SOME MEASUREMENTS OF THE LONGITUDINAL ELASTIC FREQUENCIES OF CYLINDERS USING A MAGNETOSTRICTION OSCILLATOR

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Abstract

Cylindrical rods cut from two samples of stainless steel are excited to longitudinal vibration in a magnetostriction oscillator and the resonant frequencies at which the rods control the oscillator are measured by beating with a crystal oscillator of known frequency. Many rods of each material are measured in order to test the theoretical relation given by Lord Rayleigh between the natural longitudinal frequency and the dimensions, in particular the effect of the diameter of the rod on its frequency.

Evidence is presented to show that, in general, the frequency measurements are good to 0.01 percent or better. The lengths of the rods are known to 0.02 percent or better and the diameters to from 0.05 percent to 0.5 percent according to the size of the rod.

A very brief theoretical discussion is given to show that if Rayleigh's frequency equation for the "free-free" longitudinal vibration is sound then the theory of G. W. Pierce for the magnetostriction oscillator leads to approximate frequency equation of the same form but with slightly different constants for the case where the cylinder is driven in a magnetostriction oscillator at resonance.

The relation between the measured frequencies and the dimensions is shown graphically for each of the two kinds of stainless steel. On the same graphs are plotted curves for the theoretical formula with constants chosen to give the best fit with the experimental points. Agreement of 0.2 percent or better is obtained for one of the samples and 0.1 percent or better for the other. It is pointed out that in all probability a large part of this deviation from the theoretical curves is due to the lack of uniformity of consistancy in the alloys used.

THE magnetostriction oscillator, invented by Professor G. W. Pierce,[1] offers a means of maintaining and accurately measuring the frequencies of longitudinal elastic vibrations in bars and rods. The method is, of course, limited to the ferromagnetic metals and only such of those as show strong magnetostrictive properties, but the frequencies can be measured to one part in ten thousand so that it has been interesting to change the dimensions of a cylinder of stainless steel in small steps and determine its longitudinal frequencies after each cutting. By this process it has been possible to examine the nature of the dependency of frequency on the length and diameter of the cylinder over a certain range of dimensions. It turns out, as will be seen, that this relation between the frequencies and the dimensions is very nearly that predicted for the "free-free" vibrations of a cylinder by elastic theory.[2]

[1] G. W. Pierce, Proc. Am. Acad. Sci. **63**, (1928).

[2] "Mathematical Theory of Elasticity" by A. E. H. Love. Cambridge University Press.

Experimental

The apparatus used consisted of a magnetostriction oscillator with one stage of amplification, a piezoelectric oscillator also with one stage of amplification, and an audiofrequency meter with telephones. These three pieces were connected together as shown in Fig. 1. The outputs of the two oscillators were joined and connected to the input of the frequency meter so that any audible beat frequencies could be measured.

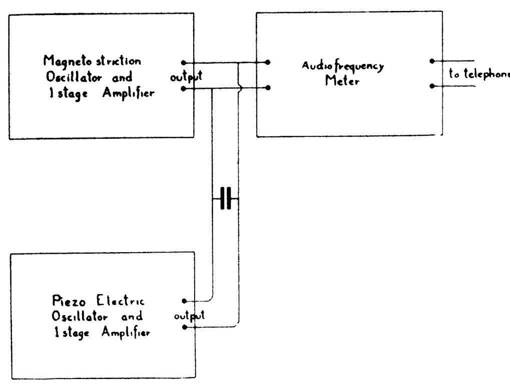

Fig. 1. Showing the arrangement of the apparatus.

