THE VALUE OF e/m BY DEFLECTION EXPERIMENTS

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Abstract

To clarify the discussion as to whether or not a quantum mechanical treatment of the motion of electrons in magnetic fields will lead to a formula for e/m, differing from that obtained by classical electro-dynamics the following problem was solved. A uniform magnetic field in the z-direction exists in the half-space $x>0$. A plane monochromatic de Broglie wave, travelling in the positive x-direction representing electrons of arbitrary energy, impinges normally on the plane $x=0$. Solutions of the wave equation were found fulfilling appropriate boundary conditions at the plane $x=0$. Currents are calculated quantum mechanically and compared with the corresponding classical expressions. It was found that for electrons possessing energies of the order of magnitude used in deflection experiments, no *observable* deviations from classical results are predicted. Another quantum mechanical effect is diffraction at slits. Simple approximate calculations show that this effect can produce a fractional error in e/m of the order of the de Broglie wave-length divided by the slit width. These results are opposite to the conclusions reached by Page. We may remark that he solved a problem of "stationary states" which does not represent the actual experiments.

In a recent issue of the Physical Review Professor Leigh Page[1] tried to explain the difference in the value of e/m obtained from deflection experiments and from spectroscopic evidence on the basis of quantum mechanics. This was later criticized by Eckart,[2] first, he doubted certain approximate calculations of Page which may, however, be shown to be correct;[3] second, he referred to articles of Kennard[4] and Darwin[5] which prove quite conclusively that a wave packet representing an electron in a uniform magnetic field moves as in the classical theory. To this we may remark that in the deflection method the measurements are not made with a single particle but are statistical since we work with beams of electrons. However the work of Page does not seem conclusive because he considers the solutions of the wave equation corresponding to the completely quantized motion of isolated electrons in a magnetic field of infinite extension; whereas in the deflection experiments we have to deal with a problem of "streaming."

§2. In trying to decide the question as to whether or not quantum mechanics will lead to results appreciably different from those of the classical theory we have considered the following problem. A homogeneous magnetic field in the x-direction fills the half-space $x>0$. Free electrons moving in the

[1] L. Page, Phys. Rev. **36**, 444 (1930).
[2] C. Eckart, Phys. Rev. **36**, 1014 (1930).
[3] L. Page, Phys. Rev. **36**, 1418 (1930).
[4] E. H. Kennard, Zeits. f Physik **44**, 347 (1927).
[5] C. G. Darwin, Proc. Royal Soc. **A117**, 258 (1927).

x-direction impinge on the plane $x=0$ and penetrate the magnetic field. Classically they will describe half-circles with centers on the plane $x=0$ and radii

$$r = \frac{cp}{eH} \tag{1}$$

where p represents the momentum of the particles. We find easily that in the magnetic field the components of the current density S are

$$S_x = 0 \quad S_y = \frac{2Ix}{(r^2 - x^2)^{1/2}} \tag{2}$$

and the total current $\int_0^r S_y\, dx = 2Ir$ where I is the current density of the incoming electrons. Quantum mechanically we have for $x>0$ the wave equation

$$\left\{ \Delta_2 + \frac{2\pi i}{\lambda r}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \left(\frac{\pi}{\lambda r}\right)^2 (x^2 + y^2) + \left(\frac{2\pi}{\lambda}\right)^2 \right\} \psi_1 = 0 \tag{3}$$

where Δ_2 is the Laplace operator for two dimensions and $\lambda = h/p$ is the de Broglie wave-length of the incident electrons. The incoming electrons are represented by a wave function $\psi_i = A \exp(2\pi i/\lambda x)$ and the outgoing electrons by $\psi_\sigma = B \exp(-2\pi i/\lambda x)$ so that for $x<0$ the total wave function $\psi_2 = \psi_i + \psi_\sigma$. We must seek solutions of (3) which fulfill the boundary conditions at $x=0$:

$$\begin{aligned} \psi_1 &= \psi_2 && \text{(a)} \\ (mv_x)_1\psi_1 &= (mv_x)_2\psi_2 && (b) \\ (mv_y)_1\psi_1 &= (mv_y)_2\psi_2 && \text{(c)} \\ \psi_1(\infty) &= 0 && \text{(d)} \end{aligned} \tag{4}$$

