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# EFFECTS OF CATHODE-RAY DIFFUSION ON INTENSITIES IN X-RAY SPECTRA

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Abstract

**Rediffusion of cathode rays.**—The available evidence is reviewed and conclusions formulated on the rediffusion constants and distribution functions needed for the calculation of effects on x-ray emission intensities.

**Thin-target x-ray intensities.**—A previous paper gave the intensity of the $K$ lines from very thin Ag supported on Be, as a function of voltage, in terms of the intensity at twice the excitation voltage as a standard. In the same terms after correction for two opposing effects, diffusion of cathode rays within the thin target and rediffusion from the beryllium, the corrected values differ from the originals by not over 2 percent. Possible sources of error are reviewed, and it is concluded that none can increase this percentage very greatly. This answers a criticism on the theoretical significance of these data.

**Comparison with theories.**—Present wave mechanics theories involve approximations not valid at low voltages, but subject to this qualification Bethe's theory compares very favorably with these data. The best classical-mechanics theory is Thomas's. Empirically, the intensity $=$ const. $\times U^{-0.8} \log U$, where $U=$ (tube voltage)/(excitation voltage).

**Thick-target effects.**—For line intensities percentage corrections for rediffusion, again taking the intensities at $U=2$ as standard, are more than twice those for thin targets at all voltages in the above range, and are of opposite sign. On continuous spectrum intensities averaged over all directions, these calculations confirm Kramers' suggestion, that most of the non-linear term in the intensity formula is due to rediffusion.

THE effects of diffusion of cathode rays on the intensities in x-ray spectra have become important recently, on account of improvements in research technique. Such improvements, in themselves, do not necessarily increase the effects of diffusion, but as the sources of larger errors of interpretation are eliminated, diffusion comes next in line for treatment. The larger errors, in this case, occur in the study of intensities as functions of cathode-ray energy, and come from two sources. One is the variety of speeds with which the cathode rays strike the atoms, even though they all enter the anode at one speed. The other is the difficulty of separating the parts of the measured radiation coming from atoms ionized by cathode-ray impact and by x-ray absorption. Both these sources of error have now been reduced to negligible amounts, by the use of targets so thin that the cathode rays penetrate them without serious loss of speed, and that x-rays penetrate them without appreciable absorption. Thus the largest errors remaining may well be those due to cathode-ray diffusion.

In a recent paper[1] on measurements of x-rays from thin targets, we estimated these errors tentatively as "a few percent"; and since the deviations of all the available theories from our data exceeded this amount, the exact calculation of the corrections for these errors was postponed until further data were available. Still more recently, however, Wisshak[2] has questioned our estimate, stating his belief that the correction would prove to be so great as to reverse completely the relations between our data and the leading theories of impact ionization. Evidently the status of these theories depends on the order of magnitude of these corrections. Therefore a calculation of them, at least with sufficient accuracy to make sure of the relation of the data to the theories, becomes imperative.

It is therefore the purpose of this paper to report the results of such a calculation, and incidentally we shall include some by-products, of interest in the interpretation of other data on x-rays and cathode rays.

## THE LAWS OF REDIFFUSION

The effects to be calculated arose from two types of diffusion of the cathode rays, one type occurring within the thin targets themselves, and the other in a block of beryllium used to support the thin targets and to conduct away the heat produced by the cathode rays. Diffusion within a thin target, making the paths of the cathode rays deviate from straight lines normal to the surfaces, causes a slight increase in the number of atoms struck by an average cathode ray. Diffusion in the beryllium must be considered in order to take account of the small percentage of the cathode rays which deviate so far as to be "rediffused," that is to return to the surface of the block and thus to make second impacts on the thin target. The effect of diffusion within the thin target is readily calculated from well-known laws, but to calculate the rediffusion effect we must first examine the available evidence and formulate as well as possible the laws of the phenomena concerned.

**The rediffusion constant.** First of all, we must find the best value for beryllium, of the "rediffusion constant" $p$, or fraction of the incident cathode rays that are rediffused. Since none of the data we have found in the literature refer directly to beryllium, we must get the value for it from the data, through the law relating $p$ to the atomic number $Z$.

To measure $p$, one must measure the current carried by the rediffused electrons, but not include any of the slow secondary electrons leaving the metal at the same time. The theoretical reason for separating these two classes is that their causes for leaving the metal are different: rediffusion and and ejection by the cathode rays, respectively. The practical reason, for the

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[1] D. L. Webster, H. Clark, R. M. Yeatman, and W. W. Hansen, Proc. Nat. Acad. Sci. **14**, 679 (1928).

[2] F. Wisshak, Ann. d. Physik **5**, 507 (1930).

present purpose, is that the secondary electrons are too slow to produce x-rays as hard as those used in most intensity measurements.

Data on $p$, for various primary-ray energies, have been obtained by several investigators: Becker,[3] at 35 kv.; Schmidt,[4] with beta rays of about 800 kv.; Schonland,[5] at 10 to 100 kv.; and Stehberger[6] at 2 to 12 kv. The term "rediffusion constant," and its symbol $p$, do not mean the same quantity in all these papers. In Schonland's it refers to incidence along the normal, in Stehberger's at 50° from it, in Schmidt's and Becker's, at least as recomputed by Lenard[7] and quoted by Stehberger, to incidence "im Normalfall," that is with the cathode rays already completely diffused before they strike the metal. For present purposes, the most important case is that of incidence along the normal, so we shall adopt the definition for that case.

One of the most striking conflicts of values is between those of Becker as recomputed and quoted (Al 0.28, Au 0.68) and those of Schonland (0.13 and 0.50 respectively). A part of this conflict is due to the difference in definition, since Lenard finds for Al along the normal the value 0.23, and this reduces the discrepancy considerably. Another part, however, is probably due to the theory underlying the recomputation, which involved the assumption of sudden absorption of cathode rays by individual atoms. Since this assumption has later been disproved,[8] we may remove this part of the discrepancy by going back to Becker's original values. Thus we obtain Table I.

TABLE I. *Rediffusion constants for incidence along the normal.*

| Metal | Al | Cu | "Blattmetall" | Ag | Au |
|---|---|---|---|---|---|
| Becker | 0.172 | — | 0.407 | 0.433 | 0.496 to 0.56 |
| Schonland | 0.13 | 0.29 | — | 0.39 | 0.50 |

Considering the general improvements in cathode-ray technique between Becker's work in 1905 and Schonland's in 1925, we must indeed congratulate Becker on the smallness of the discrepancy, even though we shall use Schonland's values of $p$.

A more serious question is presented by Stehberger's data. His values of $p$ at 50° incidence on Al are from 2.5 to 5.5 times Schonland's at normal incidence. This ratio seems far too great. Furthermore, Schonland found $p$ independent of primary voltage, whereas Stehberger found it to decrease rapidly with increasing voltage. To be sure, their ranges of voltage barely overlap; but Schonland gives theoretical reasons for believing that $p$ should be constant, and an important question is, whether Stehberger's data make Schonland's theory unreliable.

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[3] A. Becker, Ann. d. Physik **17**, 381 (1905).

[4] H. W. Schmidt, Ann. d. Physik **23**, 671 (1907).

[5] B. F. J. Schonland, Proc. Roy. Soc. **A108**, 187 (1925).

[6] K. H. Stehberger, Ann. d. Physik **86**, 825 (1928).