The magnetostriction oscillator was of the type invented by Professor Pierce and described by him in the paper referred to in footnote[1] below. It was a commercial model manufactured by the General Radio Company, of Cambridge, Massachussetts. In Fig. 2 the wiring diagram for this oscillator and its accompanying single stage of amplification is reproduced

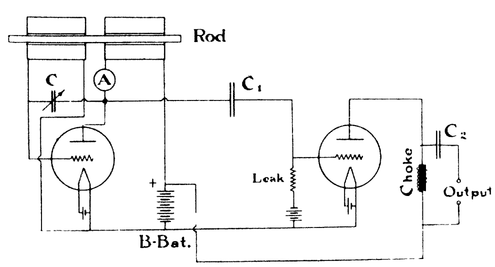

Fig. 2. The wiring diagram of the magnetostriction oscillator.

from Professor Pierce's paper. In as much as the oscillating circuit must be tuned to a natural period of the rod to start it vibrating, any one pair of coils (i.e. grid and plate coils) is useful only for frequencies falling within a certain range which is determined by the range of the variable condenser. For this research it was desirable that the oscillator be tuneable to any frequency between five thousand and one hundred thousand cycles without disturbing the rod in the coils. To effect this the grid and plate coils were made up of four coils each as is shown by the wiring diagram in Fig. 3. The two four-point switches enable one to change the coil combination without disturbing the rod. The method of mounting the rod by balancing it on a narrow wooden support placed between the grid and plate coils is also shown in Fig. 3. The plate winding is about the rod on one side of its center and the grid winding on the other side.

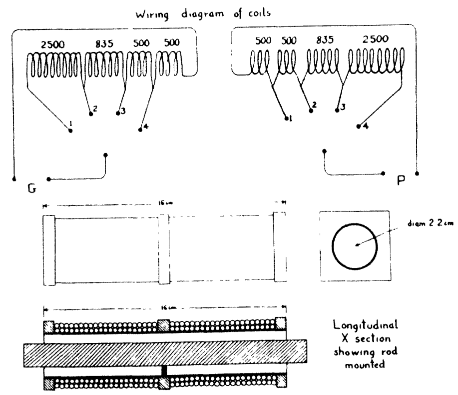

Fig. 3. The wiring diagram and mounting of the coils.

The circuit of the piezoelectric oscillator with its single stage of amplification is shown in Fig. 4. The quartz crystal was one of the Cruft Laboratory standards known as number 28. It maintained the frequency of the oscillator

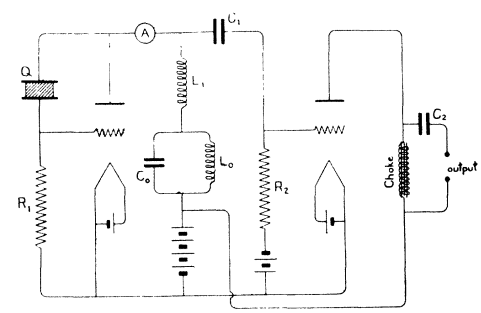

Fig. 4. The wiring diagram of the piezoelectric oscillator.

circuit at $28{,}067.5 \pm 0.2$ cycles. This frequency was determined at the start of the research by calibrating the piezoelectric oscillator in the manner described on pages 18–19 of Professor Pierce's paper "Magnetostriction Oscillators."[1] Since it was found that other factors prevented the measurement of the rod frequencies to better than one part in ten thousand this accuracy was considered sufficient in checking the crystal calibration at the close of the research. The crystal frequency was then found to be $28{,}069 \pm 2.0$ cycles. The apparatus was mounted in a "constant temperature" room whose temperature did not change by more than two degrees centigrade throughout the year and in as much as the temperature coefficient of frequency for the crystal in the mounting used had been found to be about one cycle in one hundred thousand per degree centigrade at 20°C. it was not necessary to correct the crystal frequency for temperature to ensure accuracy to one part in ten thousand.

The audiofrequency meter was identical with the one described on page 22 of "Magnetostriction Oscillators."[1] The wiring diagram given there is reproduced in Fig. 5. The instrument has three ranges covering frequencies of from 500 to 5000 cycles and is equipped with a direct reading scale.

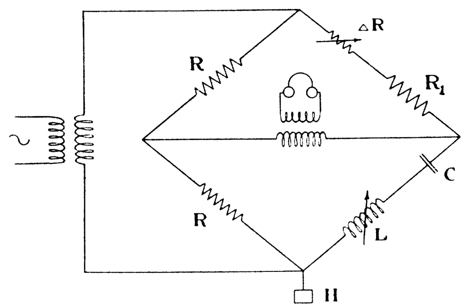

Fig. 5. The wiring diagram of the audiofrequency meter.