where

$$mv_x = \frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}A_x$$

and

$$mv_y = \frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c}A_y.$$

In our case the components of the vector potential A_x and A_y are given by $A_x = -\frac{1}{2}Hy$ and $A_y = \frac{1}{2}Hx$. Making use of results given in a recent paper by Landau[6] we find that

$$\psi_1 = \phi(x) e^{\pi i x y/\lambda r} \tag{5}$$

is a solution of (3) if $\phi(x)$ satisfies the equation

$$\phi''(x) + \left(\frac{2\pi}{\lambda}\right)^2 \left(1 - \frac{x^2}{r^2}\right)\phi(x) = 0. \tag{6}$$

[6] L. Landau, Zeits. f. Physik **64**, 629 (1930).

We may therefore write our boundary conditions (4) (conditions 4c is identically fulfilled)

$$\phi(0) = A + B \qquad (a)$$

$$\phi'(0) = \frac{2\pi i}{\lambda}(A - B) \qquad (b) \quad (7)$$

$$\phi(\infty) = 0 \qquad (d)$$

It will be shown later that the condition $\phi(\infty)=0$ will, for a given λ determine $\phi'(0)/\phi(0)$ so that we may write for convenience

$$\frac{\lambda\phi'(0)}{2\pi\phi(0)} = q(\lambda) \tag{8}$$

where $q(\lambda)$ is a real quantity.
The first and second of the relations (7) then give

$$q(A + B) = i(A - B)$$

or

$$B = \frac{i - q}{i + q}A = Ae^{\pi i\tau}. \tag{9}$$

The phase of the outgoing electrons therefore differs from that of the incoming electrons by $\pi\tau$, where

$$\cos \pi\tau = \frac{1 - q^2}{1 + q^2}\cdot \tag{10}$$

Thus

$$\phi(0) = A(1 + e^{\pi i\tau}). \tag{11}$$

§3. The current density is in general given by

$$S_x = \frac{eh}{4\pi im}\left(\psi\frac{\partial\bar{\psi}}{\partial x} - \bar{\psi}\frac{\partial\psi}{\partial x}\right) - \frac{e^2}{mc}A_x\psi\bar{\psi}$$

with a similar expression for S_y. Substituting equation (5) and the vector potentials we find immediately

$$(S_x)_1 = 0 \quad (S_x)_2 = \frac{e}{m}\,\frac{h}{\lambda}(B\bar{B} - A^2) = 0. \tag{12}$$

This latter expression is zero due to the fact that A and B are equal in absolute value from (9). The intensities of the incoming and outgoing waves are equal. Further

$$(S_y)_1 = -\frac{e}{m}\,\frac{h}{\lambda r}x\phi\bar{\phi} \quad (S_y)_2 = 0. \tag{13}$$

At $x=0$ $(S_y)_1=(S_y)_2=0$. Continuity of the currents is therefore a consequence of the boundary conditions (4). Finally we can calculate the total current

$$\int_0^\infty (S_y)_1 dx = -\frac{eh}{\lambda rm}\int_0^\infty x\phi\bar{\phi}dx.$$

By integrating partially and using the differential equation (6) for ϕ and the fact that $\phi(\infty)=0$ it is found that

$$\int_0^\infty x\phi\bar{\phi}dx = \frac{1}{2}r^2(1+q^2)\phi(0)\bar{\phi}(0).$$

Substituting (11) and noting that $I=ehA^2/m\lambda$ we find that

$$\int_0^\infty (S_y)_1 dx = 2Ir$$

which is just the classical value.

