DEFLECTION OF ELECTRONS BY A MAGNETIC FIELD ON THE WAVE MECHANICS

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Abstract

The wave equation for a stream of electrons passing through a uniform magnetic field is solved and the characteristic functions discussed. The motion at right angles to the field is found to be completely quantized.

The current density is investigated, and the mean square and mean fourth power radii of curvature found for each quantum state. The mean square radius of curvature is found to be the same for all quantum states of the same energy and identical with that predicted by the classical theory.

The mean radius of curvature is determined by interpolation and is found to be less than that predicted by the classical theory. Therefore values of e/μ calculated by means of the classical formula from measured values of the mean radius of curvature should be too large. The error is estimated and found to be of the same general magnitude as the discrepancy between the results of deflection experiments and those of spectroscopic measurements.

In his review of the probable values of the general physical constants, Birge[1] gives for the best values of the ratio of charge to rest-mass of the electron:

$e/\mu = (1.769 \pm 0.002)\ (10)^7$ abs e.m.u. from deflection experiments,
$= (1.761 \pm 0.002)\ (10)^7$ " " " from Zeeman effect,
$= (1.761 \pm 0.001)\ (10)^7$ " " " from H and He spectra.

The discrepancy between the value obtained from deflection experiments on the one hand and those obtained from spectroscopic evidence on the other is four times the probable error of the former, indicating that the difference is real and not merely the result of accidental errors. Now the value of e/μ obtained from deflection experiments is calculated on the classical electrodynamics, whereas the others are based on quantum theory. Therefore it is important to examine the Schrödinger theory of the motion of a stream of electrons in a uniform magnetic field in order to ascertain whether or not it introduces a correction into the classical formula for the radius of curvature r of the circular path of the electrons about the magnetic lines of force. This problem has been investigated by Alexandrow[2] on the basis of Dirac's theory. His second order wave Eq (10) differs from the Schrödinger equation which will be used in this paper only in the value of one of the constant coefficients—a difference which is quite negligible for the large energy values which are to be considered. Alexandrow obtains a solution of the wave equation which is ev where finite and vanishes at infinity, and concludes that the motion is

. T. Birge, Phys. Rev. Sup. **1**, 47 (1929).

[2] Alexandrow, Zeits. f. Physik **56**, 825 (1929).

not quantized, all values of the energy being allowed. His solution (11) however, contains the factor

$$e^{(k/2)\log(y+iz)} = r^{k/2}\left(\cos\frac{k}{2}\theta + i\sin\frac{k}{2}\theta\right)$$

in plane polar coordinates. Now, one of the requirements of the wave mechanics is that a solution, to be allowed, must be single valued. Therefore it would seem that $k/2$ in Alexandrow's solution must be an integer, and consequently the energy, which is a linear function of k, can assume only discrete values, indicating that the motion is quantized. In fact it will appear that Alexandrow's solution is just one of a multitude of characteristic functions corresponding to a stream of electrons of assigned energy.

Solution of the Wave Equation

The wave equation[3] for electrons moving through a magnetic field is

$$-\frac{h^2}{8\pi^2\mu}\nabla\cdot\nabla\psi - \frac{eh}{2\pi\mu ic}\mathbf{a}\cdot\nabla\psi + \frac{e^2a^2}{2\mu c^2}\psi + \frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = 0, \tag{1}$$

where $\mathbf{a}$ is the vector potential. If the field is uniform and in the X direction

$$\mathbf{a} = -j\tfrac{1}{2}Hz + k\tfrac{1}{2}Hy,$$

and if we introduce cylindrical coordinates r, θ, x the wave Eq. (1) becomes

$$-\frac{h^2}{8\pi^2\mu}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial x^2}\right\} - \frac{ehH}{4\pi\mu ic}\frac{\partial\psi}{\partial\theta} + \frac{e^2H^2}{8\mu c^2}r^2\psi + \frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = 0. \tag{2}$$

