THE RATE AT WHICH IONS LOSE ENERGY IN ELASTIC COLLISIONS

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Abstract

The rate of energy loss of ions (including electrons) moving through a gas is rigorously calculated on the assumption that the ions and molecules are smooth elastic spheres with no attraction at a distance, having Maxwellian velocity distributions corresponding to the temperatures T_i and T_m respectively. The result is

$$f = \frac{8}{3}\frac{mM}{(m+M)^2}\left(1 - \frac{T_m}{T_i}\right)$$

where m and M are the masses of ion and molecule respectively, T_i and T_m are their temperatures, and f is the average energy loss per collision expressed as a fraction of the average ionic energy.

THERE are a number of problems in which ions (including electrons) move through a gas with an approximately Maxwellian velocity distribution corresponding to a temperature higher than that of the gas, and an expression for the rate at which the ions lose energy in elastic collisions is required. For instance, Compton[1] required it for his electron mobility equation. He deduced an approximate expression as follows. First he calculated the average energy loss for ions of given velocity striking molecules at rest at all angles from grazing to head on. He then corrected this for the motion of the molecules by finding how the energy loss for head on collisions was altered by a (one dimensional) molecular velocity distribution, and multiplying the expression for molecules at rest by this factor. The velocity distribution of the ions was not considered at all. Indeed there *is* no ion velocity distribution if one is merely interested in the average fraction of *its own* energy which an ion loses per collision. What was actually needed, however, was the average energy loss in all collisions expressed as a fraction, f, of the average energy (of all the ions in a given volume). The result obtained was

$$f = 2\frac{m}{M}\left(1 - \frac{\Omega}{\omega}\right)$$

where m and M are the masses and ω and Ω the average energies of the ions and molecules respectively.

The present paper gives the rigorous calculation of the energy loss on the assumption that ions and molecules are smooth elastic spheres with no attraction at a distance, having Maxwellian velocity distributions.

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[1] Compton, Phys. Rev. **22**, 333 (1923).

The symbols which will be used are:

m = mass of ion.
M = " " molecule.
σ = sum of radii of ion and molecule.
T_i = absolute temperature of ions.
T_m = " " " molecules.
k = Boltzmann's constant.
n = number of ions per cm^3.
N = " " molecules per cm^3.
$\boldsymbol{V}$ = initial vector velocity of ion.
$\boldsymbol{U}$ = " " " " molecule.
$\mu = M/(m+M)$.
$\boldsymbol{W} = (1-\mu)\boldsymbol{V}+\mu\boldsymbol{U}$ = velocity of center of mass.
$\boldsymbol{R} = \boldsymbol{V}-\boldsymbol{W} = \mu(\boldsymbol{V}-\boldsymbol{U})$ = initial velocity of ion relative to center of mass.
$\boldsymbol{D}$ = unit vector giving direction of line of centers, from ion to molecule, at collision.
Q = total energy loss of all ions in 1 cm^3 in 1 sec.
$f = Q/\{$(no. collisions per cm^3 per sec) (average ionic energy)$\}$ = average fraction of average energy lost per collision.

To find the total energy loss, Q, we have simply to find the loss in each type of collision and then integrate over all the collisions in 1 cm^3 in 1 sec. A given type of collision is specified by given values of $\boldsymbol{V}$, $\boldsymbol{U}$, and $\boldsymbol{D}$. In a collision, that component of $\boldsymbol{R}$, the velocity of the ion relative to the center of mass, which is parallel to the line of centers, $\boldsymbol{D}$, is reversed, and the perpendicular component is unaltered. Hence the final velocity of the ion is

$$\boldsymbol{V}' = \boldsymbol{V} - 2\boldsymbol{R}\cdot\boldsymbol{D}\boldsymbol{D}$$

and the energy lost by the ion is

$$\tfrac{1}{2}m\boldsymbol{V}^2 - \tfrac{1}{2}m\boldsymbol{V}'^2 = \tfrac{1}{2}m\{4\boldsymbol{V}\cdot\boldsymbol{D}\boldsymbol{R}\cdot\boldsymbol{D} - 4(\boldsymbol{R}\cdot\boldsymbol{D})^2\} = 2m\boldsymbol{R}\cdot\boldsymbol{D}\boldsymbol{W}\cdot\boldsymbol{D}$$

where $\boldsymbol{W}$ is the velocity of the center of mass.