§4. To get a clearer idea of $(S_y)_1$, a quantity which would be measured in a deflection experiment, we must study the solutions of (6) in more detail. Introducing $\xi=2x(\pi/\lambda r)^{1/2}$, $\nu+\frac{1}{2}=\pi r/\lambda$ (6) becomes

$$\frac{d^2\phi}{d\xi^2}+\left(\nu+\frac{1}{2}-\frac{1}{4}\xi^2\right)\phi = 0.$$

This equation is treated extensively in Chapter 16, page 341 of Whittaker and Watson "Modern Analysis." In their notation we obtain for a solution vanishing at $\xi=\infty$:

$$\phi(\xi) = aD_\nu(\xi) = ae^{-\xi^2/4}\xi^\nu\left\{1-\frac{\nu(\nu-1)}{2\xi^2}+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{8\xi^4}-\cdots\right\}$$

for large ξ (a is a constant to be determined from $\phi(0)$. If ν is an integer the series breaks off and we obtain the well-known harmonic oscillator eigenfunctions. This solution "connects" with the following solution for small ξ:

$$\phi(\xi) = a\left\{\frac{\Gamma\left(\frac{1}{2}\right)2^{\nu/2+1/4}}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}\xi^{-1/2}M_{\nu/2+1/4,-1/4}\left(\frac{\xi^2}{2}\right) + \frac{\Gamma\left(-\frac{1}{2}\right)2^{\nu/2+1/4}}{\Gamma\left(-\frac{\nu}{2}\right)}\xi^{-1/2}M_{\nu/2+1/4,1/4}\left(\frac{\xi^2}{2}\right)\right\}$$

where

$$M_{k,m}(\eta) = \eta^{1/2+m}e^{-\eta/2}\left\{1+\frac{\frac{1}{2}+m-k}{1!(2m+1)}\eta+\frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m-k)}{2!(2m+1)(2m+2)}\eta^2+\cdots\right\}$$

From this solution one can show easily that

$$q = 2(2\nu + 1)^{-1/2}\frac{\Gamma\left(\frac{\nu}{2}+1\right)}{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}\tan\frac{\nu\pi}{2} \tag{14}$$

$$\cong \tan\frac{\nu\pi}{2} \text{ for } \nu \gg 1$$

thus

$$\cos \pi\tau = \frac{1-q^2}{1+q^2} = \cos \nu\pi$$

or

$$\pi\tau = \pi\nu. \tag{15}$$

Thus the phase shift between the incoming and outgoing electrons is completely determined in terms of ν and therefore in terms of the energy of the incident electrons. It may be worth while to remark that ν is very simply related to the number N of de Broglie wave-lengths on the circumference of the classical half circle

$$\nu = N - \tfrac{1}{2} \text{ or } \pi\tau = (N - \tfrac{1}{2})\pi$$

or since λ is usually of the order of 10^{-8} cm, ν is of the order of magnitude 10^8 to 10^{10}. For quantum numbers of this order of magnitude one would hardly expect to find the differences between classical and quantum mechanical predictions to be measurable.

§5. To add weight to this belief we may consider the character of the function $\phi(x)$ in more detail. This is best done by considering the Wentzel-Brillouin-Kramers[7] approximation. For $x > r$ $\phi(x)$ behaves as an increasing or decreasing exponential and to satisfy the condition $\phi(\infty) = 0$ we must choose the latter. We obtain

$$\phi(x) \cong \frac{a}{(x^2 - r^2)^{1/4}} \exp\left\{-\frac{2\pi}{\lambda r}\int_r^x (x^2 - r^2)^{1/2} dx\right\}.$$

This "connects" according to Kramers with the oscillatory solution for $x < r$

$$\phi(x) \cong \frac{2a}{(r^2 - x^2)^{1/4}} \cos\left\{\frac{2\pi}{\lambda r}\int_x^r (r^2 - x^2)^{1/2} dx - \frac{\pi}{4}\right\}.$$

We see immediately that this approximate solution determines $q(\lambda) = \lambda\phi'(0)/2\pi\phi(0)$ and we find for it again

[7] G. Wentzel, Zeits. f. Physik **38**, 518 (1926); L. Brillouin, C. R., Juli, 1926; H. A. Kramers, Zeits. f. Physik **39**, 828 (1926); A. Zwaan, Utrecht Dissertation, 1929; L. A. Young and G. E. Uhlenbeck, Phys. Rev. **36**, 1154 (1930).