The rods measured were of commercial stainless steel. For the first three series of measurements all the rods were cut from a single cylinder which was 24.33^8 cm long by 1.805^7 cm in diameter. For the first series this piece was shortened by steps which became smaller as the rod grew shorter. After each cutting the frequencies of all the modes of longitudinal vibration that could be excited were measured. For the second and third series two of the pieces left from the cuttings of the first series were reduced in diameter a little at a time, frequencies being measured for each diameter. The rods measured in the fourth and fifth series were all cut from a second piece of stainless steel obtained from another source. At the start is was machined to a length of 12.983^7 cm and a diameter of 1.905^0 cm. For the fourth series of measurements the diameter was cut down in small steps and for the fifth series the piece that remained was shortened in steps.

Stainless steel was chosen for these measurements simply because it is easily obtained and works well in the magnetostriction oscillator. Each of the samples was annealed and each rod was permanently magnetized before its frequencies were measured.

The machining of the rods was done as accurately as was possible in the shop of the Cruft Laboratory. The lengths of the rods were uniform over the end faces to within some two or three ten thousandths of an inch so that the lengths of rods that were longer than two inches were known to one part in ten thousand or better. All of the diameters were uniform to within one half of one thousandth of an inch and for most of the rods the uniformity was considerably better than this. The possible error in the value taken as the diameter of any rod varied between one percent and one tenth of one percent according to the size of the rod. Since changes in diameter have a much smaller effect on the frequencies of longitudinal vibration than changes in the length this accuracy was sufficient for the purpose of this research.

The method of procedure was as follows: The rod whose frequency was to be measured was inserted in the coils and balanced on the narrow wooden support (see Fig. 3). After time had been allowed for the rod to come to the temperature of the surrounding air the magnetostriction oscillator was turned on and tuned by means of the condenser till the needle of the plate circuit direct current milliammeter "kicked," showing resonance of the oscillator with a mode of vibration of the rod. Then the crystal oscillator was turned on and the magnetostriction oscillator readjusted so that the milliammeter needle was at the peak of its "kick." Finally the frequencies of the audible beats between the two oscillators were measured with the audio frequency meter. A record was kept of the beat frequencies, the condenser setting, the positions of the switches on the grid and plate coils (see Fig. 3), the temperature of the air surrounding the rod, and the dimensions of the rod. In order to use these beat frequencies to determine the frequency of the rod one must know beforehand the approximate value of that frequency. This was was obtained by using the simple formula given by elementary theory, namely

$$f = \frac{n}{2l}\left(\frac{E}{\rho}\right)^{1/2}$$

where

f =frequency, $n=1, 2, 3$, etc. any + integer.
l =length of the rod, E=Young's modulus
ρ density of the rod.

The density of each of the samples was measured and the value of E given in Professor Pierce's paper[1] for stainless steel was used.

With the exception of the lengths of the four longest rods all the dimensions were measured with micrometer calipers. Enough readings were taken in each case to determine the uniformity of the dimension in question. The four lengths which were too long for the largest available micrometer were measured by using a cathetometer and an auxiliary cylinder marked with a fine scratch parallel to its plane ends.

Table I below is a sample of the data taken for the frequency measurements on a single rod. The abbreviations heading the columns of this table are to be interpreted as follows:

N refers to the mode of longitudinal vibration, $N=1$ for the fundamental, $N=2$ for the next to the gravest mode etc.

P and G refer to the positions of the switches of the plate and grid coil respectively i.e. the combination of coils being used in each case. Under *Cond.* are the condenser settings. Under *Temp.* are the temperatures of the air about the rod. The beat frequencies are tabulated under *Beats*. f heads the column of computed rod frequencies and in the next column these frequencies are corrected to twenty degrees centigrade which was taken as the standard temperature. The pairs of numbers in the last column refer to the harmonics (produced in the vacuum tubes) of the rod and crystal frequencies that are causing the beat frequency measured.