The number of ions per cm^3 with velocity components between V_x and V_x+dV_x, V_y and V_y+dV_y, V_z and V_z+dV_z is

$$dn = n(m/2\pi kT_i)^{3/2}e^{-mV^2/2kT_i}dV_xdV_ydV_z$$

where n is the number of ions per cm^3 and T_i is their temperature. Similarly, the number of molecules per cm^3 with given velocity is

$$dN = N(M/2\pi kT_m)^{3/2}e^{-MU^2/2kT_m}dU_xdU_ydU_z.$$

The number of collisions per cm^3 per sec. between ions and molecules of the above velocities in which the line of centers at impact lies within the solid angle $d\omega$ about the direction $\boldsymbol{D}$ is

$$dndN(\boldsymbol{V} - \boldsymbol{U})\cdot\boldsymbol{D}\sigma^2d\omega = dndN(1/\mu)\boldsymbol{R}\cdot\boldsymbol{D}\sigma^2d\omega$$

where σ is the sum of the radii of ion and molecule. The energy loss per cm^3 per sec. in these collisions of the given type is

$$(2m\boldsymbol{R}\cdot\boldsymbol{D}\boldsymbol{W}\cdot\boldsymbol{D})\left\{n(m/2\pi kT_i)^{3/2}e^{-mV^2/2kT_i}dV_x dV_y dV_z\right\}$$
$$\times\left\{N(M/2\pi kT_m)^{3/2}e^{-MU^2/2kT_m}dU_x dU_y dU_z\right\}(\sigma^2/\mu)\boldsymbol{R}\cdot\boldsymbol{D}d\omega.$$

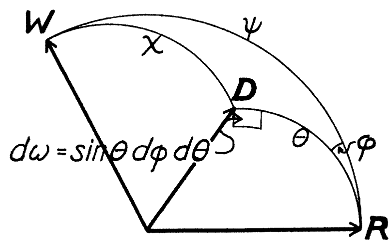

Fig. 1.

$\boldsymbol{D}$ may be expressed in polar coordinates ϕ and ϑ (Fig. 1) with the axis parallel to $\boldsymbol{R}$ and the plane $\phi=0$ containing $\boldsymbol{W}$. Then

$$d\omega = \sin\theta d\phi d\theta$$
$$\boldsymbol{R}\cdot\boldsymbol{D} = R\cos\theta$$
$$\boldsymbol{W}\cdot\boldsymbol{D} = W\cos\chi = W(\cos\theta\cos\psi + \sin\theta\sin\psi\cos\phi)$$

and the total energy loss per cm^3 per sec. becomes

$$Q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{2\pi}\int_{0}^{\pi/2}(2mnN\sigma^2/\mu)(m/2\pi kT_i)^{3/2}(M/2\pi kT_m)^{3/2}$$
$$e^{-mV^2/2kT_i - MU^2/2kT_m}R^2W\cos^2\theta\sin\theta$$
$$(\cos\theta\cos\psi + \sin\theta\sin\psi\cos\phi)d\theta d\phi dV_x dV_y dV_z dU_x dU_y dU_z$$

in which R, W, and ψ are all functions of $V_x \cdots\cdots U_z$.

The integrations with respect to θ and ϕ may be performed immediately. For the remaining six integrations, $\boldsymbol{V}$ and $\boldsymbol{U}$ are replaced by $\boldsymbol{R}$ and $\boldsymbol{W}$, expressed in polar coordinates. The result is

$$Q = 8(2\pi)^{1/2}nN\sigma^2k^{3/2}\left\{mM(mT_m + MT_i)\right\}^{1/2}(m+M)^{-2}(T_i - T_m).$$

Finally, f, the average energy loss per collision expressed as a fraction of the average energy of the ions, is equal to Q divided by the number of collisions per cm^3 per sec. all divided by the average energy:

$$f = Q\Big/\left\{\left[nN\pi\sigma^2\left(\frac{8kT_i}{\pi m}\right)^{1/2}\left(\frac{m}{M}\frac{T_m}{T_i}+1\right)^{\frac{1}{2}}\right]\left[\frac{3}{2}kT_i\right]\right\}$$
$$= \frac{8}{3}\frac{mM}{(m+M)^2}\left(1 - \frac{T_m}{T_i}\right).$$