[7] P. Lenard, "Quantitatives uber Kathodenstrahlen aller Geschwindigkeiten" ed. 1925. p. 226.

[8] W. Bothe "Handbuch der Physik" ed. 1927, Vol. XXIV, Chap. 1, See especially Sec. 25.

The answer to this question appears to us to lie in the method of separation of secondary electrons from rediffused. Both observers found a considerable fraction of the electrons to have energies below 10 or 20 equivalent volts; both adopted a voltage basis for the separation, Stehberger drawing the line at 36 volts and Schonland at 200; and both agreed that any such basis was somewhat arbitrary, and might be incorrect.

On this point, we may use a line of reasoning similar to that of Bothe,[8] about the distribution of velocities among cathode rays transmitted through metal films. Here the Thomson-Whiddington law of retardation makes it impossible for many of the rays to emerge with very small energy, simply because the range of a cathode ray of small energy is so short. For example, if an electron has been retarded until its kinetic energy is only 10 percent of that at incidence, it has only 1 percent of its range still before it, and the chances are very much against that short section of its range intersecting the surface. The same reasoning applies to rediffusion. Many cathode rays rediffuse with half their initial energy or more, so there is no question of any grave difficulty for a cathode ray to return to the surface with a long range still before it, and it is therefore highly improbable that any large fraction of the rediffusion will be in the last 1 percent of the range.

Drawing the line between secondary and rediffused electrons at 36 volts, however, with a primary voltage of 8 or 12 kv., Stehberger found over 30 percent of his "rediffused" electrons to have less than 10 percent of their initial energy. Evidently in view of these considerations of range, there is good ground for belief that these were not rediffused, but secondary.

Furthermore, the distribution of energy among secondary electrons, as calculated by Bothe,[9] should be nearly independent of the primary energy; but their number should decrease rapidly with increasing primary energy, as in fact Stehberger found it did. Thus it may well be that Stehberger's data are not inconsistent, after all, with Schonland's data, or with his theory of the constancy of the rediffusion constant. And so long as we are not concerned with secondary electrons in the x-ray problems at hand, we shall use the values of $p$ obtained at the higher voltages, where secondary electrons were less abundant and the probable error from them was therefore smaller.

Incidentally, if we had adopted the alternative of accepting Stehberger's values of $p$ at low voltages and Schonland's at high, with a graded compromise for the intermediate region, the corrections to the x-ray data would not have been as greatly affected as the values of $p$, since the influence of the derivative of $p$ with voltage would have largely offset that of the increase in the values of $p$. The chief difference would have been in the complexity of the problem.

For the thin-target problem, we shall need the rediffusion constant of beryllium, the material of the block supporting the thin targets in some of our own experiments. Since Schonland finds for atomic number $Z=29$, $p=0.29$, and for $Z=13$, $p=0.13$, a reasonable guess for $Z=4$ is $p=0.04$. But for a

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[9] W. Bothe, reference 8, Section 42.

more careful estimate, we shall use an equation from Wentzel's[10] theory, which predicts that

$$p = b_1\left(\frac{Z^2\rho}{A\alpha_0}\right) + b_2\left(\frac{Z^2\rho}{A\alpha_0}\right)^2 \tag{1}$$

where $b_1$ and $b_2$ are constants, $Z$ and $A$ are atomic number and weight, and $\alpha_0/\rho$ is Lenard's "pure" mass absorption coefficient. Since this theory depends on the hypothesis of absorption, now disproved, this equation must be considered on a semi-empirical basis; but at that, since Lenard found $\alpha_0/\rho$ practically constant for light elements, for which also $A=2Z$, we may extrapolate to low atomic numbers by assuming that

$$p = p_1 Z + p_2 Z^2. \tag{2}$$

This fits all of Schonland's data pretty well, with $p_1=0.011$ and $p_2=-6.0\times10^{-5}$, and for Be it predicts $p=0.043$.

**The rediffusion energy distribution.** The next problem is to find the best formula for the distribution of the rediffused electrons on a scale of kinetic energy. The data available are far from complete or accurate, but fortunately the answers to the x-ray problems at hand are changed so little by relatively great changes in this distribution function that even a very rough approximation will prove sufficient.

First let us consider some theoretical evidence. Schonland explained the constancy of his rediffusion constants by noting that for cathode rays of all speeds, in any one element, the distance required for a probable deflection of one degree, and the distance required for a loss of kinetic energy of one percent, bear a constant ratio. Thus, although any two cathode rays of different speeds of incidence must have different ranges, the shapes of their paths may be geometrically similar, and the probabilities that they come back to the surface must be the same. Extending this reasoning, their probabilities of coming to the surface with any given fraction of their initial kinetic energy still left, must also be the same. Or in algebraic terms, calling this fraction $W$, and this probability $pF(W)dW$, with

$$\int_0^1 F(W)dW = 1, \tag{3}$$

$F(W)$ should be independent of $V$.

Another theoretical point about $F(W)$ is the one discussed above, about the scarcity of electrons of very low energy. Quantitatively, as one may readily prove from the Thomson-Whiddington law, this means that $F(0)=0$, and that its first derivative is finite.

Turning now to the experimental evidence, we have some velocity spectra photographed by Wagner[11] for Al, Cu, Ag and Au at primary voltages from 16 to 40 kv. While an exact translation of the densities in these photographs into

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[10] G. Wentzel, Ann. d. Physik **70**, 561 (1923).

[11] P. B. Wagner, Phys. Rev. **35**, 98 (1930).

numbers of electrons is impossible, they do appear to give at least a qualitative confirmation to the theoretical deductions just stated. And without any exact translation, they prove some other points more definitely: first, $F(W)$ is a continuous function, without any sharp peaks; second, there are no electrons rediffused with full primary energy, so that $F(1)=0$; third, reducing $W$ from 1, $F(W)$ rises rapidly to a maximum and then declines, probably more slowly than it rose; and fourth, that the value $W_0$, where the maximum occurs, is higher for heavy elements than for light. Taking the densities literally, as a rough approximation, $W_0$ for Al would be about 0.85, for Cu 0.90, and for Ag and Au 0.94.

Confirmatory evidence appears in the spectra of the x-rays produced by the rediffused electrons, investigated by Nicholas[12] with copper and Lorenz[13] with tungsten. While this evidence is not so direct, it indicates the same general sort of preponderance of high energies as Wagner's photographs, and again with no notable change in $F(W)$ with $V$.

A third line of evidence is on $\overline{W}$, the mean of the values of $W$ for all the rediffused electrons. This may be calculated from measurements by Wisshak[2] on the ratio of the power absorbed by the anode of a gas-filled tube to the whole power input. If it is permissible to neglect the power given to positive ions, this ratio is $(1-p\overline{W})$. According to Wisshak, it is independent of $V$, from about 3 to 35 kv., having the value 0.8 for Co and Cu, and 0.6 for Mo and Ag. These data, with Schonland's $p$'s, would make $\overline{W}=0.7$ for Co and Cu, and nearly 1 for Mo and Ag. The correction for positive-ion power would reduce these values of $\overline{W}$.