Putting

$$\psi \equiv X(x)R(r)e^{-im\theta}e^{-2\pi i(W/h)t},$$

where m must be an integer in order that the solution may be single valued, we are led to the two ordinary differential equations

$$\frac{d^2X}{dx^2} + \frac{8\pi^2\mu}{h^2}W_xX = 0, \tag{3}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left\{\frac{8\pi^2\mu}{h^2}W_{r\theta} - \frac{2\pi eH}{hc}m - \frac{m^2}{r^2} - \frac{1}{4}\left(\frac{2\pi eH}{hc}\right)^2 r^2\right\}R = 0, \tag{4}$$

W_x being the energy associated with the motion along the lines of force and $W_{r\theta}$ that with the motion in the plane at right angles to the magnetic field.

Equation (3) admits a continuous manifold of allowable solutions[4] representing a uniform current parallel to the field. Our interest being confined to

[3] Condon and Morse, Quantum Mechanics, p. 28.

[4] Condon and Morse, p. 42.

the motion in the plane at right angles to the field, we may put $W_z=0$, $W_{r\,\theta}=W$ without limiting the generality of the solution. Our problem, then, is to find the allowed solutions of (4).

If we put

$$\rho^2 \equiv \frac{2\pi e H}{hc} r^2, \qquad w \equiv \frac{4\pi\mu c}{heH} W,$$

(4) simplifies to

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left\{ w - m - \frac{m^2}{\rho^2} - \frac{1}{4}\rho^2 \right\} R = 0. \tag{5}$$

Making the substitution

$$R \equiv \rho^m e^{-\rho^2/4} V(\rho)$$

in (5), where we are supposing m to be positive, we get

$$\frac{d^2V}{d\rho^2} + \left(\frac{2m+1}{\rho} - \rho\right)\frac{dV}{d\rho} + (w - 2m - 1)V = 0. \tag{6}$$

This equation can be simplified by changing the independent variable from ρ to x where $x \equiv \rho^2$. Then

$$4x\frac{d^2V}{dx^2} + 2\{2(m+1) - x\}\frac{dV}{dx} + \{w - 2m - 1\}V = 0. \tag{7}$$

Assuming a series solution of the form

$$V = \sum_p A_p x^p$$

we find on substitution in (7)

$$4p(p+m)A_p = \{2p - w + 2m - 1\}A_{p-1}.$$

Therefore the two independent solutions of the differential equation are ascending series starting with x^0 and x^{-m}, that is, with ρ^0 and ρ^{-2m}. The second is inadmisssible since it makes R infinite at the origin. Therefore we have to consider only the series which starts with a constant term. The successive coefficients are related by the equation

$$A_p = \frac{2p - w + 2m - 1}{4p(p+m)} A_{p-1}. \tag{8}$$

If the series fails to terminate, this relation may be written

$$A_p = \frac{1}{2p} A_{p-1}$$

for large values of p. Hence the function V approaches infinity for x infinite in the same way as

$$A_0\left[\cdots\frac{\left(\frac{x}{2}\right)^p}{p!}+\frac{\left(\frac{x}{2}\right)^{p+1}}{p+1!}+\cdots\right]=A_0e^{\rho^2/2},$$

making R become infinite of the order $e^{\rho^2/4}$. Consequently the only allowable solutions are those for which the series terminates. We see from (8), then, that

$$w = 2s + 1, \tag{9}$$

where s is a positive integer equal to or greater than m. The motion, therefore, is completely quantized, the energy values being

$$W = \left(s + \frac{1}{2}\right)h\left(\frac{eH}{2\pi\mu c}\right), \quad s = 0, 1, 2 \cdots. \tag{10}$$

The last factor in (10) is just the frequency of revolution about the lines of force predicted by the classical theory.

