MULTIPLE SCATTERING IN THE COMPTON EFFECT

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Abstract

In the experimental study of the spectral distribution of x-radiation scattered by light elements it has always, up to the present, been assumed that multiple scattering could be neglected. Recent improvements in experimental technique however make it possible to suppose that multiple (especially double) scattering may now be detectable from large scattering bodies. Multiple scattering may (1) affect the breadth of the modified line, (2) change the structure of the modified line, (3) distort the background in such a way as to render measurements of shift unreliable. It is therefore valuable to analyze the effect of multiple scattering in case some of the recent mutually discordant experimental results may be explained and harmonized in this way.

Assuming initially monochromatic radiation, scattering of any degree of multiplicity contributes a spectral band whose wave-length limits are here determined and discussed. The Breit, Compton, Jauncey formula taking polarization into account is adopted for the purposes of calculation of modified intensity. The results are thus fairly accurate for hard radiation scattered from very light elements. For softer radiation modified scattering at small scattering angles is greatly reduced below the values given by the Breit, Compton, Jauncey formula and unmodified scattering appears. The effect of this on the results of this paper is discussed in a qualitative way.

The case of double scattering from a spherical scatterer is computed in complete detail and a formula for the ratio of double to single scattering is derived. Curves of the spectral distributions due to double scattering are shown. The dependence of the total doubly scattered intensity on the primary scattering angle is plotted.

The natural width of the modified line is neglected throughout these calculations. Absorption in the scatterer is also neglected. The ratio of double to single scattering for a spherical scatterer observed under any given angle is proportional to the radius of the scatterer (neglecting absorption) and is given by

$$\frac{\text{Doubly scattered energy}}{\text{Singly scattered energy}}=\frac{9}{32}\sigma r R(\theta) \text{ where}$$

σ is the linear scattering coefficient for the material of the scattering sphere, r the radius of the sphere, θ the angle under which single scattering occurs. $R(\theta)$ never differs greatly from 2.5.

Triple scattering is negligible in comparison to single scattering.

Twice modified doubly scattered radiation may contribute a faint asymmetric line or edge at the shifted position

$$\Delta\lambda=2(h/mc)(1+\cos\tfrac{1}{2}\theta)$$

For hard radiation when the Thompson formula (with Breit correction) applies with fair accuracy to modified scattering alone twice modified doubly scattered radiation contributes a spectral band of breadth

$$4(h/mc)\cos\tfrac{1}{2}\theta$$

Once modified doubly scattered radiation may cause a slight broadening of the Compton line except in regions near $\theta=0$ and $\theta=180°$.

Introduction

IN THE experimental study of scattered x-radiation it is impossible completely to eliminate multiple scattering. X-radiation whose direction is defined more or less closely within some solid angle $\delta\Omega_1$ is incident upon a scattering body. The apparatus (a spectrograph, ionization spectrometer or what not) is arranged so as to receive scattered radiation whose direction again is more or less closely defined within some solid angle $\delta\Omega_2$. The angle θ between the initial and final directions has thus assignable limits of inhomogeneity and is called the angle of primary scattering. A ray cannot be prevented, however, from suffering any number of scattering processes between its entry along the first direction and its exit along the last. It is the purpose of this paper to discuss theoretically the effect of such processes on the spectral distribution of the scattered radiation. The path of a multiply scattered ray is a broken line (in space) of $(M+1)$ straight segments, where M is the multiplicity of the scattering. The first and last segments have the fixed directions of the incident and scattered beams just mentioned. The other segments will be called *intermediate rays*. Their directions may be any whatever. In double scattering there is but one intermediate ray. The directions of the entry, exit and intermediate rays are to be plotted upon a sphere of unit radius by means of unit vectors originating at the center of the sphere and terminating on its surface. These vectors have the respective directions of propagation of the radiation along entry, exit and intermediate rays for their directions. The terminii of these vectors thus locate $M+1$ points on the surface of the sphere and represent the $(M+1)$ directions taken by the M-tuply scattered ray. The first and last of these points are fixed upon the sphere and separated by the angle of primary scattering θ. The remaining $(M-1)$ points may have any position on the entire surface of the sphere. The problem of M-tuple scattering thus involves the $(2M-2)$ degrees of freedom of these points.

Each process of scattering splits an initially monochromatic ray into modified and unmodified components as regards wave-length. The modified ray suffers an increase of wave-length given by $\Delta\lambda = h/mc(1-\cos\theta_i)$ where θ_i is the angle of scattering. There exist therefore $(M+1)$ cases in m-tuple scattering for the ray may be *scattered* M times but only *modified* 0, 1, 2, 3, etc. up to M times. Thus in double scattering the ray may be scattered twice without modifications, or modified at the first scattering but not at the second, or modified at the second scattering but not at the first, or modified twice successively. The two cases of singly modified doubly scattered radiation will be identical.