TABLE I. *Sample of data: Frequency measurements on rod No. 6 of the first series, beating with crystal No. 28, f=28,067.6 March 13, 1929.*

			Data				Results	
N	P	G	Cond.	Temp.	Beats	f	f at 20°	C–R
1	2	2	28	23.5	1893	23,788	23,799	6–7
1	2	2	28	23.5	4275	23,793	23,804	1–1
1	2	2	28	23.5	2388	23,788	23,799	5–6
1	3	2	53	23.5	4280	23,788	23,799	1–1
1	3	2	53	23.5	2386	23,787	23,798	5–6
1	3	3	62	23.5	1898	23,787	23,798	6–7
1	3	3	62	23.5	4280	23,788	23,799	1–1
1	3	3	62	23.5	1420	23,786	23,797	17–20
2	3	3	23	23.5	1861	47,399	47,422	5–3
2	4	3	48	23.5	1898	47,412	47,434	5–3
2	4	4	94	23.5	1830	47,389	47,412	5–3
3	4	3	23	23.5	747	70,543	70,576	5–2
3	4	4	45	23.5	744	70,541	70,574	5–2

In the above table are the data from just one of the sets of measurements made on rod #6. Every rod was subjected to two or more independent sets of measurements with intervals ranging from several hours to several days between them. The temperature was often different at the times of these repetitions so that the agreement of the frequencies found for the standard temperature furnished a check on the value of the temperature coefficient of of frequency used. (This coefficient was taken from Professor Pierce's paper "Magnetostriction Oscillators."[1]) An examination of the complete data shows that in many of the measurements two or more beat frequencies were observed and recorded for the same mode of vibration of the rod and with the same coil combination and condenser setting. An example of this may be seen in the sample data above for the fundamental mode of rod #6. Separate computations of the rod frequency from these beat frequencies served as a check on the calibration and readability of the audiofrequency meter. Furthermore, in many cases, a vibration-mode of a rod could be excited with two or more different combinations of coils in the grid and plate circuits of the oscillator. This may also be noted in the sample above. The results of these measurements with different coils furnished a check on the assumption that the frequencies might be considered peculiar to the rods and the effects of the coils neglected.

With a very few exceptions, 19 in the 133 measurements made, all the frequencies obtained for any one mode of any one rod under these different conditions agreed to better than one part in ten thousand when reduced to the standard temperature. The second gravest mode of rod #6 was one of the exceptions to this as may be seen from the sample data above. This experimental fact is the foundation for the statement that the frequencies could be measured to one part in ten thousand or better. In plotting the results the mean of the several frequencies found for each mode of vibration of each rod was taken as the frequency of that mode.

THEORETICAL

The equations for the elastic vibration of a "free-free" cylindrical rod have been solved approximately by Pochhammer[3] and by Chree.[4] The solution is approximate in that it does not satisfy the boundary condition that the tractions or tangential forces on the ends of the rod must be zero. As Love[5] points out, the tractions on the ends introduced by this solution are small if the radius of the rod is small compared to its length. The frequencies of the rod are given by an equation in Bessel's functions and their derivatives but if one neglects the fourth and higher powers of na/l one may obtain the following expression for the frequency: (see Love, pages 289–290)

$$f = \frac{n}{2l}\left(\frac{E}{\rho}\left(1 - \frac{n^2\pi^2\sigma^2a^2}{2l^2}\right)\right)^{1/2} \tag{1}$$

where f = frequency, $n = 1, 2, 3$, etc. i.e. a positive integer, l = length of rod, E = Young's modulus, ρ = density of rod, σ = Poisson's ratio, a = radius of rod.

Lord Rayleigh[6] has obtained the same result much more simply by assuming the standing waves in the rod to be made up of plane waves normal to the longitudinal axis. Under this assumption all the boundary conditions can be fulfilled and the frequency is given by

$$f = \frac{n}{2l}\left(\frac{E}{\rho\left(1 + \frac{n^2\pi^2\sigma^2a^2}{2l^2}\right)}\right)^{1/2} \tag{2}$$

where the letters have the same meanings as above. This equation becomes identical with (1) if $(na/l)^4$ is negligible with respect to 1.