Putting (9) into (8)

$$A_p = -\frac{(s-m)-(p-1)}{2p(p+m)}A_{p-1}. \tag{11}$$

It is more convenient to express the wave functions in terms of the quantum numbers s and k where $k \equiv s-m$, in place of s and m. The energy is a function of s alone, and for a given s, k may have any of the values 0, 1, 2 $\cdots s$. For $k=0$ the polynomial V is a constant, for $k=1$ it consists of two terms, and so on. In terms of s and k (11) becomes

$$A_p = -\frac{k-p+1}{2p(s-k+p)}A_{p-1}$$

giving

$$V_{s,k} = \sum_{p=0}^{k}\frac{(-1)^p x^p}{2^p p!(k-p)!(s-k+p)!}, \quad k = 0, 1, 2 \cdots s, \tag{12}$$

and

$$R_{s,k} = A_{s,k}X^{(s-k)/2}e^{-x/4}V_{s,k}, \quad k = 0, 1, 2 \cdots s. \tag{13}$$

Alexandrow's solution of the wave epation is $R_{s,0}e^{-is\theta}$.

The V Polynomials

Denote differentiation with respect to x by D. Then by differentiating (12) we find

$$V_{s,k} = -2DV_{s,k+1}. \tag{14}$$

Next we easily prove

$$V_{s,k} = \frac{1}{k!s!}x^{-(s-k)}e^{x/2}D^k[x^s e^{-x/2}], \tag{15}$$

which we can put in neater form if we write

$$X_s \equiv x^s e^{-x/2}.$$

Then (15) becomes

$$V_{s,k} = \frac{1}{k!s!} x^k X_s^{-1} D^k X_s. \tag{16}$$

From (14) and (15) we get the recursion formula

$$[2(s-k)-x]V_{s,k} = 2(k+1)V_{s,k+1} + xV_{s,k-1}. \tag{17}$$

The first four V polynomials are

$$V_{s,0} = \frac{1}{s!},$$

$$V_{s,1} = -\frac{1}{2s!}\{x-2s\},$$

$$V_{s,2} = \frac{1}{2^2 2!s!}\{x^2-4sx+4s(s-1)\},$$

$$V_{s,3} = -\frac{1}{2^3 3!s!}\{x^3-6sx^2+12s(s-1)x-8s(s-1)(s-2)\}.$$

On account of (15) the zeros of $V_{s,k}$ between 0 and ∞ are the zeros of $D^k[x^s e^{-x/2}]$. The latter function vanishes at 0 and ∞ for all k's less than s. Now $x^s e^{-x/2}$ has one turning point (maximum at $x=2s$). Consequently $D[x^s e^{-x/2}]$ has one zero between 0 and ∞, and as it vanishes at both limits, it has two turning points. This requires $D^2[x^s e^{-x/2}]$ to have two zeros and three turning points and so on. Consequently $V_{s,k}$ has k zeros between 0 and ∞ for any k less than or equal to s. This means that all the roots of the equation $V_{s,k}=0$ are real and positive.

The Radial Function

In terms of the quantum numbers s and m the differential Eq. (5) for R is

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left\{2s-m+1-\frac{m^2}{\rho^2}-\frac{1}{4}\rho^2\right\}R = 0. \tag{18}$$

If we put $u \equiv \rho^{1/2} R$ this becomes

$$u'' + \left\{2s-m+1+\frac{1-4m^2}{4\rho^2}-\frac{1}{4}\rho^2\right\}u = 0.$$

Denoting solutions for the same value of m but different values of s by u_{s_1} and u_{s_2},

$$2(s_2-s_1)\int_0^\infty u_{s_1} u_{s_2} d\rho = 0 \tag{19}$$

showing that the u functions are orthogonal.