Unfortunately the law governing the partition of the scattered radiation between modified and unmodified rays is complicated and not very well known. For sufficiently hard radiation scattered by sufficiently light atoms however the unmodified radiation is negligible at nearly all scattering angles. In this paper only this limiting case will be discussed analytically in detail and therefore the total shift of the original wave-length will be the sum of all the separate shifts occurring at the successive scattering processes. The breadth of the shifted line will be neglected.

Wave-Length Limits of Multiply Scattered Radiation

It is immediately possible to make the following obvious generalizations if M is both the number of scatterings and the number of shifts of a given ray:

1. When M is odd the maximum shift occurs for a primary scattering angle of $\theta = 180°$.
2. When M is even the maximum shift occurs for $\theta = 0°$.
3. Minimum shift for any multiplicity is always zero and always occurs at a primary scattering angle $\theta = 0$.
4. Maximum shift is always $2M(h/mc)$ where M is the multiplicity of scattering.

Double Scattering

Referring to Fig. 1, on the sphere, A is the direction of the entry ray, B the direction of the intermediate ray and C the direction of the scattered ray. The arc AC measures the fixed primary scattering angle θ and is bi-

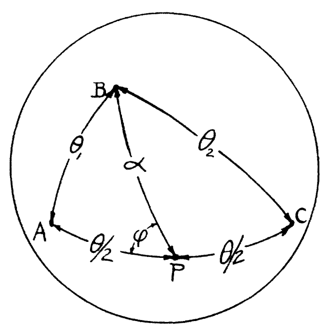

Fig. 1. Illustrating spherical coordinate system (α, ϕ) appropriate to the problem of double scattering.

sected at P. P is taken as the pole of a system of spherical polar coordinates (α, ϕ) which describe the location of the intermediate ray B whose two degrees of freedom are the independant variables of the problem.

That such a system of coordinates (α,ϕ) is appropriate to the problem will be seen from the following considerations:

The total shift

$$\Delta\lambda = \frac{h}{mc}\{2 - (\cos\theta_1 + \cos\theta_2)\} \tag{1}$$

If the letters $\boldsymbol{A}$, $\boldsymbol{B}$ and $\mathbf{C}$ represent unit vectors, then

$$\begin{aligned} \Delta\lambda &= h/mc\{2 - (\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{B}\cdot\mathbf{C})\} \\ &= h/mc\{2 - \boldsymbol{B}\cdot(\boldsymbol{A} + \mathbf{C})\} \\ &= 2h/mc(1 - \cos\tfrac{1}{2}\theta\cos\alpha) \\ &(\text{since}\,|\,\boldsymbol{A} + \mathbf{C}\,| = 2\cos\tfrac{1}{2}\theta) \end{aligned} \tag{2}$$

Hence regions of constant total shift $\Delta\lambda$ correspond to directions of B lying in zones on the sphere described about P as a pole.

Wave-Length Limits of Doubly Scattered Doubly Shifted Radiation

Since cos α may range from 1 to -1, doubly scattered radiation may have a range of shifts defined by the inequalities

$$2h/mc(1 - \cos \tfrac{1}{2}\theta) < \Delta\lambda < 2h/mc(1 + \cos \tfrac{1}{2}\theta) \qquad (3)$$

The limits of this range are evidently equidistant from the point $\Delta\lambda = 2h/mc$ and the total spectral width of the range is $4h/mc \cos \frac{1}{2}\theta$. Thus for a primary scattering angle of zero degrees double scattering contributes shifted radiation over a range of shifts from zero to $4h/mc$ while for a primary scattering angle of 180° double scattering contributes shifted radiation falling exactly at $2h/mc$ and extending over no range whatever. It is convenient to describe the spectral distributions caused by double scattering in terms of a variable x, which is defined as $x = (\Delta\lambda)/(2h/mc)$. This variable which is proportional to the shift in wave-lengths is a pure number.

The Law Governing Intensity of Scattering

In order to compute the distribution of intensity due to double scattering over the range determined for any particular primary scattering angle, some law must be adopted to express the scattered intensity as a function of the angle of scattering and of the polarization of the radiation. Kallman and Mark[1] have shown in a beautiful experiment that both shifted and unshifted radiations are completely polarized by scattering at 90°. In double scattering therefore the beam becomes partially polarized in the first scattering process and this fact is of importance in determining the intensity of scattering in the second scattering process.

We therefore assume in this computation that each scattering process gives modified radiation having the intensity and polarization dictated by the classical Thompson theory of scattering and then apply the Breit correction to this by multiplying by a factor $(1+\alpha \text{ vers } \theta_1)^{-3}$ $(1+\alpha \text{ vers } \theta_2)^{-3}$ (c.f. Compton "X-Rays and Electrons" pp. 304–305). Fortunately $(1+\alpha \text{ vers } \theta_1)^{-3}$ $(1+\alpha \text{ vers } \theta_2)^{-3}$ can be replaced by $(\lambda_0/\lambda_1)^3$ $(\lambda_1/\lambda_2)^3$ and hence by $(\lambda_0/\lambda_2)^3$ where λ_0 is the primary wave-length, λ_1 the wave-length after the first scattering, and λ_2 the wave-length after the second scattering. It is thus a very fortunate fact that the Breit correction factor depends only on the ratio of the initial and final wave-lengths in double scattering and not at all on the intermediate wave-length. Thus after the doubly scattered spectral distribution has been computed by the classical Thompson formula and the distribution is obtained as a function of $\Delta\lambda$ or x the Breit correction can be applied to the curves by simply multiplying each ordinate by the ratio $(\lambda_0/\lambda_2)^3$ applicable to its abscissa.