Finally, there is evidence from the total intensity of the x-rays from the back and stem of the target of a Coolidge tube. Assuming the $V^2$ law of total x-ray intensity, the ratio of this "stem-radiation" to the radiation from the whole target, including the focus, should be $p\overline{W^2}$. Coolidge,[14] with one of his standard tungsten tubes, found this ratio to be 2/11, both with a 2-inch alternative spark gap and with a 10-inch; Ledoux-Lebard and Dauvillier[15] found 20 percent; and Rump,[16] on the basis of this latter figure and some measurements of his own, considered 20 percent correct for any voltage from 43 to 150 kv. at least. For tungsten at normal incidence, equation (2) would make $p=0.48$, so that $\overline{W^2}$ would be about 0.4, and therefore $\overline{W}$ probably about 0.6; but for these 45° targets, $p$ must be somewhat greater and $\overline{W^2}$ correspondingly less. This estimate is lower than those given above, but it agrees in making quantities dependent on $p$ and $F(W)$ independent of $V$, as required by theory.

All this evidence is still far from a quantitative formula for $F(W)$, but all that is really needed for present purposes is a rough approximation, in a reasonably convenient mathematical form. To satisfy this latter requirement

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[12] W. W. Nicholas, Phys. Rev. **29**, 619 (1927).

[13] E. Lorenz, Proc. Nat. Acad. Sci. **14**, 582 (1928).

[14] W. D. Coolidge, G. E. Review **20**, 272 (1917).

[15] P. Ledoux-Lebard and A. Dauvillier, "Physique des Rayons X" p. 45. (quoted by Rump).

[16] W. Rump, Zeits. f. Physik **43**, 254 (1927).

we shall neglect that about the behavior of $F(W)$ at $W=0$, where we shall never need to use it. All other requirements are then satisfied by either of the following formulas:

$$\left.\begin{aligned} F_1(W) &= G(1-W)e^{-g(1-w)}, \\ \text{with } G &= g^2/\{1-(g+1)e^{-g}\}; \end{aligned}\right\} \tag{4}$$

and

$$\left.\begin{aligned} F_2(W) &= Q(1-W)W^q, \\ \text{with } Q &= (q+1)(q+2). \end{aligned}\right\} \tag{5}$$

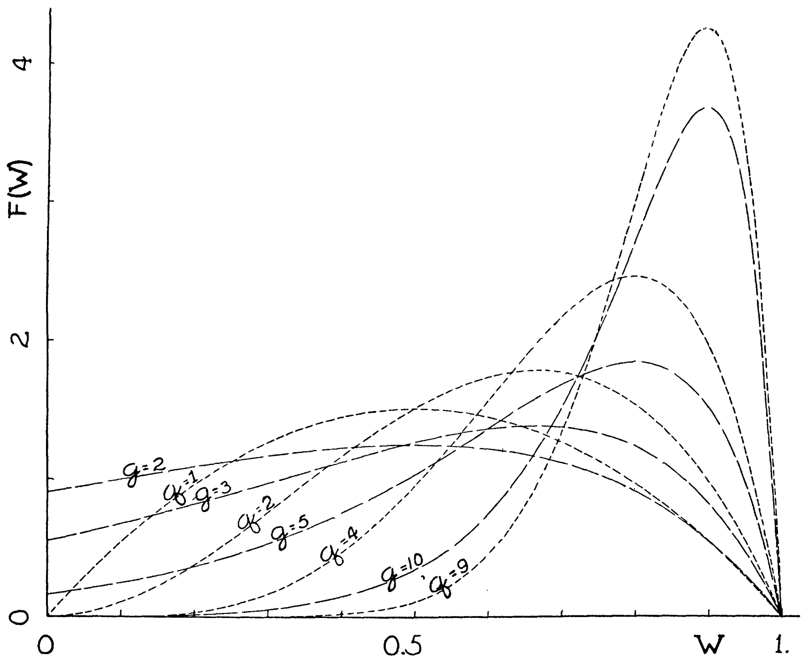

Fig. 1. Graphs of $F_1(W)$ and $F_2(W)$ from equations (4) and (5) respectively.

TABLE II. *Parameters of $F(W)$ in Fig. 1.*

| $W_0$ | $g$ | $q$ | $\overline{W}$ | $\overline{W^2}$ |
|---|---|---|---|---|
| 0.90 | 10 | — | 0.800 | 0.660 |
| 0.90 | — | 9 | 0.833 | 0.706 |
| 0.80 | 5 | — | 0.635 | 0.454 |
| 0.80 | — | 4 | 0.714 | 0.536 |
| 0.67 | 3 | — | 0.520 | 0.333 |
| 0.67 | — | 2 | 0.600 | 0.400 |
| 0.50 | 2 | — | 0.455 | 0.272 |
| 0.50 | — | 1 | 0.500 | 0.300 |

Graphs of these functions, for several values of each parameter, $g$ or $q$, are shown in Fig. 1; and corresponding values of $W_0$, $\overline{W}$ and $\overline{W^2}$, for use in selecting the best value of $g$ or $q$, are listed in Table II.

Briefly, for heavy elements, the evidence on $W_0$ and $\overline{W}$ seems to favor the higher values listed here for these parameters, and that from $\overline{W^2}$ the lower ones; for light elements, the values must be lower than for heavy. In view of this uncertainty, it is indeed fortunate that the quantities we must calculate for the comparison of x-ray data with theories change so slowly with these parameters that even such rough information as this will suffice to place them within probable limits of experimental error.

**The directional distribution of rediffused cathode rays.** In the problem of the effect of rediffusion on the intensity of x-rays from a thin target supported on a block of another metal, it is important to know the ratio of the average number of atoms in the thin target, struck by a rediffused ray on its way out, to the number struck by a primary ray on its way in. Since all the corrections for diffusion and rediffusion effects are reasonably small, we shall neglect small quantities of the second order, i.e. in this case, the effect of diffusion within the thin target itself on the ratio of these numbers of atoms. Thus we shall assume that for cathode rays rediffusing at the angle $\Theta$ from the normal, this ratio is simply sec $\Theta$. The directional distribution need then be found only with enough accuracy to get a reasonably accurate estimate of the mean value of sec $\Theta$, which we shall call $S$.

Rediffusion can happen only to cathode rays deflected through more than a right angle, and indeed considerably more unless the paths after deflection through a right angle are much longer than those before. Thus rediffusion comes fairly near to involving complete diffusion, and the assumption of complete diffusion should give a first approximation for $S$. With this assumption, the probability that any infinitesimal segment of the path of a ray will intersect the surface of the block would be proportional to the sine of the angle this segment makes with the surface, and so the number escaping in a given solid angle $d\Omega$ would be proportional to $\cos \Theta d\Omega$. This deduction was indeed verified, for beta rays transmitted through reasonably thick aluminum, by Kovarik and McKeehan.[17] Assuming it to hold for rediffused cathode rays, a simple integration gives $S=2$.

For a better approximation, still with normal incidence, but with incomplete diffusion, we may assume that at any point within the metal, the number of paths in a solid angle $d\Omega$ is a function $f(\Theta)d\Omega$. Then the number emerging in this solid angle will be proportional to $f(\Theta) \cos \Theta d\Omega$, and $f(\Theta)$ may be measured directly by catching the emergent rays from an area varying inversely as $\cos \Theta$.

It was in this way, but with incidence at 30° from the normal, that Kovarik and McKeehan verified the simple cosine law for transmitted rays. Their rediffused rays, however, showed a minimum in the direction backward toward the source. From a typical graph which they published, it seems prob-

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[17] A. F. Kovarik and L. W. McKeehan, Phys. Rev. 6, 426 (1915).

able that if the incidence had been along the normal, the rediffused rays would have been represented well enough for present purposes by the formula

$$f(\Theta) = \text{const.} \times (1 + a\Theta^2), \tag{6}$$

with $1+a(\pi/2)^2=1.4$.