To normalize these functions we must determine the arbitrary coefficient $A_{s,k}$ in (13) so that

$$\int_0^\infty u_s^2 d\rho = \tfrac{1}{2}\int_0^\infty R_{s,k}^2 dx = 1, \tag{20}$$

where x, as before, stands for ρ^2. Squaring (13) and using (15)

$$R_{s,k}^2 = A_{s,k}^2 x^{s-k} e^{-x/2} V_{s,k}^2 = \frac{A_{s,k}^2}{k!s!} V_{s,k} D^k [x^s e^{-x/2}].$$

Integrating by parts, with the aid of (14),

$$\int_0^\infty R_{s,k}^2 dx = \frac{A_{s,k}^2}{k!s!}\int_0^\infty V_{s,k} D^k [x^s e^{-x/2}] dx = \frac{2^{s-k+1}}{k!s!} A_{s,k}^2.$$

To satisfy (20) it is necessary that

$$A_{s,k} = \frac{1}{2^{(s-k)/2}}(k!s!)^{1/2} \tag{21}$$

and the normalized radial function is

$$R_{s,k} = \frac{(k!s!)^{1/2}}{2^{(s-k)/2}} x^{s-k/2} e^{-x/4} V_{s,k}$$

$$= \frac{1}{2^{(s-k)/2}(k!s!)^{1/2}} x^{-(s-k)/2} e^{x/4} D^k [x^s e^{-x/2}]. \tag{22}$$

Also

$$R_{s,k}^2 = \frac{1}{2^{s-k}} V_{s,k} D^k [x^s e^{-x/2}]. \tag{23}$$

The Current

The quantity actually measured experimentally is the deflection of the current of cathode rays or electrons. The current density[5] is given by

$$j = \frac{eh}{4\pi\mu i}(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}) - \frac{e^2}{\mu c} a\psi\bar{\psi}$$

$$= -\frac{e^2 H}{2\mu c}\left\{1 + 2m\left(\frac{hc}{2\pi e H}\right)\frac{1}{r^2}\right\} R^2 r \theta_1$$

where $\boldsymbol{\theta}_1$ is a unit vector at right angles to r in the direction of increasing θ. It is clear from the form of this expression that the current lines are circles about the lines of force for all values of the quantum number m. The current between cylinders of radii r and $r+dr$ is

$$j_{s,k} dr = -\frac{eh}{8\pi\mu}\left\{1 + 2\frac{s-k}{x}\right\} R_{s,k}^2 dx \tag{24}$$

[5] Condon and Morse, p. 30.

in terms of the quantum numbers s, k and $x \equiv \rho^2$.

Since the constant factor appearing in (24) is of no significance for our purposes, let us consider the function

$$J_{s,k} = \frac{1}{2} x^{1/2} \left\{ 1 + 2\frac{s-k}{x} \right\} R_{s,k}^2$$
$$= \frac{1}{2^{s-k+1}} x^{1/2} \left\{ 1 + 2\frac{s-k}{x} \right\} V_{s,k} D^k X_s \qquad (25)$$

and

$$J_{s,k} d(x)^{1/2} = \frac{1}{4} \left\{ 1 + 2\frac{s-k}{x} \right\} R_{s,k}^2 dx$$
$$= \frac{1}{2^{s-k+2}} \left\{ 1 + 2\frac{s-k}{x} \right\} V_{s,k} D^k X_s dx. \qquad (26)$$

First we will calculate the total current

$$I_{s,k} = \int_0^\infty J_{s,k} d(x)^{1/2} = \frac{1}{4} \int_0^\infty R_{s,k}^2 dx + \frac{s-k}{2^{s-k+1}} \int_0^\infty \frac{1}{x} V_{s,k} D^k X_s dx$$

corresponding to each normalized quantum state. The value of the first term on the right is 1/2 from (20). To evaluate the second we note that if we integrate by parts with the aid of (14)

$$\int_0^\infty V_{s,k-p} D^k X_s dx = \frac{1}{2^{k-p} s!} \left| D^{p-1} X_s \right|_0^\infty = 0 \qquad (27)$$

provided p is an integer greater than zero and less than k. Now, from the recursion formula (17),

$$V_{s,k} = \frac{s-k+1}{k} V_{s,k-1} - \frac{x}{2k} (V_{s,k-1} + V_{s,k-2}). \qquad (28)$$

Consequently

$$\int_0^\infty \frac{1}{x} V_{s,k} D^k X_s dx = \frac{s-k+1}{k} \int_0^\infty \frac{1}{x} V_{s,k-1} D^k X_s dx$$
$$= \frac{s(s-1)\cdots(s-k+1)}{k!s!} \int_0^\infty \frac{1}{x} D^k X_s dx. \qquad (29)$$