The above outlined method should give fairly accurate results for hard radiation and light scattering atoms where the presence of unmodified radiation and diminished modified radiation for small scattering angles can be neglected.

[1] **Kallman and Mark, "Uber einige Eigenschaften der Comptonstrahlung Zeits. f. Physik 36, 120–142 (1926).**

Unfortunately the case of say Mo$K\alpha$ radiation scattered by graphite is not such a one. The author has not succeeded in obtaining an analytic solution for double scattering rigorously applicable to this case taking account of the unmodified scattering and diminished modified scattering. Moreover the really great analytical difficulties incident to such a solution seem scarcely worth overcoming as a good qualitative idea of the spectral intensity distribution can be obtained by a rough examination of the effect of reduced modified scattering on the case which is here solved.

The shape of the scattering body also plays a part in determining the spectral distribution of doubly scattered radiation. It is of course practically impossible to take account of this factor and accordingly the solution here given can again only be applied to real cases qualitatively. In order to construct an ideal case in which the shape of the scatterer does not complicate the analysis we consider first only one electron at the center of a spherical scattering body. This electron receives primary radiation along the direction A and scatters in various directions B to all the other electrons in the scatterer which in turn scatter the radiation along the exit direction C. It is the spectral distribution of this radiation as affected by two wave-length modifications which we propose to investigate. Subsequently the reasoning is readily extended to initial scattering by *all* the electrons in the spherical scattering body.

Computation of Spectral Distributions Due to Double Scattering

Referring to Fig. 2, let E_{n1} and E_{a1} be the primary component electric intensities E_{n1} being normal to the plane of the first angle of scattering θ_1

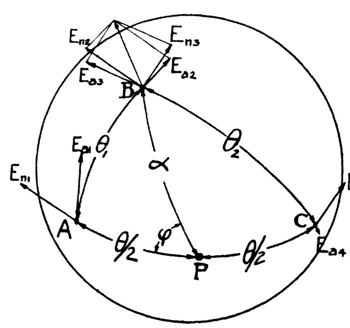

Fig. 2. Illustrating the components of the electric vector of the incident, intermediate and exit rays.

and E_{a1} parallel to that plane. At a distance r_1 from the first scattering electron along the intermediate direction B the scattered radiation will have electric components E_{n2} and E_{a2} again normal and parallel respectively to the plane of θ_1 given by the Thompson classical theory as:

$$E_{n2} = -\frac{E_{n1}e^2}{r_1 mc^2} \tag{4}$$

$$E_{a2} = -\frac{E_{a1}e^2}{r_1 mc^2}\cos\theta_1. \tag{5}$$

Let B represent the supplement of the angle B of the spherical triangle $A\ B\ C$. We now resolve the electric vector whose components are E_{n2} and E_{a2} along two new directions normal and parallel respectively to the plane of the second angle of scattering θ_2 and obtain E_{n3} and E_{a3}

$$E_{n3} = E_{n2} \cos B + E_{a2} \sin B \tag{6}$$

$$E_{a3} = E_{a2} \cos B - E_{n2} \sin B. \tag{7}$$

The scattered radiation having these components E_{n3} and E_{a3} is now scattered again by an electron at distance r_1 from the first, the new scattering angle being θ_2. At distance r_2 from the second scattering electron along the direction C the doubly scattered radiation will have electric components E_{n4} and E_{a4}

$$E_{n4} = -\frac{E_{n3}e^2}{r_2mc^2} \tag{8}$$

$$E_{a4} = -\frac{E_{a3}e^2}{r_2mc^2} \cos \theta_2. \tag{9}$$

We now express E_{n4} and E_{a4} in terms of E_{n1} and E_{a1} and evaluate the square of the final electric intensity E_4^2

$$\begin{aligned} E_4^2 = E_{n4}^2 + E_{a4}^2 = \frac{e^8}{r_1^2r_2^2m^4c^8}[&E_{n1}^2 \cos^2 B + E_{a1}^2 \cos^2 \theta_1 \sin^2 B \\ &+ 2E_{n1}E_{a1} \cos B \sin B \cos \theta_1 + \cos^2 \theta_2(E_{a1}^2 \cos^2 \theta_1 \cos^2 B \\ &+ E_{n1}^2 \sin^2 B - 2E_{a1}E_{n1} \cos \theta_1 \cos B \sin B)]. \end{aligned} \tag{10}$$