A simple integration then gives

$$S = 2 \cdot \frac{1 + 1.14a}{1 + 0.73a} \cdot \tag{7}$$

If the diffusion had been complete, $a$ would have been zero, making $S=2$, as above; with $a$ as in equation (6), $S=2.11$. For present purposes, the difference is immaterial, and as the foil from which these data were taken was not thick enough for complete diffusion of its transmitted rays, the difference for a thick block may be even less. So in all probability $S$ is somewhere between 2.0 and 2.1.

## THE THIN-TARGET EFFECTS

We are now in a position to solve the problem of the effects of diffusion and rediffusion on the intensities of x-rays, and we shall consider first a thin target supported by a thick block. Since targets of this type are useful primarily for line spectra, the intensity will be considered to be that of a line, for example one belonging to a $K$ series, of excitation potential $V_K$, and we shall write all functions of the tube potential $V$ in terms of the ratio $U=V/V_K$. Letting $X_0$ stand for the thickness of the target, the intensity of the line, with neither diffusion nor rediffusion, is $X_0i_0(U)$, where $i_0(U)$ is the intensity per unit length of path. Then with rediffusion, the intensity due to the rediffused rays alone will be called $X_0r(U)$, and the total intensity will be called $X_0i(U)$. Now with diffusion within the thin target, the paths of the cathode rays are no longer straight, and the mean length of path of an incident ray within the target is slightly greater than $X_0$. As it is a function of voltage, we shall call it $X(U)$, and we shall replace $X_0$ in each of the above expressions by $X(U)$. Strictly speaking, this is not quite accurate for the rediffused rays, but as the error is a small quantity of the second order, we shall neglect it.

**Calculation of the effect of rediffusion.** With these definitions and the formulas on rediffusion deduced in the preceding section, it is evident that

$$r(U) = pS\int_{1/U}^{1} F(W)i_0(UW)dW. \tag{8}$$

We do not know $i_0(UW)$ accurately, of course, until after $r(U)$ has been calculated. But as will be proved later, we may safely replace it here by either of two functions, used previously in empirical formulas for $i(U)$, and shown graphically in Fig. 2, along with some data. Strictly speaking, these data refer to the function $j(U)=X(U)i(U)/X(2)i(2)$, but $X$ changes very little with $U$, so this is nearly $i(U)/i(2)$; thus an empirical formula for $j(U)$ will do well enough for $i(U)$, if we can use it with an unknown constant factor. One[1] of the formulas thus obtained is

$$i_1(U) = k_1\frac{1 - 1/U}{m_1 + U} \text{ if } U > 1, \text{ or } 0 \text{ if } U < 1, \tag{9}$$

with $k_1$ the unknown constant and $m_1$ about 3.0. The other[18] is

$$i_2(U) = k_2 U^{-m_2} \log U \text{ if } U > 1, \text{ or } 0 \text{ if } U < 1, \tag{10}$$

with $k_2$ unknown and $m_2$ about 0.77.

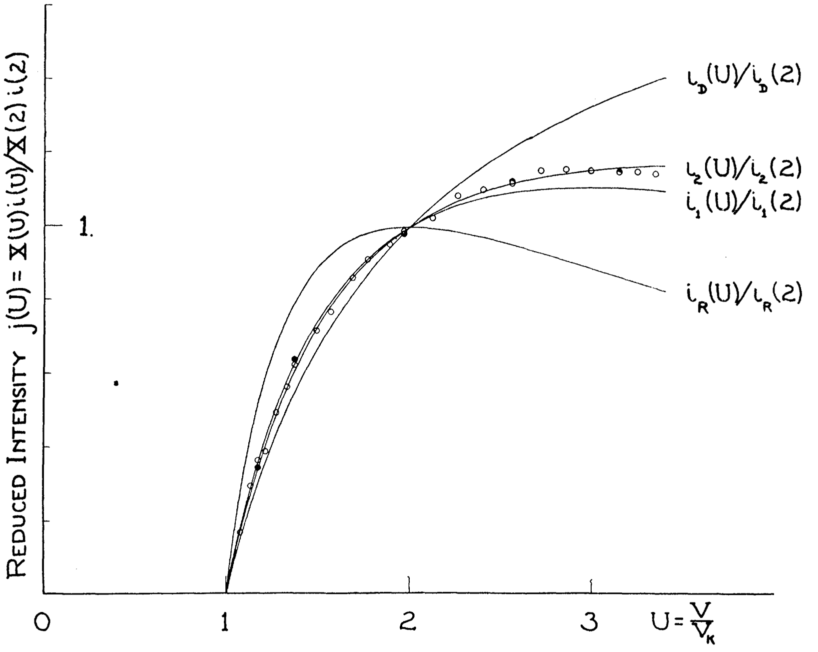

Fig. 2. Uncorrected data and empirical formulas. Black circles, 280A Ag film; white circles, 170A film; black and white circles, data from these films practically coincident.

To see whether the departures of these formulas from $i_0$ are of any importance here, we have also calculated $r$ with two other functions, departing farther from the corrected graph than either $i_1$ or $i_2$, and in opposite directions. One of these functions, suggested by Davis' theory,[19] is

$$i_D(U) = k_D\left(1 - \frac{1}{U}\right); \tag{11}$$

the other, from Rosseland's theory,[20] is

$$i_R(U) = k_R\left(\frac{1}{U} - \frac{1}{U^2}\right). \tag{12}$$

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[18] D. L. Webster and W. W. Hansen, Phys. Rev. **33**, 635 (1929).

[19] B. Davis, Phys. Rev. **11**, 433 (1918).

[20] S. Rosseland, Phil. Mag. **45**, 65 (1923).

The resulting values of $r$ agree with those from $i_1$ and $i_2$ to better than half of one percent of $i$, for any value of $U$ in the range covered. So $i_1$ and $i_2$ are well within the limits of accuracy needed for this purpose.

For $F(W)$ we may use either of the functions $F_1$ and $F_2$ obtained in the preceding section, Eqs. (4) and (5), and tests with the simple intensity functions $i_R$ and $i_D$ show that it makes no notable difference which $F$ is used. For convenience in integration, however, we have used $F_1$ with $i_1$, and $F_2$ with $i_2$. Thus we obtained two formulas for $r$, namely

$$\begin{aligned} r_1(U) = pSk_1 \frac{G}{m_1 g U}\big[(m_1+1) y e^{-y}\{Ei\,y - Ei\,z\} \\ - g e^{-g}\{Ei\,g - Ei(g/U)\} - m_1\{1 - e^{z-y}\}\big], \end{aligned} \tag{13}$$

where $y = g(m_1 + U)/U$ and $z = g(m_1 + 1)/U$, and

$$\begin{aligned} r_2(U) = pSk_2 Q\, U^{-m_2}\Bigg[\frac{\log U}{(q+1-m_2)(q+2-m_2)} \\ - \frac{1 - U^{-q-1+m_2}}{(q+1-m_2)^2} + \frac{1 - U^{-q-2+m_2}}{(q+2-m_2)^2}\Bigg]. \end{aligned} \tag{14}$$

For comparison of x-ray data with theories, we should like to find the function

$$i_0(U) = i(U) - r(U); \tag{15}$$

but as we have already seen, the data give $i$, and therefore $r$ also, only with an unknown constant coefficient. To eliminate this coefficient, in our first paper this subject,[1] we plotted the "reduced intensity"

$$j(U) = \frac{X(U)i(U)}{X(2)i(2)} \tag{16}$$

as an empirical function; and now likewise we can compare with theoretical predictions the function

$$j_0(U) = i_0(U)/i_0(2). \tag{17}$$

As an intermediate step, correcting for rediffusion only, we shall make use of the function

$$\begin{aligned} j_r(U) &= \frac{X(U)i_0(U)}{X(2)i_0(2)} = j(U)\cdot\frac{1 - \{r(U)/i(U)\}}{1 - \{r(2)/i(2)\}}, \\ &= j(U)\,\frac{1 + \{r(2)/i_0(2)\}}{1 + \{r(U)/i_0(U)\}}, \end{aligned} \tag{18}$$

and this coefficient of $j(U)$ will be called $c_r(U)$.