Since the effects of "viscosity" in the rod have been neglected in what has been said above, the resonance frequencies for "forced-free" vibration will be given to the same degree of approximation by these same equations. However, in this research the rods were not "forced" in the usual sense of the word for the reaction of the vibrations of the cylinders on the oscillator was an important part of the operation of the apparatus and may not be neglected. Briefly, the changing magnetic induction in the rod due to the changing current in the coils forces vibration of the rod, which vibration, in turn, changes the magnetic flux in the rod and sets up E.M.F.'s in the coils. When the electric circuits are tuned to resonance with a mode of

[3] L. Pochhammer, J. für Math., Crelle, **881**, 324 (1876).

[4] C. Chree, Quart. Jour. of Math. **21**, (1886).

[5] "The Mathematical Theory of Elasticity," by A. E. H. Love, fourth ed., Camb. Univ. Press.

[6] "Theory of Sound," by Lord Rayleigh, Vol. 1.

vibration of the rod there is a sudden change in the impedance of the coils due to the "motional" impedance of the rod. This sudden change in the impedance of the coils produces a change in the average plate current and therefore kicks the needle of the D.C. plate circuit milliammeter.

Now one can put into the equations of motion for the rod (assuming a plane wave after Rayleigh,—see Love[5] page 428) terms to take care of the forcing of the rod by magnetostriction as well as the magnetostrictive reaction of the vibrations of the rod on the magnetic induction through the coils. (see Pierce's "Magnetostriction Oscillators"[1]). If the amplitude of the alternating magnetic induction in the rod is small enough so that one may say that the longitudinal "pressure" on a small piece of the rod is proportional to the induction in that piece, i.e. $p = qB$, where q is a constant; and if the strains produced in the rod are small encugh so tnat one may say that the magnetic induction caused by them (longitudinal strains) is proportional to them i.e. $B' = q'(\delta\xi/\delta x)$, where q' is a constant and ξ is the displacement in the x direction which is along the longitudinal axis of the rod, then the equation of motion is linear and the frequencies for resonance are given by

$$f = \frac{n}{2l}\left(\frac{E'}{\rho\left(1 + \dfrac{n^2\pi^2\sigma^2a^2}{2l^2}\right)}\right)^{1/2} \tag{3}$$

which is the same as formula (2) except that E is replaced by E' where

$$E' = E + qq'.$$

Assuming that E', ρ, and σ are the same for all the rods of the same material this formula gives a relation between the dimensions and frequencies of the rods that can be tested by the results of the measurements described above. Actualy E and ρ may vary slightly from one rod to another because the materials used were commercial alloys and the original cylinders may have varied slightly in consistency. Furthermore the values of q and q' depend on the magnetic polarization of the rod,[7] though less in the case of stainless steel than in that of some of the other magnetostrictive metals that might have been used for the research.[8] This polarization was not the same for the different rods for they were all magnetized in the same field wherefore the shorter ones were left with less residual magnetism than the longer ones because of the demagnetizing effect of their ends.

It should also be pointed out that formula (3) applies only when the surfaces of the rods are entirely free from forces. Actually, in these experiments the rods were surrounded by the atmosphere and supported at their midpoints on a narrow piece of wood. To test the effect of the air about the rod the first four modes of rod #2 were measured both in the atmosphere and with rod and coils inside a large brass cylinder in which the pressure was reduced to five millimeters. The frequencies were found to

[7] Honda, Shimizu, and Kusakabe, Phil. Mag. **4,** 459 (1902) and 537 (1902).
[8] K. C. Black, Proc. Am. Acad. Sci. **63,** 49 (1928).

be the same to the limit of accuracy of the measurements i.e. one part in ten thousand. This, of course, does not show that under certain conditions of resonance of the rod's frequency with a natural period of the air column formed by the tubing on which the coils are wound the frequency of the rod will not be affected. In connection with the support, it was found that substituting two supports at the ends of the rod for the one at the center did change the frequencies of the lower modes of vibration by several hundredths of a percent while hanging the rod on a linen thread at its mid-point did not change the frequencies. Furthermore it was noticed that if a rod was slightly unbalanced so that one of its ends touched the wall of the surrounding tube the frequency was not measureably changed.