For unpolarized primary radiation we average over all possible orientations of E_1 about the primary beam giving all orientations equal weights. This means that $\{E_{n1}^2\} = \{E_{a1}^2\} = \frac{1}{2}E_1^2$ and $\{E_{n1}E_{a1}\} = 0$, the curly bracket here being used to indicate the average. The average doubly scattered intensity is then given by

$$\begin{aligned} \{E_4^2\} = \frac{1}{2}E_1^2\frac{e^8}{r_1^2r_2^2m^4c^8}[&\cos^2 B \\ &+ (\cos^2 \theta_1 + \cos^2 \theta_2) \sin^2 B + \cos^2 \theta_1 \cos^2 \theta_2 \cos^2 B] \end{aligned} \tag{11}$$

Now from a well-known formula applying to spherical triangles we can express B in terms of θ_1, θ_2, and θ

$$\cos B = \frac{\cos \theta - \cos \theta_1 \cos \theta_2}{\sin \theta_1 \sin \theta_2} \cdot \tag{12}$$

By appropriate substitutions employing this formula we obtain the doubly scattered intensity in terms of θ_1, θ_2 and θ as

$$\begin{aligned} \{E_4^2\} = \frac{1}{2}E_1^2\frac{e^8}{r_1^2r_2^2m^4c^8}[&\cos^2 \theta_1 + \cos^2 \theta_2 \\ &+ \cos^2 \theta - 2 \cos \theta \cos \theta_1 \cos \theta_2 + \cos^2 \theta_1 \cos^2 \theta_2]. \end{aligned} \tag{13}$$

We now transform this to an expression in α and ϕ the spherical coordinates of our problem by the use of the relations from spherical trigonometry

$$\cos\theta_1 = \cos\alpha\cos\tfrac{1}{2}\theta + \sin\alpha\sin\tfrac{1}{2}\theta\cos\phi \tag{14}$$

$$\cos\theta_2 = \cos\alpha\cos\tfrac{1}{2}\theta - \sin\alpha\sin\tfrac{1}{2}\theta\cos\phi. \tag{15}$$

This gives

$$\begin{aligned}\{E_4{}^2\} = \frac{1}{2}E_1{}^2\frac{e^8}{r_1{}^2r_2{}^2m^4c^8}\big[&2\cos^2\alpha\cos^2\tfrac{1}{2}\theta + 2\sin^2\alpha\sin^2\tfrac{1}{2}\theta\cos^2\phi + \cos^2\theta \\ &- 2\cos\theta\cos^2\alpha\cos^2\tfrac{1}{2}\theta + 2\cos\theta\sin^2\alpha\sin^2\tfrac{1}{2}\theta\cos^2\phi \\ &+ (\cos^2\alpha\cos^2\tfrac{1}{2}\theta - \sin^2\alpha\sin^2\tfrac{1}{2}\theta\cos^2\phi)^2\big]\end{aligned} \tag{16}$$

or:

$$I_4 = \frac{1}{2}I_1\frac{e^8}{r_1{}^2r_2{}^2m^4c^8}[y(\alpha,\theta,\phi)]. \tag{17}$$

I_4 is the doubly scattered intensity. It is the energy per cm.² per second at a distance r_2 from the second scattering electron which in turn is at a distance r_1 from the first scattering electron.

The exit ray is confined to some solid angle $\Delta\Omega_3$ by the construction of the apparatus for studying the scattered radiation. If we multiply the last expression by the area $r_2{}^2\Delta\Omega_3$ we obtain the *energy* scattered per second into the solid angle $\Delta\Omega_3$

$$I_4r_2{}^2\Delta\Omega_3 = \frac{1}{2}I_1\frac{e^8\Delta\Omega}{r_1{}^2m^4c^8}[y(\alpha,\theta,\phi)]. \tag{18}$$

Consider now a spherical shell of radius r_1 and thickness dr_1 described about the first scattering electron as a center. Let the volume density of electrons in the scatterer be ρ. In the solid angle between α and $\alpha+d\alpha$, ϕ and $\phi+d\phi$ there will be $\rho r_1{}^2\sin\alpha\, dr_1d\alpha d\phi$ electrons. The energy per second entering the solid angle $\Delta\Omega_3$ due to all the electrons just mentioned will be

$$I_4r_2{}^2\Delta\Omega_3\rho r_1{}^2\sin\alpha dr_1d\alpha d\phi = \frac{1}{2}I_1\frac{e^8\Delta\Omega_3\rho dr_1}{m^4c^8}[y(\alpha,\phi,\theta)\sin\alpha d\alpha d\phi]. \tag{19}$$

Integrating this from $r=0$ to $r=r$, the distance from the first scattering electron to the furthermost electron in the body in the direction (α, ϕ), we have

$$\frac{1}{2}I_1\frac{e^8\Delta\Omega_3\rho r_{\alpha\phi}}{m^4c^8}[y(\alpha,\phi,\theta)\sin\alpha d\alpha d\phi]. \tag{20}$$