To compute $c_r$ we need numerical values of $pS$ and either $g$ or $q$. It appeared above that $p$ for beryllium was probably about 0.043, and $S$ 2.0 to 2.1, Thus $pS$ is probably about 0.086 to 0.090, though it may well be outside these limits. So we shall tentatively use the rounder number, 0.090, and if a better value of either $p$ or $S$ is found later, new values for $c_r$ may be found by changing $(c_r-1)$ in proportion to $pS$.

For $g$ and $q$, since there is considerable doubt as to the best values to assign to them, we have left room for later improvements in accuracy by making calculations with all the values used in Fig. 1 or Table II. The results are given in Table III.

TABLE III. *Values of 100* $(c_r-1)$.

| $W_0$ | $g$ | $q$ | $U=1.0$ | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 |
|---|---|---|---|---|---|---|---|---|
| 0.90 | 10 | — | +7.4 | +2.2 | 0 | −1.0 | −1.6 | −2.0 |
| 0.90 | — | 9 | +7.9 | +2.0 | 0 | −0.8 | −1.2 | −1.5 |
| 0.80 | 5 | — | +4.5 | +1.9 | 0 | −1.3 | −2.0 | −2.9 |
| 0.80 | — | 4 | +5.6 | +2.5 | 0 | −1.3 | −2.2 | −2.8 |
| 0.67 | 3 | — | +3.1 | +1.6 | 0 | −1.2 | −2.2 | −3.0 |
| 0.67 | — | 2 | +3.7 | +2.0 | 0 | −1.4 | −2.5 | −3.3 |
| 0.50 | 2 | — | +2.3 | +1.4 | 0 | −1.0 | −2.0 | −2.8 |
| 0.50 | — | 1 | +2.5 | +1.5 | 0 | −1.3 | −2.2 | −3.0 |

The changes actually made in the reduced intensities, by the use of these corrections, are shown in Table IV, for values of $U$ from 1 to 3.3 the limits of the data in Fig. 1. Here, for simplicity, we have averaged the corrections for all of the values of $g$ and $q$ from Table III, that have any reasonable chance of applying to such a light element as beryllium, that is, for $W_0=0.80$, 0.67 and 0.50. And to give a rough indication of the probable error due to the uncertainty in these parameters, we are tabulating with each averaged correction the mean of the deviations from it, given by the six values of $g$ or $q$.

TABLE IV. *Reduced intensities, corrected for rediffusion.*

| $U$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.3 |
|---|---|---|---|---|---|---|
| Average of 100 $(c_r-1)$ | +3.6 | +1.8 | 0 | −1.3 | −2.2 | −2.7 |
| $j(U)$ from empirical graph | 0 | +0.727 | +1 | +1.117 | +1.148 | +1.144 |
| Average of corrections | 0 | +0.013 | 0 | −0.014 | −0.025 | −0.031 |
| Mean deviation of corrections | 0 | 0.002 | 0 | 0.001 | 0.002 | 0.002 |
| Ave. corrected intensity $j_r(U)$ | 0 | +0.740 | +1 | +1.103 | +1.123 | +1.113 |

From this table it is evident that it makes no appreciable difference what value of $g$ or $q$ we use, within the range covered here; and even if the distribution of kinetic energy is not exactly like any of these functions $F(W)$, one could probably represent it pretty accurately by some linear combination of them, and even such a change would make no notable change in the corrections to be applied to $j(U)$.

**Calculation of the effect of diffusion within the thin target.** To calculate the reduced intensity $j_0(U)$, of Eq. (17), corrected for both rediffusion and diffusion, we must not only correct the factor $i(U)/i(2)$ in $j(U)$ for rediffusion, but also the factor $X(U)/X(2)$ for diffusion. The basis for this correction is a formula given by Bothe[21] "zum praktischen Gebrauch," namely

$$\lambda = \frac{8.0}{V}\cdot\frac{V+511}{V+1022}\cdot Z\left(\frac{\rho x}{A}\right)^{1/2}. \tag{19}$$

Here $\lambda$ is the most probable deflection of a cathode ray by multiple scattering in a thickness $x$ of an element of atomic number $Z$, atomic weight $A$ and density $\rho$; and $\lambda$ must be expressed in radians, $V$ in kilovolts, $x$ in microns and $\rho$ in gm/cc.

Letting $\Theta$ represent the deflection of any individual cathode ray at the depth $x$, the contribution of a layer $dx$ to the mean path $X$ will be the mean value for all cathode rays, of sec $\Theta\,dx$. For a thin target, therefore, since sec $\Theta$ is nearly enough $(1+\frac{1}{2}\Theta^2)$, and the mean value of $\Theta^2$ is $2\lambda^2$, we have

$$X(U) = \int_0^{X_0}(1+\lambda^2)dx = X_0(1+\tfrac{1}{2}\lambda_0^2), \tag{20}$$

where $\lambda_0$ is the value given by equation (19) for $\lambda$ at $x = X_0$.

The correction factor for diffusion, analogous to $c_r(U)$ for rediffusion, is now obtained from Eqs. (16), (17) and (18), as

$$c_d(U) = X(2)/X(U), \tag{21}$$

so that

$$j_0(U) = j(U)c_r(U)c_d(U). \tag{22}$$

As an illustration, and also for more definite use in the next section, the effect of diffusion in one of our own Ag targets is shown in Table V.

TABLE V. *Effect of diffusion in a 280A Ag target.*

| $U$ | = | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 |
|---|---|---|---|---|---|---|---|
| $X(U)/X_0$ | = | 1.076 | 1.034 | 1.019 | 1.013 | 1.009 | 1.006 |
| $100\,(c_d-1)$ | = | −5.2 | −1.4 | 0 | +0.6 | +1.0 | +1.3 |

**Results.** Combining the corrections for diffusion and rediffusion, as applied to our data on thin silver, we obtain the reduced intensities in Table VI. Here as in Table IV, the values of $j(U)$, for the round-number values of $U$ used in the computations, were read from the empirical graph in our previous paper, and the values of $j_0(U)$ were computed from them.

TABLE VI. *Reduced intensities, corrected for diffusion and rediffusion.*

| $U$ | = | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.3 |
|---|---|---|---|---|---|---|---|
| $j(U)$ | = | 0 | 0.727 | 1 | 1.117 | 1.148 | 1.144 |
| $100\,(c_rc_d-1)$ | = | −1.8 | +0.4 | 0 | −0.7 | −1.2 | −1.4 |
| $j_0(U)$ | = | 0 | 0.730 | 1 | 1.109 | 1.134 | 1.128 |

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[21] Bothe, reference 8, Section 9.

