+\psi^l$	$-\psi^l$

We obtain

(a)
$$\partial_{\dot{\sigma}\lambda}\left(\psi^{\dot{\sigma}}\psi^\lambda + \chi^{\dot{\sigma}}\chi^\lambda\right) = 0. \tag{6}$$

This is Eq. (6) for the current.

(b)
$$\partial_{\dot{\sigma}\lambda}\left(\psi^{\dot{\sigma}}\psi^\lambda - \chi^{\dot{\sigma}}\chi^\lambda\right) + 2i\frac{mc}{h}\left(\psi^\lambda\chi_\lambda + \psi_{\dot{\sigma}}\chi^{\dot{\sigma}}\right) = 0 \tag{11}$$

(c) (d) Adding and subtracting the results of process (c) and (d) we have, using (7) and (7a) Chapter I

[15] W. Gordon, Zeits. f. Physik **50**, 630 (1928).
[16] G. Darwin, Proc. Roy. Soc. **A120**, 621 (1928).

$$\chi^{\lambda}\partial_{\dot\sigma\lambda}\psi^{\dot\sigma} - \psi^{\dot\sigma}\partial_{\dot\sigma\lambda}\chi^{\lambda} = 0 \tag{12}$$

$$\chi^{\dot\sigma}\partial_{\dot\sigma\lambda}\psi^{\lambda} - \psi^{\lambda}\partial_{\dot\sigma\lambda}\chi^{\dot\sigma} = 0. \tag{13}$$

It is clear, that these are the only relations between the wave functions which are independent of the potentials $\phi_{\dot m l}$, because we have eight equations (the four Dirac equations and their complex conjugates) in which only four potentials $\phi_{\dot m l}$ occur.[17] It is curious, that three of these relations are also independent of the mass.

A few words may be added concerning the quadratic invariants which do not involve differentiation. Due to (7a) Chapter I the only two invariants are:

$$\left.\begin{aligned}\Delta &= \psi_{\lambda}\chi^{\lambda}\\ \overline{\Delta} &= \psi_{\dot\sigma}\chi^{\dot\sigma}\end{aligned}\right\}. \tag{14}$$

It is easily verified[18] that the square of the current

$$j_{\dot\sigma\lambda}j^{\dot\sigma\lambda} = 2\Delta\overline{\Delta} \tag{15}$$

and the square of the polarization spinor (10):

$$\left.\begin{aligned}m_{\dot\rho\dot\sigma}m^{\dot\rho\dot\sigma} &= 2\left(\frac{h}{mc}\right)^2\overline{\Delta}^2\\ \text{and}\qquad m_{\alpha\beta}m^{\alpha\beta} &= 2\left(\frac{h}{mc}\right)^2\Delta^2\end{aligned}\right\} \tag{16}$$

Introducing the abbreviation

$$k_{\dot s l} = \psi_{\dot s}\psi_l - \chi_{\dot s}\chi_l$$

for the spinor whose divergence occurred in (11), it is readily seen that

$$\begin{aligned}k^{\dot\rho}{}_{l}m_{\dot\rho\dot s} &= -\frac{h}{imc}j_{\dot s l}\overline{\Delta}\\ j^{\dot\rho}{}_{l}m_{\dot\rho\dot s} &= -\frac{h}{imc}k_{\dot s l}\overline{\Delta}.\\ k_{\dot\sigma\lambda}j^{\dot\sigma\lambda} &= 0\end{aligned} \tag{17}$$

§11. Darwin[19] has shown how to derive both the Maxwell and the Dirac equations from a variation principle. The analogous development using spinor analysis runs as follows. We start with the Lagrangian function

[17] Relations (11), (12) and (13) were found more or less accidentally by the authors. The point of view described in the text was supplied by Professor G. Y. Rainich, who found them independently. Professor Rainich further communicated to us a rigorous proof of the fact that the Dirac equations possess only two algebraic quadratic invariants, which are simply our Δ and $\overline{\Delta}$. Also relations (17) are due to him. The authors are greatly indebted to Professor Rainich for several discussions on the subject.

[18] Compare C. G. Darwin, Proc. Roy. Soc. **A120,** 621 (1928). See esp. p. 627.

[19] C. G. Darwin, Proc. Roy. Soc. **A118,** 654 (1928).

$$L = \frac{h}{i}\left\{\psi^{\dot\mu}\partial_{\dot\mu}{}^{\lambda}\psi_{\lambda} - \chi^{\dot\mu}\partial_{\dot\mu}{}^{\lambda}\chi_{\lambda} - \psi^{\lambda}\partial^{\dot\mu}{}_{\lambda}\psi_{\dot\mu} + \chi^{\lambda}\partial^{\dot\mu}{}_{\lambda}\chi_{\dot\mu}\right\} + 2mc(\Delta + \bar{\Delta}) + \phi^{\dot\mu\lambda}j_{\dot\mu\lambda} - \tfrac{1}{2}g^{\dot\mu\dot\rho}g_{\dot\mu\dot\rho} \tag{18}$$

where the meaning of Δ, $\bar{\Delta}$, $j_{\dot m l}$ and $g_{\dot m \dot n}$ in terms of ψ, χ, and ϕ are given in Eq. (14) (7) Chapter III and (9a) Chapter II respectively. L is to be considered a function of ψ, χ, ϕ and their derivatives.

1°. Varying ψ_l we obtain as Euler-Lagrange equation

$$\partial_{\dot\mu}{}^{l}\left\{\frac{\partial L}{\partial(\partial_{\dot\mu}{}^{\alpha}\psi_{\alpha})}\right\} - \frac{\partial L}{\partial\psi_l} = 0$$

which is identical with (1a).

2°. Varying $\chi_{\dot m}$ we obtain

$$\partial^{\dot m}{}_{\lambda}\left\{\frac{\partial L}{\partial(\partial^{\dot\sigma}{}_{\lambda}\chi_{\dot\sigma})}\right\} - \frac{\partial L}{\partial\chi_{\dot m}} = 0$$

which is identical with (1b).

3°. Varying $\psi_{\dot m}$ we get (2a) and varying χ_l we get (2b).

4°. Varying $\phi_{\dot m l}$ we obtain

$$\partial_{\dot\sigma}{}^{l}\left\{\frac{\partial L}{\partial(\partial_{\dot\sigma\alpha}\phi_{\dot m}{}^{\alpha})}\right\} - \frac{\partial L}{\partial\phi_{\dot m l}} = 0$$

that is to say

$$\partial_{\dot\sigma}{}^{l}g^{\dot\sigma\dot m} = 2j^{\dot m l} = 2(\psi^{\dot m}\psi^{l} + \chi^{\dot m}\chi^{l})$$

which are the Maxwell equations with the Dirac current.