The total shift after two scatterings is independant of ϕ as has already been pointed out. We therefore obtain the total energy going into a zone between α and $\alpha+d\alpha$, all of which contributes to one and the same shifted position in the doubly scattered spectrum by integrating around the zone from $\phi=0$ to $\phi=2\pi$. We thus obtain the energy per second passing out through the solid angle $\Delta\Omega_3$ after first scattering by one electron at the center of the *spherical* scattering body of radius, r, and then scattering by all other electrons in the body between the cones of half angle α and $\alpha+d\alpha$. It is

$$\frac{1}{2}I_1\frac{e^8\Delta\Omega_3\rho r}{m^4c^8}[y(\alpha,\, d\alpha,\, \theta)] = \frac{1}{2}I_1\frac{e^8\Delta\Omega_3\rho r}{m^4c^8}\Big[2\pi(2-2\cos\theta)\cos^2\alpha\cos^2\tfrac{1}{2}\theta + \pi(2+2\cos\theta)\sin^2\alpha\sin^2\tfrac{1}{2}\theta + 2\pi\cos^2\theta + 2\pi(\cos^2\alpha\cos^2\tfrac{1}{2}\theta)^2 - 2\pi\cos^2\alpha\cos^2\tfrac{1}{2}\theta\sin^2\alpha\sin^2\tfrac{1}{2}\theta + \frac{3}{4}\pi(\sin^2\alpha\sin^2\tfrac{1}{2}\theta)^2\Big]\sin\alpha d\alpha. \tag{21}$$

We must now transform this expression to one in terms of x and dx the independant variable used to express the shift in the description of spectral distribution. $x=(\Delta\lambda)/(2h/mc)$. The shift is given by

$$\Delta\lambda = 2h/mc(1-\cos\tfrac{1}{2}\theta\cos\alpha)$$

hence

$$x = (1-\cos\tfrac{1}{2}\theta\cos\alpha) \tag{22}$$

$$dx = \cos\tfrac{1}{2}\theta\sin\alpha d\alpha \tag{23}$$

$$\sin\alpha d\alpha = dx/\cos\tfrac{1}{2}\theta \tag{24}$$

$$\cos^2\alpha = \frac{(1-x)^2}{\cos^2\frac{1}{2}\theta};\quad \sin^2\alpha\sin^2\tfrac{1}{2}\theta = \sin^2\tfrac{1}{2}\theta - (1-x)^2\tan^2\tfrac{1}{2}\theta.$$

Making these substitutions we obtain the total energy per second in the spectral region between x and $x+dx$; it is

$$I_4dx = \frac{1}{2}I_1\frac{\pi e^8\Delta\Omega_3\rho r}{m^4c^8}[y(x,\theta)]dx = \frac{1}{2}I_1\frac{e^8\Delta\Omega_3\rho r\pi}{m^4c^8}\Big[\Big(2\cos^2\theta + \sin^2\theta + \frac{3}{4}\sin^4\tfrac{1}{2}\theta\Big) + \Big(2\sin^2\tfrac{1}{2}\theta - \frac{3}{2}\sin^2\tfrac{1}{2}\theta\tan^2\tfrac{1}{2}\theta\Big)(1-x)^2 + \Big(2+2\tan^2\tfrac{1}{2}\theta + \frac{3}{4}\tan^4\tfrac{1}{2}\theta\Big)(1-x)^4\Big]\frac{dx}{\cos\frac{1}{2}\theta}. \tag{25}$$

This expression refers to one electron only at the center of the spherical scattering body as the first scattering agent. This same electron will by *single scattering* into the solid exit angle $\Delta\Omega_3$ scatter energy

$$I_2 = \frac{1}{2}I_1\frac{e^4\Delta\Omega_3}{m^2c^4}(1+\cos^2\theta). \tag{26}$$

The ratio of double to single scattering is then

$$\frac{I_4}{I_2}dx = \pi\frac{e^4\rho r}{m^2c^4}\,\frac{[y(x,\theta)]}{1+\cos^2\theta}dx \tag{27}$$

but the "scattering coefficient" is;

$$\sigma = \frac{8}{3}\pi\frac{e^4\rho}{m^2c^4}$$

$$\frac{I_4}{I_2}dx = \frac{3}{8}\sigma r\frac{[y(x,\theta)]}{1+\cos^2\theta}dx. \tag{28}$$

The ratio of total doubly scattered energy to singly scattered energy is

$$\frac{\int_{1-\cos 1/2\theta}^{1+\cos 1/2\theta} I_4 dx}{I_2} = \frac{3}{8}\sigma r\frac{\int [y(x,\theta)]dx}{1+\cos^2\theta} = \frac{3}{8}\sigma r R. \tag{29}$$

Evaluating the integral in this expression we obtain the ratio of double to single scattering, it is

$$\frac{3}{8}\sigma r\frac{1}{1+\cos^2\theta}\left[2A + \frac{2}{3}B\cos^2\tfrac{1}{2}\theta + \frac{2}{5}C\cos^4\tfrac{1}{2}\theta\right] \tag{30}$$

where $A = (2\cos^2\theta + \sin^2\theta + \frac{3}{4}\sin^4\frac{1}{2}\theta)$
$B = (2\sin^2\frac{1}{2}\theta - \frac{3}{2}\sin^2\frac{1}{2}\theta\tan^2\frac{1}{2}\theta)$
$C = (2 + 2\tan^2\frac{1}{2}\theta + \frac{3}{4}\tan^4\frac{1}{2}\theta)$

Both values of the bracket in the last expression and values of R are plotted in Fig. 3.