For the other values of $U$ at which data are given in Fig. 2, we have got the correction factors by interpolation, using Table VI for the 280A target, and a similar table for the 170A, differing only in the smaller correction for diffusion. The results, with the theoretical graphs with which they are to be compared, are shown in Fig. 3.

With regard to the empirical intensity functions, $i_1$ and $i_2$ of eqs. (9) and (10) resp., these corrections are so small that they come within the limits of error of $i_1$, but for $i_2$ it is best to change $m_2$ from 0.77 to 0.80.

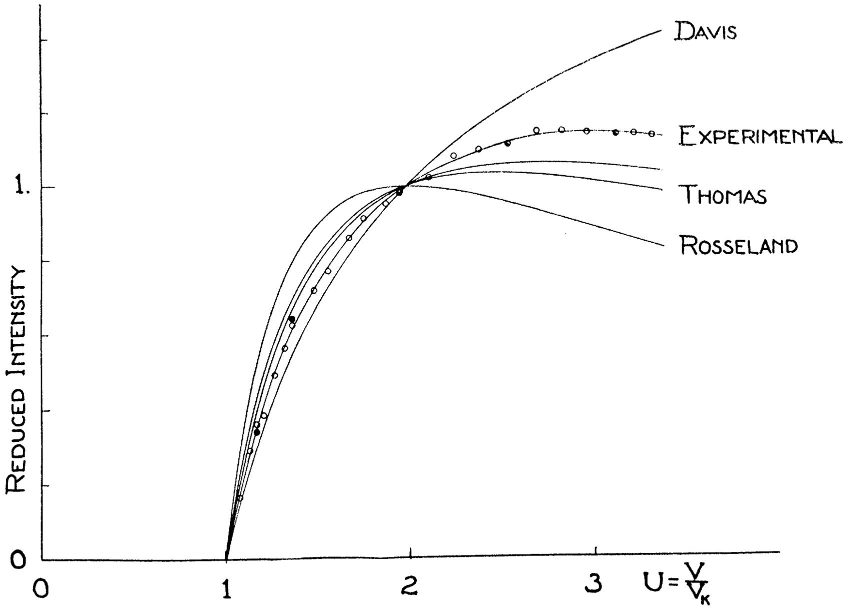

Fig. 3. Corrected data and theoretical graphs.

**Reliability of these corrections.** In a recent paper on thick-target x-ray measurements, Wisshak[2] reported data differing radically from our thin-target data in their relations to the theories. To compare our data with his he calculated a thick-target intensity function from our thin-target function $i_1(U)$, of Eq. (9), by the same formula that Rosseland had applied to his own theoretical function. The relation between this new thick-target function for Ag and Rosseland's was therefore much like that between the corresponding thin-target functions, shown in Figs. 2 and 3, i.e. the empirical graph was below Rosseland's at low voltages and above at high. Wisshak's empirical graph for the average of Cr and Cu, on the contrary, crossed Rosseland's theoretical graph in the opposite direction, at an angle of the same order of magnitude. Therefore Wisshak concluded, not only that there was a serious discrepancy between Rosseland's theory and his data, but also that the still larger difference between our graph and his must have been due to rediffusion in our target.

The present calculations, however, do not alter the relation between our graph and Wisshak's by anything like the amount indicated by this conclusion. In fact, to bring our graph into agreement with Rosseland's would require the correction, at $U=3$ for example, to be about 10 times as large as our calculations indicate, and for agreement with Wisshak's graph this factor would have to be nearer to 20. This raises the question: Can any of the uncertainties in the quantities or functions we have used be such as to make possible an increase of this order in our corrections?

Any change of $g$ or $q$, in either direction, would reduce the corrections. Any other change in the energy distribution, keeping within the requirements of the evidence reviewed above, could hardly be such as to render impossible a good approximation by some linear combination of the functions we have used, and this would keep the corrections much as calculated here. A change in the parameter $a$, of the directional distribution, even to $a=\infty$, which would be so extreme as to make the probability of normal emergence zero, would still introduce only a factor of about 1.5. A change in $p$, keeping it independent of $V$, would certainly be limited by the fact that $p$ for Be must be considerably less than for Al, and thus it could not go beyond a factor of 2 or 3. And finally, if $p$ were assumed not independent of $V$, it could only decrease with increasing $V$, and this would reduce the correction. Altogether, therefore, any great increase in these corrections appears highly improbable. Evidently we must look elsewhere for the cause of the disagreement between our conclusions and Wisshak's, and in view of the great difficulties in the interpretation of thick-target data, it is our opinion that the cause is probably there.

This conclusion is of especial importance for the major question, which of the theoretical graphs in Fig. 3 is the best; but there is also the question, whether any of them is exact, so it may be well to note here some additional points about the accuracy of these data, not related to these corrections. Three possible sources of systematic error were mentioned in our first paper on these data. One was rediffusion (we had not realized the importance of diffusion within the thin target); another was the presence of some fluorescence radiation, never more than 0.2 percent and therefore negligible; the third was an unexplained darkening of the focal spot during use. This has since been explained, through the kindness of Dauvillier, who pointed out to us by letter an investigation of his,[22] proving such darkening to be due to carbon, which in our case could have come from the carbonization of adsorbed grease vapor by cathode rays (not by heat, because the block was oil-cooled and the power input was low). If this carbon were thick enough to retard the cathode rays appreciably, it might cause serious errors, especially at low voltages. Fortunately, however, after finishing the measurements shown in Fig. 2, we made a very careful study of the intensity function in the first 2 kilovolts above $V_K$. Here we found effects exactly such as one would predict for a 280A silver target assuming that it retarded the cathode-rays at 25 kv. by about 200

[22] A. Dauvillier, Journ. d. Physique **8**, 1 (1927).

volts, as predicted by the Thomson-Whiddington law, and that an infinitely thin target would give an intensity starting from zero at $U=1$, with a finite derivative. These measurements will be reported in more detail later, along with others with improved apparatus. For the present, the important point is, that if there had been any appreciable retardation in the carbon, it would almost surely have been evident here. So there can hardly have been any serious error from that source.

A similar possibility is that of a deposit of tungsten. But this is ruled out, not only by the above test, but also by the absence of the tungsten $K$ lines at 80 kv.

There may of course be sources of systematic error still unsuspected, and there are the erratic errors, whose magnitude can be judged by the departures of the points from the empirical graph in Fig. 3. But for the present, we shall assume that Fig. 3 is accurate enough to serve as a basis for conclusions about the theories now at hand.

## Comparison with Theories

Of the theoretical graphs in Fig. 3, those for Davis and Rosseland are plotted from Eqs. (11) and (12) respectively, and that for Thomas[24] from

$$i_T(U) = k_T \frac{1 - 1/U + (2/3)m_T(1 - 1/U^2)}{1 + U + m_T}, \tag{23}$$

where the parameter $m_T$ is the ratio of the kinetic energy of an atomic electron in a Bohr orbit to its ionization energy.

While these theories based on classical mechanics give definite formulas, the wave mechanics, so far as we know, has been applied only through approximations that become invalid at voltages near the ionization voltage. On this basis Bethe[23] gives a formula, in which the dependence of intensity on $U$ may be expressed by

$$i_B(U) = k_B U^{-1} \log (4U/B) \tag{24}$$

where $BeV_K$ is a quantity of energy, "of the order of magnitude of the ionization energy," $eV_K$.