For convenience the frequency equation (3) above may be re-arranged to give

$$\frac{2fl}{n} = \left(\frac{K}{1 + C(n^2a^2/l^2)}\right)^{1/2} \tag{4}$$

where

$$K = \frac{E'}{\rho} \quad \text{and} \quad C = \frac{\pi^2\sigma^2}{2}$$

and since the theory on which this frequency equation is based assumes plane standing waves in the rods one may call $2l/n$ the wave-length in the rod, whence

$$\frac{2fl}{n} = \text{wave velocity}$$

and

$$\frac{na}{2l} = \frac{\text{radius}}{\text{wave-length}}.$$

Results

The results of these frequency measurements are shown graphically in Figs. 6 to 10. The ordinates in each case are the values of $2fl/n$ (the "wave velocity") while the abscissa are the common logarithms of $100\,na/l$. (na/l is the ratio of diameter of the rod to "wave-length").

In Fig. 6 are plotted the results for the first series of measurements in which rod #1 (length 24.338 cm, diameter 1.8057 cm) was shortened in fourteen steps to a final length of 3.2299 cm. In this series eight modes of vibration could be measured for the longest rod, six for the second longest rod, four for each of the next two in order of length, three for each of the next three, and the fundamental mode only for the remaining shorter rods. The numbers beside the points refer to the rods. For instance, there are eight points numbered 1 which are the plottings of the eight modes of the longest rod of the series. The curve drawn out in the figure is the graph of

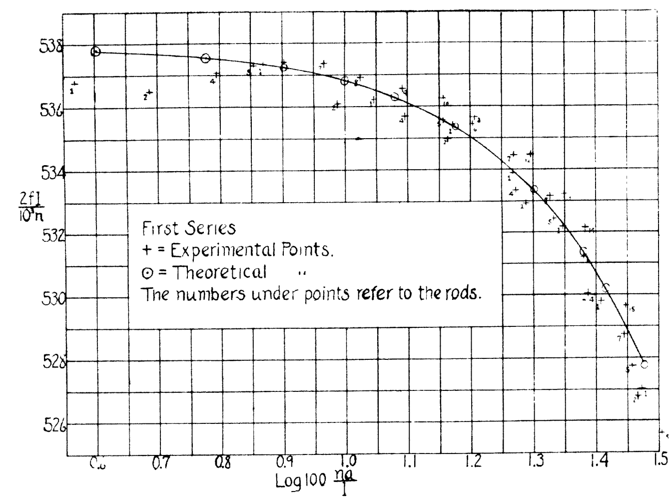

Fig. 6. Graph showing relation between the velocity of the standing wave components and the ratio of diameter to wave-length for the first series.

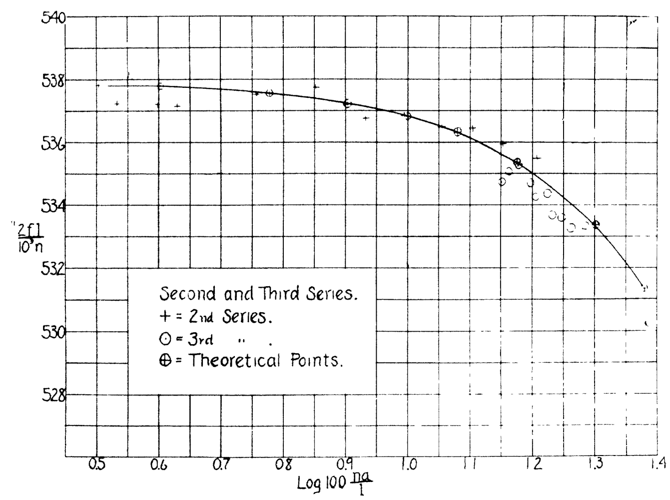

Fig. 7. Graph showing relation between the velocity of the standing wave components and the ratio of diameter to wave-length for the second and third series.