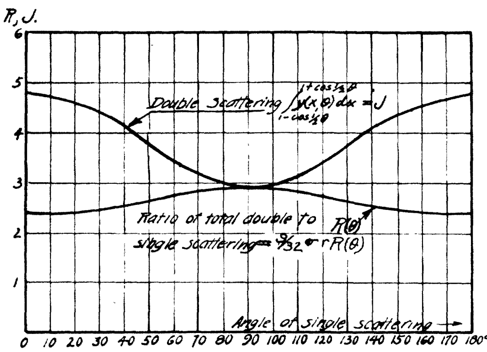

Fig. 3. Curves of the integral in Eq. (29) and also of the ratio R in that equation.

For scattering from the *central* electron in the spherical scatterer to all other electrons, the value of $r_{\alpha\phi}$ is the radius r of the spherical scatterer. For initial scattering from *all* electrons in the scatterer however an average[2] value of $\{r_{\alpha\phi}\} = \frac{3}{4}r$ must be taken. To obtain this we average the distance along any fixed arbitrary direction from each elementary volume in the spherical scatterer to the boundary of the sphere throughout the entire volume. Taking as volume elements of the sphere the space between two coaxial cylinders of radii r' and $r'+dr'$ with axis through the center of the

[2] The bracket { } is used to denote "average value."

sphere, the average distance in question is given by $1/(4/3)\pi r^3\int_0^r(r^2-r'^2)4\pi r'dr'=\frac{3}{4}r$. Now $\sigma=0.4$ per cm for Mo radiation scattered from graphite.

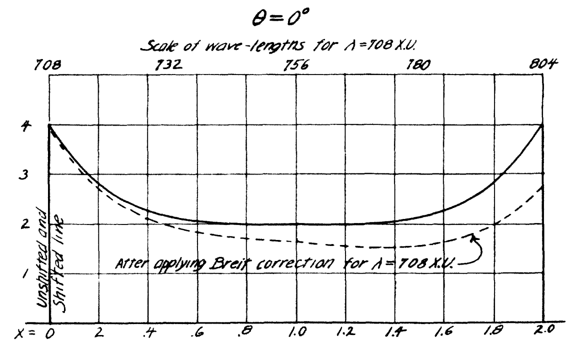

Fig. 4. Spectral distribution due to doubly modified double scattering for primary scattering angle of zero degrees.

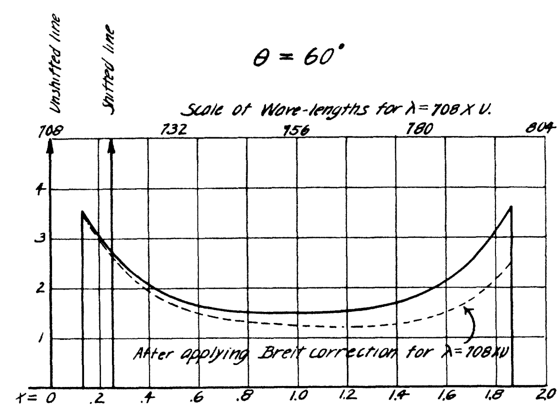

Fig. 5. Spectral distribution due to doubly modified double scattering for primary scattering angle of 60°.

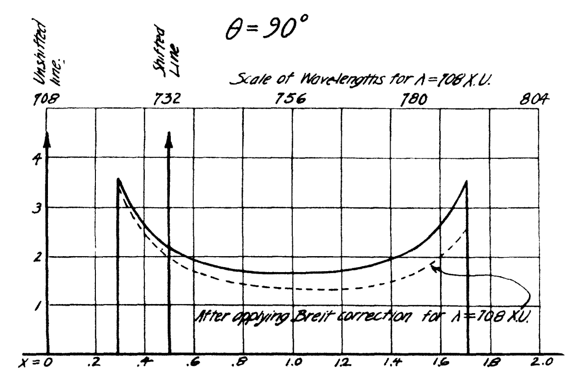

Fig. 6. Spectral distribution due to doubly modified double scattering for primary scattering angle of 90°.

It appears from Fig. 3 that R never differs very greatly from 2.5. The ratio then of total double scattering to single scattering for a spherical carbon

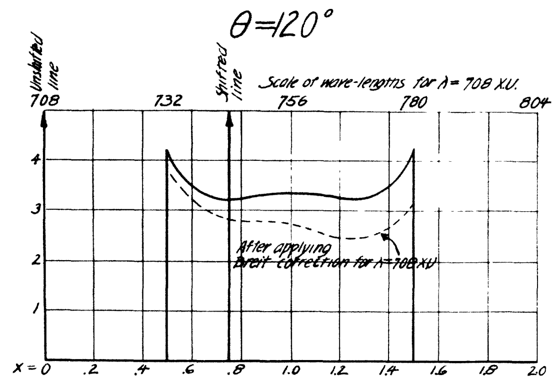

Fig. 7. Spectral distribution due to doubly modified double scattering for primary scattering angle of 120°.