But

$$\int_0^\infty \frac{1}{x} D^k X_s dx = \int_0^\infty \frac{1}{x^2} D^{k-1} X_s dx = \cdots = k! \int_0^\infty x^{s-k-1} e^{-x/2} dx$$
$$= k!(s-k-1)! 2^{s-k}, \qquad (30)$$

and

$$\int_0^\infty \frac{1}{x} V_{s,k} D^k X_s dx = \frac{2^{s-k}}{s-k} \cdot \tag{31}$$

Therefore the total current corresponding to each normalized quantum state is

$$I_{s,k} = \tfrac{1}{2} + \tfrac{1}{2} = 1 \tag{32}$$

independent of the quantum numbers s and k.

We are primarily interested in the radius of curvature r in the magnetic field of a current of electrons of kinetic energy W. From the defining equation for $x \equiv \rho^2$ and (10) we have

$$r^2 = \frac{2\mu c^2 W}{e^2 H^2} \frac{x}{2s+1} \tag{33}$$

which is to be compared with the formula

$$r^2 = \frac{2\mu c^2 W}{e^2 H^2} \tag{34}$$

of classical electrodynamics. In the experiments of Wolf,[6] which Birge considers to be the most accurate for the determination of e/μ by the deflection method, s is of the order of magnitude of $(10)^8$. Hence we are concerned only with states for which s is very large.

$J_{s,0}$ AND $J_{s,1}$.

Let us examine more closely the current densities $J_{s,0}$ and $J_{s,1}$ of the first two states of energy corresponding to the quantum number s. The first of these is

$$J_{s,0} = \frac{1}{2^{s+1} s!}\left(1 + 2\frac{s}{x}\right) x^{s+1/2} e^{-x/2}. \tag{35}$$

Its only zeros are at $x=0$ and $x=\infty$. It has one maximum, at $x=2s$. The radius of curvature of the maximum current density, then, is given by

$$r_m^2 = \frac{2\mu c^2 W}{e^2 H^2} \frac{2s}{2s+1} \tag{36}$$

agreeing almost exactly for large s with the classical value (34). The maximum current density is

$$(J_{s,0})_{\max} = \frac{2^{1/2}}{s!} s^{s+1/2} e^{-s} \tag{37}$$

and in the neighborhood of the maximum we have approximately

$$\frac{J_{s,0}}{(J_{s,0})_{\max}} = e^{-(x-2s)^2/8s} \tag{38}$$

[6] F. Wolf, Ann. d. Physik **83**, 849 (1927).

for large s. Consequently the current density falls to $1/e^{\text{th}}$ of its maximum value at a point $\Delta x = 2(2s)^{1/2}$ to either side of the maximum. For $s = (10)^8$

$$\frac{\Delta x}{x_{\max}} = \left(\frac{2}{s}\right)^{1/2} = 2^{1/2}(10)^{-4}$$

showing that the peak is extremely narrow.

The current density of the next state is

$$J_{s,1} = \frac{1}{2^{s+2}s!}\left\{1 + 2\frac{s-1}{x}\right\}(x - 2s)^2 x^{s-1/2} e^{-x/2}. \tag{39}$$