$$q = \tan\frac{\nu\pi}{2}\cdot$$

The constant a has to be determined from the value $\phi(0)$. We find from (11)

$$a = \frac{A(1 + e^{\pi i r})r^{1/2}}{\cos\frac{\pi\nu}{2}}\cdot$$

Calculation of the current gives then for $x < r$

$$(S_y)_1 = \frac{4Ix}{(r^2 - x^2)^{1/2}}\cos^2\left\{\frac{2\pi}{\lambda r}\int_x^r (r^2 - x^2)^{1/2}dx - \frac{\pi}{4}\right\}. \tag{16}$$

The current, therefore, is "oscillatory" with an amplitude just twice the classical current. The distance between the maxima is, for small x, just $\lambda/2$ but increases as x approaches r. The distance Δ of the last maximum from $x = r$ is given by

$$\frac{2\pi}{\lambda r}\int_{r-\Delta}^r (r^2 - x^2)^{1/2}dx = \frac{\pi}{4}$$

or

$$\Delta = \frac{3^{2/3}}{8}\lambda^{2/3}r^{7/3}. \tag{17}$$

For $x > r$ the current is given by

$$(S_y)_1 = \frac{2Ix}{(x^2 - r^2)^{1/2}}\exp\left\{-\frac{4\pi}{\lambda r}\int_r^x (x^2 - r^2)^{1/2}dx\right\}. \tag{18}$$

This current falls off very rapidly with increasing x, in fact in going a distance $n\Delta$ beyond $x = r$ the ratio of the current to the current at $x = r - \Delta$ is given by

$$\frac{1}{2(n)^{1/2}}\exp\left(-2n^{3/2}\right). \tag{19}$$

This ratio is equal to 0.001 for $n \cong 2.1$.

§6. These last results are perhaps doubtful since in the neighborhood of $x = r$ the approximation from which they were derived becomes invalid. We can, however, derive them more rigorously by observing that near $x = r$,

$$1 - \frac{x^2}{r^2} \cong 2\left(1 - \frac{x}{r}\right).$$

Instead of (6) we have then

$$\phi''(x) + 2\left(\frac{2\pi}{\lambda}\right)^2\left(1 - \frac{x}{r}\right)\phi(x) = 0.$$

This equation has been thoroughly discussed by Kramers (reference 7) and the solution which "connects" properly with the exponential and oscillatory solutions of (6) is found immediately to be

$$\phi(x) = \omega\left\{2\left(\frac{\pi r}{\lambda}\right)^{2/3}\left(1 - \frac{x}{r}\right)\right\} \tag{20}$$

$\omega(z)$ has been tabulated by van der Held (see Kramers, reference 7) and we find from his table

$$\Delta = \tfrac{1}{2}\pi^{-2/3}\lambda^{2/3}r^{1/3} \tag{17'}$$

and in traveling a distance 2.5Δ the current drops to 0.001 of its maximum value.

§7. Returning now to the value of e/m we must distinguish between two quantum mechanical effects. The first is the one discussed above. If we observed the maximum current we would find it displaced a distance Δ from the classical position. If we calculated e/m from

$$\frac{e}{m} = \frac{cv}{H}\cdot\frac{1}{r}$$

we would obtain too *large* a value. This is in the right direction to explain the discrepancy between the deflection and spectroscopic values of e/m but quantitatively the relative error due to this is of the order $(\lambda/r)^{2/3}$ which is about 10^{-6}. The observed relative discrepancy is 4×10^{-3}.

A second quantum mechanical effect in an actual deflection experiment would be diffraction at the entrance slit. A rough calculation based on the uncertainty principle shows that this can give a relative error in e/m no larger than λ/d where d is the slit-width. For actual cases this is of the order 10^{-6}. We conclude therefore that the quantum mechanical analysis of the deflection experiments cannot explain the observed discrepancy. One might still think of relativistic or spin effects as a possible explanation of the e/m paradox, but we have a feeling that the resulting corrections will also be extremely small.

Note added in proof: We regret that, in preparing this manuscript we overlooked the article of Charlotte T. Perry and E. L. Chaffee (Phys. Rev. **36**, 904; 1930) which seems to show quite conclusively that the discrepancy between the two values of e/m is due to an error in the experimental determination of the velocity of the electrons in the deflection experiments.