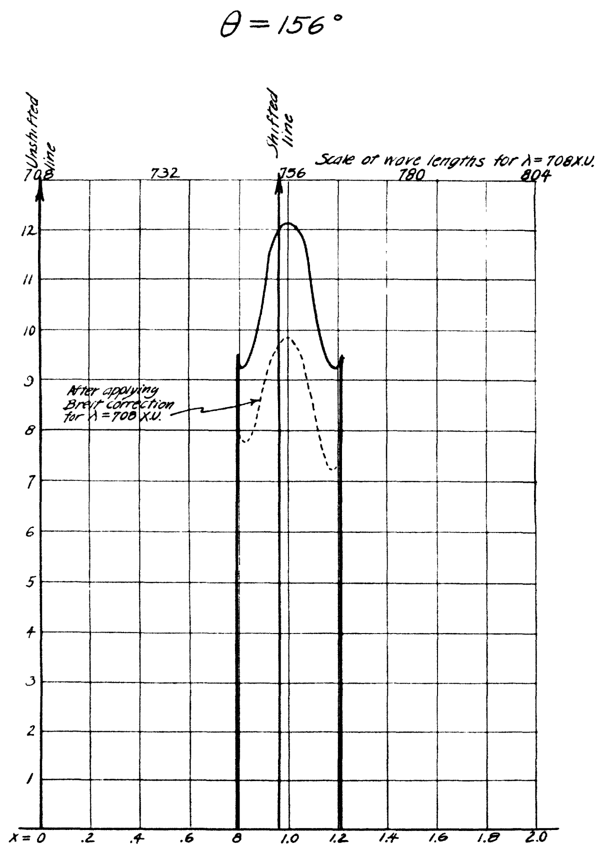

Fig. 8. Spectral distribution due to doubly modified double scattering for primary scattering angle of 156°.

scatterer 0.5 cm in radius scattering Mo radiation is about 0.140. This would increase in direct proportion to the radius of the spherical scatterer were it not for absorption. Except at $\theta=180°$ however the doubly scattered energy is distributed over a considerable wave-length band.

The spectral distribution of double scattering is obtained from expression (25) for scattering angles θ between initial and final beams of 8°, 60°, 90°, 120°, 156° and 180°. These are exhibited in Figures 4 to 9. The curves both without and with the Breit correction computed for MoK radiation are shown. An interesting feature of these curves occurs when the angle of single scattering is 180°. At this angle the spectral distribution, due to doubly

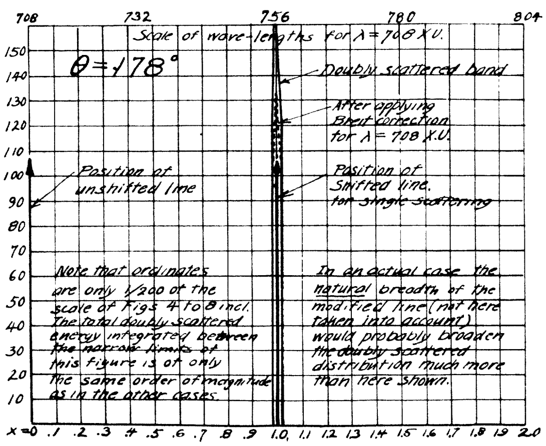

Fig. 9. Spectral distribution (with reduced ordinate scale) of doubly modified double scattering for primary scattering angle of 178°.

shifted scattering, narrows down to a sharp line (neglecting the natural breadth of the Compton shifted line itself). The presence of the term $(\cos \frac{1}{2}\theta)^{-1}$ in the expression (25) makes that expression indefinitely large for $\theta=180°$. The energy associated with double scattering corresponding to the area under this infinitesimally narrow infinitely tall curve is of course finite.

Qualitative Discussion of the Effect of Unmodified and Decreased Modified Scattering

Scattering of x-radiation of intermediate hardness (e.g. MoK radiation scattered by graphite) is of two well-known types; modified and unmodified. The total scattered energy follows the Thompson classical law with the Breit correction to a fair degree of approximation except at small scattering angles where the phenomenon of excess scattering occurs. Excess scattering is without doubt a phenomenon of interference of scattered x-radiation and applies therefore only to unmodified scattering (since modified scattering is incoherent). The total scattered energy is divided between the modified and unmodified types in proportions that depend on the scattering angle. Unfortunately the exact analytic form of this dependence is unknown. It is doubtless very complex since it depends in an intimate way on the mechanics of the scattering atom. Jauncey has treated the problem theoretically for an approximate Bohr atom model. In Fig. 10 the work of Ross and Woo relative to Mo*K* radiation scattered from graphite is collected in one curve showing the ratio of modified scattered radiation to unmodified scattered radiation as a function of scattering angle. Fig. 11 shows Hewlett's observations on total scattering. Applying the ratio of the previous curve to this we