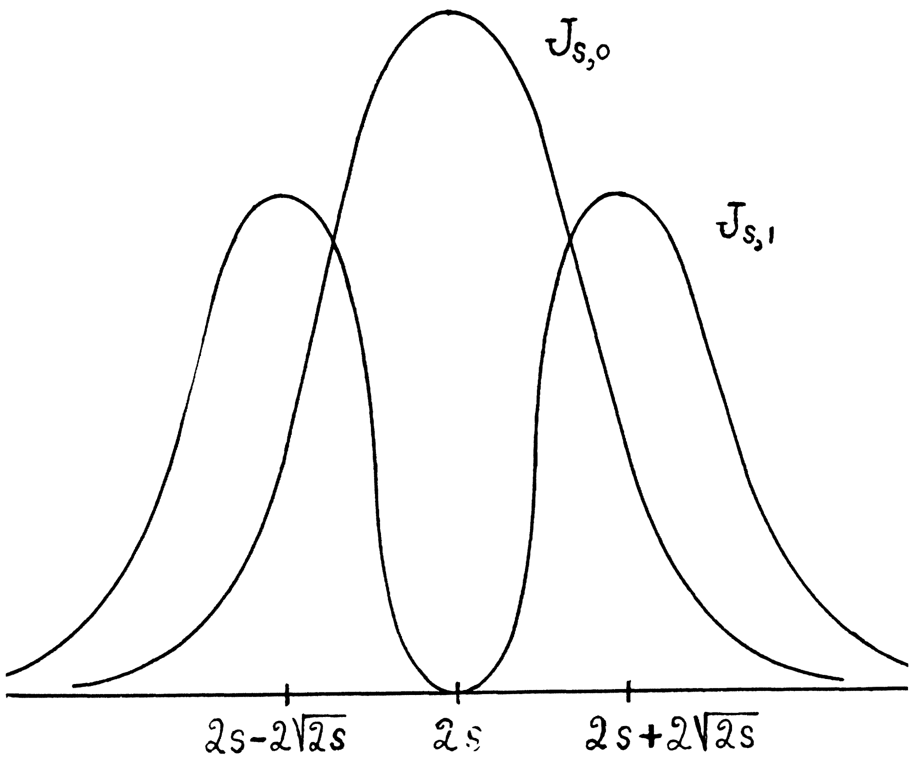

Fig. 1.

Zeros occur at $x=0$, $2s$, and ∞. There are two maxima, at approximately $x = 2s \pm 2(2s)^{1/2} + 1$. The maximum current densities are approximately the same and equal to

$$(J_{s,1})_{\max} = \frac{2^{3/2}}{s!} s^{s+1/2} e^{-(s+1)}. \tag{40}$$

The graphs of the two functions, $J_{s,0}$ and $J_{s,1}$ are sketched in Fig. 1 for large s, the origin being a great distance to the left of the center of the figure.

Evidently the zeros of $J_{s,k}$ between 0 and ∞ are identical with those of $V_{s,k}$. Thus $J_{s,k}$ has k zeros in this interval.

Mean Radii of Curvature

While the mean radius of curvature of each current cannot be evaluated easily on account of radicals appearing in the integrand, the mean square and the mean fourth power radius of curvature are readily obtained. From (26) and (32)

$$\begin{aligned}\overline{\rho^2} = \bar{x} &= \tfrac{1}{4}\int_0^\infty \{x + 2(s-k)\} R_{s,k}^2 dx \\ &= (s-k) + \frac{1}{2^{s-k+2}}\int_0^\infty xV_{s,k}D^kX_s dx,\end{aligned} \tag{41}$$

and

$$\begin{aligned}\overline{\rho^4} = \overline{x^2} &= \frac{1}{4}\int_0^\infty \{x^2 + 2(s-k)x\} R_{s,k}^2 dx \\ &= \frac{s-k}{2^{s-k+1}}\int_0^\infty xV_{s,k}D^kX_s + \frac{1}{2^{s-k+2}}\int_0^\infty x^2V_{s,k}D^kX_s dx.\end{aligned} \tag{42}$$

To evaluate the integrals involved, we have, if we integrate by parts with the aid of (14),

$$\begin{aligned}\int_0^\infty x^pV_{s,k}D^kX_s dx &= \frac{1}{2^k}\int_0^\infty x^pV_{s,0}X_s dx - \frac{kp}{2^{k-1}1!}\int_0^\infty x^{p-1}V_{s,1}X_s dx \\ &+ \frac{k(k-1)p(p-1)}{2^{k-2}2!}\int_0^\infty x^{p-2}V_{s,2}X_s dx - \cdots .\end{aligned} \tag{43}$$