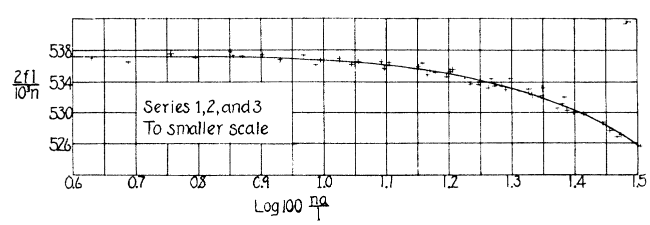

Fig. 8. Graph for the first second and third series to smaller scale to aid in fitting a curve to the points.

Eq. (4) with E'/ρ taken equal to 28.945×10^{10} and σ taken as 0.30. It will be noticed that the experimental values of $2fl/n$ fall within two tenths of a percent of the values given by this curve.

In Fig. 7 are plotted the results of the second and third series. The second series of measurements was made on a piece left from the cuttings

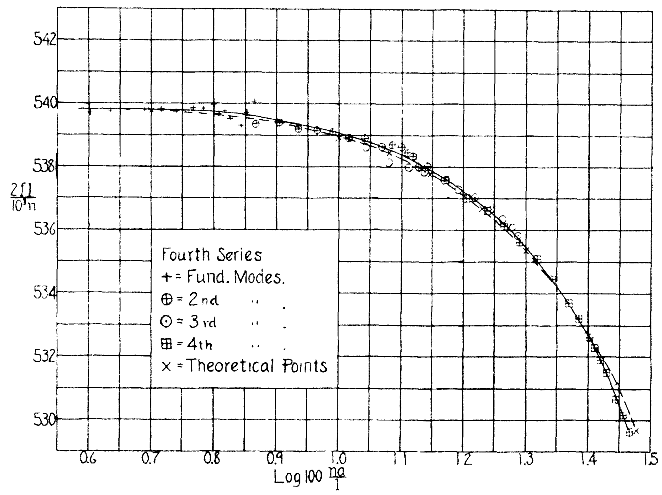

Fig. 9. Graph showing the relation between the velocity of the standing wave components and the ratio of diameter to wave-length for the fourth series.

of the first series (length 5.5946 cm) which was turned down till it became too small to control the oscillator. The third series was made by turning down a shorter piece (length 3.6995 cm) from the first series. The curve drawn is the graph of Eq. (4) using the same values for the constants as in Fig. 6.

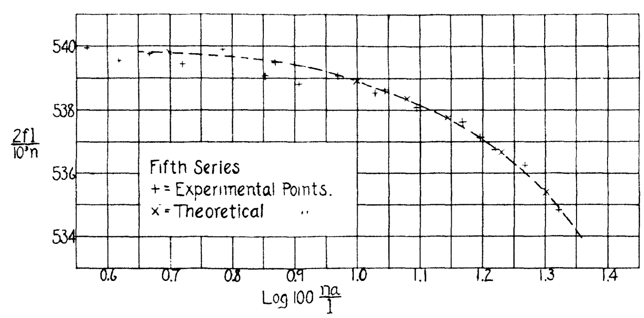

Fig. 10. Graph showing the relation between the velocity of the standing wave components and the ratio of diameter to wave-length for the fifth series.

The method of obtaining the constants for plotting this curve of Eq. (4) was to plot the points of the first three series all together and to a smaller scale so that the best fitting curve might be drawn through them. (See Fig. 8.) The coordinates of two points chosen at random from the opposite ends of this curve were substituted into Eq. (4) to obtain the constants for plotting the theoretical curves shown in Figs. 6 and 7.

In Fig. 9 are plotted the results of the fourth series of measurement in which a new sample of stainless steel (length 12.9837 cm, diameter 1.9050 cm) was turned down in sixteen steps to a diameter of 0.9588 cm. For most of the rods of this series it was possible to measure the first four modes of vibration the exceptions being cases in which the frequency of the third mode was such that good audible beats with the crystal standard could not be obtained. In this case the plotted points indicate clearly the path of the experimental curve which was drawn in without replotting to a smaller scale. (The solid curve in Fig. 9.) By trial it was found that putting E'/ρ equal to 29.171×10^{11} and σ equal to 0.30 made the graph of Eq. (4) fit this experimental curve best. The two curves coincide except for the large values of na/l, where the theoretical curve is shown by the dashed line. It will be noted that for this series all the experimental points fall within one tenth of one percent of the theoretical curve.