Comparing this function with Wisshak's data, which cover about the same range of $U$ as ours, Bethe states that it is probably unsatisfactory for this range, "because the rise of x-ray intensity with increasing excess of the bombardment energy over the ionization energy must be even steeper than in the Thomson [or Rosseland] theory, whereas experimentally it is somewhat more level." Comparing the theories with our experimental graph, however, we should exactly reverse this last clause. In view of the approximations in Bethe's theory, it must of course become invalid as $U$ is reduced toward unity, but in most of this range it apparently agrees with the thin-target data better than with the thick.

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[23] H. Bethe, Ann. d. Physik **5**, 325 (1930).

[24] L. H. Thomas, Proc. Camb. Phil. Soc. **23**, 829 (1927).

In the absence of any theoretically calculated value of $B$, beyond the order-of-magnitude estimate quoted above, we may treat it tentatively as an empirical parameter, and choose a value such as to make the equation fit the data as far down the scale of $U$ as possible. Since $i$ must be zero at $U=1$, this is $B=4$. We have therefore plotted a graph on this basis in Fig. 3, but in view of the arbitrary character of this assumption, we cannot call it exactly a graph for Bethe's theory, and so we are not giving it any name.

A suggestion as to the direction in which a true graph for Bethe's theory might differ from this one is given by his calculation of the effective cross-section of a hydrogen atom for inelastic collisions (including excitation as well as ionization). Here there is a quantity $b$, analogous to $B$. At high voltages, $b$ is constant at 0.638; but inspection of the equations from which it is obtained shows that with reduction of voltage it must begin to increase. If $B$ behaves likewise, the true graph for Bethe's theory may run somewhat steeper than the nameless graph of Fig. 3, and thus perhaps agree even better with our experimental graph.

Another interesting aspect of Bethe's equation is its agreement in mathematical form with $i_2$, at least if $m_2$ is set equal to 1, as well as $B$ to 4. This function $i_2$ was suggested first by a study of an empirical relationship between line and continuous spectra; and there some of the data indicated that $m_2=1$. Thus perhaps Bethe's theory may suggest theoretical connections, to match the empirical one between these spectra.

## THICK-TARGET EFFECTS

**Effect of rediffusion on line intensities.** In a thick target, the effect of rediffusion, removing some of the cathode rays from the target before they have made their quotas of impacts, is to diminish the intensity of the x-rays. Let $I_0(U)$ be the intensity of the rays that would be emitted in a $K$-series line if there was no such loss, and $I(U)$ the actual intensity. Then the loss,

$$I_0(U) - I(U) = R(U), \tag{25}$$

is equal to the intensity of the rays that the rediffused electrons would emit if they struck another target of the same element and did not rediffuse from it. Thus

$$R(U) = p\int_{1/U}^{1} F(W)I_0(UW)dW. \tag{26}$$

To calculate $R(U)$ we shall proceed much as with the analogous equation, (8), for $r(U)$. The chief difference in the two cases is that the sign of the correction for rediffusion here is opposite to what it was there, because the effect is a loss of impacts rather than a gain.

For $I_0$ in this integrand, we shall use either of two simple empirical formulas. One[25] of these is

$$I_1(U) = K_1(U-1)^n, \tag{27}$$

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[25] D. L. Webster and H. Clark, Proc. Nat. Acad. Sci. **3**, 181 (1917); A. Jonsson, Zeits. f. Physik **43**, 845 (1927); S. K. Allison, Phys. Rev. **30**, 245 (1927).

where $n$ is between 1.3 and 2.1, and $K_1$ is a constant. The other[26] is

$$I_2(U) = K_2(U^2 - 1), \tag{28}$$

where $K_2$ also is constant.

Neither of these formulas is very accurate, but in view of other corrections, less definitely known than the one for rediffusion, we shall go no farther with this case than to find the order of magnitude of this correction. And for that, we may use either function, simplifying matters still further by letting $n$ = either 1 or 2, although $I_1$ with $n = 1$ is very far from correct.

The constants $K_1$ and $K_2$ are not so little known as those of the thin-target functions, but there is enough uncertainty here also to make it better for most purposes to use a "reduced intensity," which we shall call

$$J(U) = I(U)/I(2). \tag{29}$$

The corrected reduced intensity is now

$$J_0(U) = J(U)\cdot\frac{1 + \{R(U)/I(U)\}}{1 + \{R(2)/I(2)\}} = J(U)\cdot\frac{1 - \{R(2)/I_0(2)\}}{1 - \{R(U)/I_0(U)\}}, \tag{30}$$

and this correction factor for $J(U)$ will be called $C(U)$.

Values of $C(U)$, corresponding to the $c_r(U)$ of Table III, are shown in Table VII.

TABLE VII. *Values of 100* $(C(U)-1)$ *for Ag,* $p=0.39$ *and* $W_0=0.90$.

| $U=$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 |
|---|---|---|---|---|---|---|
| $I_1$, n=1, $g=10$ | −23.8 | −7.2 | 0.0 | +3.3 | +5.0 | +6.1 |
| $I_1$, $n=1$, $q=9$ | −26.0 | −7.4 | 0.0 | +3.0 | +4.5 | +5.5 |
| $I_2$, $g=10$ | −21.5 | −5.9 | 0.0 | +2.3 | +3.4 | +4.1 |
| $I_2$, $q=9$ | −23.7 | −6.1 | 0.0 | +2.2 | +3.2 | +3.8 |
| $I_1$, $n=2$, $g=10$ | −17.0 | −6.0 | 0.0 | +3.1 | +4.9 | +6.1 |
| $I_1$, $n=2$, $q=9$ | −19.0 | −6.6 | 0.0 | +3.2 | +5.0 | +6.2 |
| Averages | −21.8 | −6.5 | 0.0 | +2.8 | +4.3 | +5.3 |
| 100 $(c_r-1)$ from Table 4 | 3.6 | 1.8 | 0.0 | −1.3 | −2.2 | −2.7 |
| $(C-1)/(c_r-1)$ | −6.1 | −3.6 | −2.6 | −2.2 | −2.0 | −2.0 |

From this table it is evident that the rediffusion correction factors for thick-target line intensities are about 2 to 4 times as far from unity as those for thin. This raises anew the question of the cause of the disagreement between our conclusions and Wisshak's,[2] about the status of the theories of ionization by impact. In Wisshak's interpretation of his data, the allowance for rediffusion consisted in substituting for the measured tube current, the current computed from the voltage and the heat in the target. But as the ratio of these currents was found to be constant, the difference affects only the constant coefficient $K$, in $I(U)$, and does not affect $J(U)$ at all. So it appears to us that the correction factor $C(U)$ would be useful here. And just as the cor-

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[26] Simplified from a formula used by G. Kettmann, Zeits. f. Physik **18**, 359 (1923); and E. C. Unnewehr, Phys. Rev. **22**, 529 (1923).

rection $c_r$ for thin targets shifted our graph toward Wisshak's, this opposite correction $C$ shifts his graph toward ours. Nevertheless, both corrections together move the graphs only about a seventh as far as is necessary to bring them into agreement. Evidently there is some other cause for the difference.