For positive integral values of p the series consists of a finite number of terms. Thus for $p=1$,

$$\begin{aligned}\int_0^\infty xV_{s,k}D^kX_s dx &= \frac{1}{2^k}\int_0^\infty \{xV_{s,0} - 2kV_{s,1}\} X_s dx \\ &= \frac{1}{2^k s!}\left[(k+1)\int_0^\infty x^{s+1}e^{-x/2}dx - 2ks\int_0^\infty x^s e^{-x/2}dx\right] \\ &= 2^{s-k+2}[(k+1)(s+1) - ks] = 2^{s-k+2}(s+k+1).\end{aligned} \tag{44}$$

Hence (41) becomes

$$\overline{\rho^2} = \bar{x} = (s-k) + (s+k+1) = 2s+1 \tag{45}$$

independent of the quantum number k. Putting this in (33) we find

$$r^2 = \frac{2\mu c^2 W}{e^2H^2}, \tag{46}$$

which is identical with the classical formula (34). Hence we conclude that the mean square radius of curvature is the same for all quantum states of the same energy and identical with the square of the radius of curvature given by the classical theory.

Next we shall evaluate (42) for the mean fourth power radius of curvature. In this case we need in addition to (44) the integral (43) for $p=2$. The latter is

$$\int_0^\infty x^2 V_{s,k} D^k X_s dx = \frac{1}{2^k}\int_0^\infty \{x^2 V_{s,0} - 4kxV_{s,1} + 4k(k-1)V_{s,2}\} X_s dx$$
$$= \frac{1}{2^k s!}\left[\left(1+\frac{3}{2}k+\frac{1}{2}k^2\right)\int_0^\infty x^{s+2}e^{-x/2}dx - 2ks(k+1)\int_0^\infty x^{s+1}e^{-x/2}dx \right.$$
$$\left. + 2ks(k-1)(s-1)\int_0^\infty x^s e^{-x/2}dx\right] \tag{47}$$
$$= 2^{s-k+2}[(k+1)(k+2)(s+1)(s+2) - 2k(k+1)s(s+1) + (k-1)k(s-1)s].$$

Consequently (42) becomes

$$\overline{\rho^4} = \overline{x^2} = 2(s-k)[(k+1)(s+1) - ks]$$
$$+ [(k+1)(k+2)(s+1)(s+2) - 2k(k+1)s(s+1) + (k-1)k(s-1)s] \tag{48}$$
$$= (2s+1)^2 + 2(2k+1)(2s+1) + 1.$$

The mean fourth power of the radius of curvature, therefore, increases with increasing k and is greater than the square of the mean square radius of curvature for all quantum states of a given energy. If we average over all k's form 0 to s we find

$$\overline{\overline{\rho^4}} = \overline{\overline{x^2}} = 8s^2 + 10s + 4. \tag{49}$$

Discussion of Results

The usual experimental method of measuring the deflection of a stream of electrons by a magnetic field involves the determination of the mean radius of curvature rather than the mean square. On account of the slit or slits used to define the stream, we should expect the greatest number of electrons to be in the states for which k is small, for the increase of spread of J with increase in k would prevent electrons in the higher quantum states from passing through the slit system. In the absence of knowledge of the effect of slit width on the distribution of the electrons, no exact quantitative correction to the classical formula can be given. Certain qualitative conclusions may be drawn, however.

To obtain an estimate of the order of magnitude of the mean radius of curvature of the current for quantum number k we can make use of the known values of $\overline{\rho^0}$, $\overline{\rho^2}$ and $\overline{\rho^4}$. Putting $S\equiv 2s+1$, $K\equiv 2k+1$ we have from (32), (45) and (48)

$$\overline{\rho^0} = 1,$$
$$\overline{\rho^2} = S,$$
$$\overline{\rho^4} = S^2\left[1 + 2\frac{K}{S} + \frac{1}{S^2}\right].$$