In Fig. 10 are shown the results of the fifth series in which the last rod of the fourth series was shortened in six steps to a final length of 5.9375 cm. The curve shown in the figure is the graph of Eq. (4) with the same constants as were used for the theoretical curve on Fig. 9. Here again the experimental points have ordinates within one tenth of a percent of the ordinates of the theoretical curve.

The average density for the first sample of stainless steel (first three series) was 7.65 grams per cc hence the value of E' for this sample was 22.1×10^{11} dynes per cm^2. The second sample (fourth and fifth series) had an average density of 7.69 grams per cc and hence an E' of 22.4×10^{11} dynes per cm^2. These values for E' are about what one would expect for the Young's modulus of steel showing that the product of the magnetostrictive constants q and q' is small. ($E'=E+qq'$.)

Summary and Discussion of Results

The lower modes of longitudinal vibration of magnetized cylinders of stainless steel have been measured with an accuracy of one part in ten thousand. These cylinders varied in length from twenty four centimeters down to slightly more than three centimeters and in diameter from nearly two centimeters to less than three tenths of a centimeter. The lengths were known to one or two parts in ten thousand and the diameters with varying accuracy, but none to better than one part in five thousand. (Some of the small diameters were not known to better that 0.5 percent.) The product of frequency by length was therefore known to some two or three parts in ten thousand. Nevertheless, the experimental values of $2fl/n$ when plotted against log 100 na/l varied as much as one or two parts in one thousand from the curve of the theoretical formula with values for the constants chosen to give the best fit.

This theoretical formula (Eq. (3) above) is based on Lord Rayleigh's theory[6] for the longitudinal vibrations of a cylinder taking account of the radial motion and G. W. Pierce's theory[1] for the magnetostriction oscillator.

As has been pointed out, it assumes the simple form presented above only because of several assumptions beyond those ordinarily made in applying the elastic theory. It was assumed that the standing waves in the rods were plane and that the action and reaction between the alternating magnetic field and the vibrating rod were small enough in magnitude that the relations between them might be considered linear. Furthermore the cylinders were treated as though free of all body and surface forces and of uniform consistency throughout. The actual experimental conditions may have been sufficiently different from these ideal ones to account for the deviations of the experimental points from the theoretical curves. It is not possible to say which factor or factors caused these deviations and all that one is justified in concluding is that, for stainless steel cylinders of the dimensions used, the frequencies are given by equation (3) to within one or two tenths of one percent.

The fact that the first sample of stainless steel showed variations from the expected curve of as much as 0.2 percent while the second sample agreed with the theory to better than 0.1 percent may mean that the first sample was not as uniform in consistency as the second and that therefore a large part of the deviation is due to the fact that the samples were merely pieces of the commercial alloy. This supposition is born out by the further fact that density measurements made on two pieces cut from opposite ends of the first sample differed by almost one percent.

There is no reason why other magnetostrictive metals subjected to these same measurements should not yield as good agreement with the theory, provided they come as near to being uniform, homogeneous, isotropic substances and provided the rods of different sizes are so magnetized that the permanent magnetic polarizations in each will be nearly the same. This last precaution was not taken in this research because of the small effect of polarization on frequency for stainless steel reported by K. C. Black.[8] As a matter of fact, when all the rods are magnetized in the same field as in these measurements the first order effect due to the different permanent polarizations will not change the form of the frequency Eq. (3) but only the apparent value of Poisson's ratio (σ). This is because the actual field in the rod may be expressed approximately as a function of the ratio of the diameter to the length.[9]

This research has also yielded values of the E' and the σ of Eq. (3) for the two samples of stainless steel. These constants are respectively Young's modulus and Poisson's ratio, each with an added term or terms arising from the fact that the rods are driven by magnetostriction and are mangnetically polarized. It turns out however that these constants have values that are very close to those of Young's modulus and Poisson's ratio for steel (as far as we know) so that the added terms must be small.

The measurements described in this paper were carried out in the Cruft Laboratory at Harvard University under the direction of Professor G. W. Pierce.

[9] J. C. Maxwell, Electricity and Magnetism, Vol. I.