The most obvious suggestion is that there is a real difference between the elements used here by Wisshak and by us, namely between his average of Cr and Cu and our Ag. Another possibility is that the other uncertain factors in the interpretation of thick-target data are responsible for the discrepancy. So it is significant, though ambiguous, that recent data of ours (not yet published) on thick targets of Ag with corrections for target absorption by Kulenkampff's method and for fluorescence radiation by ours,[27] give a far better agreement with the function calculated by Wisshak from our thin-target formula for Ag than do his data on Cr and Cu.

To put this statement into a more quantitative form, we may express all these intensity functions with enough accuracy in terms of the function $I_1(U)$, of Eq. (27), namely $K_1(U-1)^n$. Wisshak's graphs for Cr and Cu make $n$ between 1.3 and 1.4, and his function calculated from our thin Ag makes $n$ about 1.85. Our data on thick Ag make $n$ about 1.65 before correction for rediffusion, and about 1.73 after. The remainder from 1.73 to 1.85 is in the direction one would expect from Kulenkampff's[28] comparison of his continuous-spectrum intensities for thin Al with the intensities calculated from his thick Al by a method the inverse of that which Wisshak used here. So this discrepancy may be ascribed tentatively to the errors in the Thomson-Whiddington law of retardation of the cathode rays, which has been used, for lack of anything more accurate, in all such calculations. The net result, then, is that the apparent discrepancy between the thick and thin-target data may well be no greater than can be expected from the uncertainties in the relations between them.

Details of these comparisons would be beyond the scope of this paper, however, and will therefore be reserved for later consideration, in connection with more data. In the meantime, the values of $n$ given here for thick Ag must be regarded as somewhat provisional, as we are making still further improvements in our apparatus, but they are fairly good up to about 100 kv ($4\ V_K$); and for rays corrected only for target absorption, an exponent 1.6 will serve moderately well up to 180 kv.

**Continuous spectra.** Turning from the line spectra of thick targets to the continuous, we are rid of the complications of fluorescence, but we run into another complication, arising from the fact that the intensity emitted by a single atom (or a thin target) is not the same in all directions, as proved by Kaye,[29] Duane,[30] Kulenkampff[28] and Nicholas.[31] As it concerns us here, this means that the diffusion of the cathode rays within a thick target will

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[27] D. L. Webster, Proc. Nat. Acad. Sci. **14**, 330 (1928).

[28] H. Kulenkampff, Ann. d. Physik **87**, 597 (1928).

[29] G. W. C. Kaye, Proc. Camb. Phil. Soc. **15**, 269 (1909).

[30] W. Duane, Proc. Nat. Acad. Sci. **15**, 805 (1929).

[31] W. W. Nicholas, Bureau of Standards Jour. Res. **2**, 837 (1929).

tend to annul the directional differences that would have occurred without it. This effect has been discussed by Kulenkampff and Nicholas, who have shown that it is practically complete except near the high-frequency limit of the spectrum, and that the directional effects near the high-frequency limit are such as one might expect from the data on thin targets. Having no new contribution to make on this point, we may treat the effect of rediffusion alone by neglecting the directional effects and comparing theoretical predictions, not with intensities in any particular direction, but with the average, so far as it is known, for all directions.

This treatment of the rediffusion effect amounts practically to putting on a somewhat more nearly quantitative basis an idea advanced by Kramers[32] in a paper on his theory of the continuous spectrum. First neglecting rediffusion as well as all directional effects, Kramers found for the thick-target intensity in a range $d\nu$.

$$I_0(V,\nu)d\nu = K_0 Z(\nu_0 - \nu)d\nu = K_0 Z\nu_0 u, \tag{31}$$

where $K_0$ is a constant, $\nu_0 = eV/h$, and $u = (\nu_0 - \nu)/\nu_0$. Observed intensities are not far from obeying this formula, but they are given more accurately by adding a non-linear term,[33] so that the empirical formula is something like

$$I(V,\nu) = K'Z\nu_0[u + BZ(1 - e^{-Au})], \tag{32}$$

where $A$ is a function of $Z$, $\nu$ and $\nu_0$, and $B$ of $\nu$ and $\nu_0$, though perhaps not of $Z$, and both $A$ and $B$ change only slowly. $K'$ will be defined more explicitly below. Kramers explained the presence of a non-linear term of this general sort, as an effect of rediffusion.

A calculation of $I$, based on Kramers' theoretical intensity $I_0$ and the rediffusion formulas used in the first part of this paper, gives, when $F_1(W)$ is used,

$$I_1(V,\nu) = K_0 Z\nu_0[(1-p)u + p\{(1-\overline{W})(1-e^{-gu}) - ue^{-gu}\}], \tag{33}$$

at least if $e^{-g} < < 1$, and when $F_2(W)$ is used,

$$I_2(V,\nu) = K_0 Z\nu_0[(1-p)u + p\{(1-\overline{W})(1-v^{q+3}) - uv^{q+2}\}], \tag{34}$$

where $v = 1 - u$.

Each of these functions is much like the empirical function of Eq. (32), in that the larger term, $(1-p)u$, is proportional to $u$, while the smaller, non-linear, term increases from zero at $u=0$, more or less like $(1-e^{-Au})$. Furthermore the ratio of these terms, as a function of $Z$, contains $p/(1-p)$, which is not far from proportional to $Z$, like the corresponding ratio in Eq. (32).

With regard to the dependence of the linear term on $Z$, it is usually said to be simply proportional to $Z$, with $K'$ in Eq. (32) constant. The best evidence on this point, however, comes from the data of Wagner and Kulen-

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[32] H. A. Kramers, Phil. Mag. **46**, 836 (1923).

[33] D. L. Webster, Phys. Rev. **9**, 220 (1917); E. Wagner and H. Kulenkampff, Phys. Zeits. **23**, 503 (1922); D. L. Webster and A. E. Hennings, Phys. Rev. **21**, 312 (1923); W. W. Nicholas, Phys. Rev. **29**, 619 (1927).

kampff,[34] who made an approximate allowance for rediffusion by defining intensity, not as x-ray energy per incident cathode ray, but per unit current as measured by heat in the target. It is in these terms only, that the $K'$ of Eq. (32) is approximately constant. To convert to the intensity definition used here, we must let

$$K' = K(1 - p\overline{W}), \tag{35}$$

with $K$ constant. This new factor $(1-p\overline{W})$, is not far from the $(1-p)$ of Eqs. (33) and (34).

Likewise on integrating to get the total intensity, unresolved, we find

$$E_1 = E_2 = \tfrac{1}{2}K_0\, Z\nu_0^2(1 - p\overline{W^2}), \tag{36}$$

and from Eq. (32), neglecting the second term,

$$E = \tfrac{1}{2}K'Z\nu_0^2 = \tfrac{1}{2}KZ\nu_0^2(1 - p\overline{W}), \tag{37}$$

agreeing fairly well with $E_1$ or $E_2$.

Thus these formulas describe the main facts of thick-target continuous spectra to a first approximation, confirming Kramers' qualitative predictions. We cannot expect great accuracy here, because the basic formula for $I_0(V, \nu)$ was derived by the Thomson-Whiddington law from a theoretical thin-target formula which agrees only approximately with Kulenkampff's data, and so $I_0(V, \nu)$ itself may be as good as it is, only because of a cancellation of errors. An approximate verification of these thick-target formulas is therefore as much as one can expect.

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[34] E. Wagner and H. Kulenkampff, reference 33.