If the current distribution were perfectly sharp, we would have

$$\log \overline{\rho^n} = An. \tag{50}$$

If K is small compared to S, the values of $\overline{\rho^0}$, $\overline{\rho^2}$ and $\overline{\rho^4}$ show that the spread of the current is small. Hence we can determine the first order correction to (50) by writing

$$\log \overline{\rho^n} = An + Bn^2$$

and determining the constants A and B to fit the values found for the three means. This gives

$$\overline{\rho^n} = S^{n/2}\left[1 + 2\frac{K}{S} + \frac{1}{S^2}\right]^{n(n-2)/8} \tag{51}$$

Consequently interpolation gives for the mean radius of curvature

$$\bar{\rho} = S^{1/2}\left[1 + 2\frac{K}{S} + \frac{1}{S^2}\right]^{-1/8}$$
$$\doteq S^{1/2}\left[1 - \frac{1}{4}\frac{K}{S}\right].$$

As this is less than $(2s+1)^{1/2}$ the square of the mean radius of curvature is less than the mean of the square. Hence, as e/μ varies inversely with r^2, the calculation of e/μ by means of the classical formula will give too large a value of the specific charge of the electron. Therefore the error indicated by theory is of the right sign to explain the discrepancy between the results of deflection experiments and spectroscopic measurements.

To estimate the magnitude of the error we need the mean value of K for the electrons passing through the slit. The spread of the current is given by

$$\Delta\rho^2 = \int_0^\infty J(\rho - \bar{\rho})^2 d\rho = \overline{\rho^2} - \bar{\rho}^2 = \tfrac{1}{2}K.$$

The quantity $\Delta\rho$ measures the effective distance of the current either side of its mean position. It seems reasonable to infer that half the current passes through the slit when $\Delta\rho/\rho$ is equal to the ratio of the half width of the slit to the radius of curvature. Now the ratio of the half width of the slit to the radius of curvature in Wolf's experiments was 0.04 and s was approximately $1.7(10).^8$ So when

$$\frac{\Delta\rho}{\rho} = \frac{1}{2}\left(\frac{K}{S}\right)^{1/2} = 0.04 \quad \text{or} \quad \frac{K}{S} = 0.0032$$

half the current passes through the slit. Putting $S=3.4(10)^8$, $K=10.9(10)^5$ or $k=5.4(10)^5$. Consequently the ratio of the magnitude of each partial current to that of the next of lower index may be taken to be

$$\alpha = (\tfrac{1}{2})^{1/5.4(10)^5}.$$

The mean value of K, then, is

$$\overline{K} = \frac{1 + 3\alpha + 5\alpha^2 \cdots + (2s+1)\alpha^s}{1 + \alpha + \alpha^2 \cdots + \alpha^s} = 1 + \frac{2\alpha}{1-\alpha}\,\frac{1-\alpha^s}{1-\alpha^{s+1}} - \frac{2s\alpha^{s+1}}{1-\alpha^{s+1}}$$

$$\doteq \frac{2}{1-\alpha} = 1.56(10)^6.$$

The error in e/μ is that in $1/\bar{\rho}^2$, that is, $(1/2)\overline{K}/S$. Hence the order of magnitude of the error is

$$\frac{1}{2}\,\frac{\overline{K}}{S} = 0.0023.$$

As the actual discrepancy between the results of Wolf's deflection experiments and of spectroscopic observations is 0.0045 ± 0.0011, *we conclude that the error indicated by theory is of the correct order of magnitude as well as of the correct sign to account for the observed discrepancy.* In view of the very rough calculation of the mean K, the numerical value obtained above cannot be considered as more than an estimate of the magnitude of the error. The agreement with the observed discrepancy is therefore as good as the method of calculation warrants.

The results of this investigation indicate that in measuring e/μ by the deflection method the classical formula is applicable only if (a) the ratio of slit width to radius of curvature is very small, or (b) the method is one in which the mean square of the radius of curvature is measured. In the latter case it makes no difference in what quantum states the electrons may be, for the mean square of the radius of curvature is the same for all states of the same energy and the value of e/μ is given correctly by the classical formula.