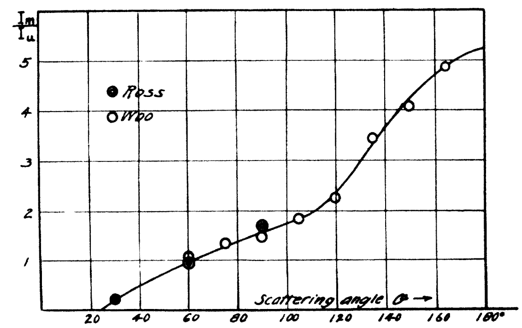

Fig. 10. Combined observations of P. A. Ross and Y. H. Woo of the ratio of modified to unmodified scattered intensity as a function of scattering angle.

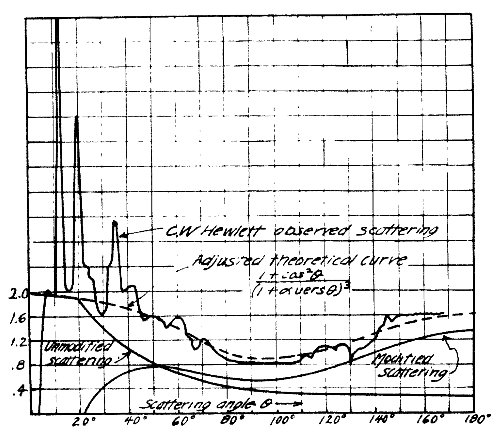

Fig. 11. Hewlett's observations of total scattering and the partition of this into modified and unmodified types in accord with Fig. 10.

obtain the two other curves which give the modified and unmodified scattered energies as a function of scattering angle. Note that below 90° modified scattering diminishes rapidly and disappears completely at about 20°. In this same region unmodified scattering becomes very strong.

To take account analytically of the effect of these phenomena on double scattering would doubtless be an extremely difficult task and one which would certainly not be warranted at present by the utility of its results. The main purpose of the present investigation is to obtain a qualitative idea of the effect of double scattering regarded as an unavoidable impurity in the experimental study of single scattering. In this paper therefore we discuss only in a brief qualitative way the effect of excess unmodified scattering and diminished modified scattering on the results we have already obtained.

Once Modified Double Scattering

Referring to Fig. 12 to which the same nomenclature of entry, exit and intermediate beams (respectively A, C, B) applies as before we obtain a rough approximation to the effect of once unmodified and once modified doubly scattered radiation by noting that the bulk of unmodified scattering will occur in the region of excess single scattering for θ_1 represented by the heavy shading around A. The zones of constant shift however are now no longer described around the bisector point P as before but around C the exit direction since the shift now occurs wholly in the second scattering process and depends only on θ_2. From these considerations it is evident that

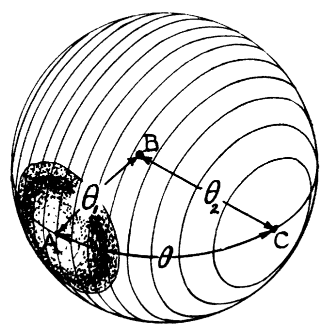

Fig. 12. Coordinate system for double scattering, unmodified at angle θ_1 but modified at angle θ_2.

the only cases of importance are those where the intermediate ray point, B, lies in the shaded regions of excess unmodified scattering. The zones of constant shift have been so spaced in the drawing that they represent equal increments of shift. One can see immediately that the greatest intensity in the spectral distribution of once modified doubly scattered radiation occurs closely adjacent to and about symmetrically on either side of the position of the modified line for single scattering, and that there will also be a considerable contribution between these two positions. Since excess unmodified scattering has been observed to be many times as strong as ordinary unmodified scattering, there is just a chance that some of the observed excess breadth of the modified Compton line may be attributed to this effect. This argument however will not apply to explain the breadth of the Compton line for primary scattering angles of nearly 180° because here the entire shaded region is included in a very small range of shifts. Singly modified double scattering will be absent near $\theta=0°$ because in this case when B is in the shaded area θ will have values for which modified scattering is practically absent. If then the effect in question is the sole cause of the observed excess breadth of the Compton line we should expect this excess breadth to be least for small and large angles of primary scattering, θ, and greatest for $\theta = 90°$.

Twice Modified Double Scattering

The effect of decreased modified scattering at small angles on the already derived spectral distributions of twice modified doubly scattered radiation occurs principally in the left hand portions of the curves. We can approximate the facts roughly by imagining circles each about 45° in radius described about A and also C from which the point B is supposed to be excluded (see Fig. 13). For scattering angles, θ, up to 90° these excluded regions will lie entirely in the low shift hemisphere. We conclude then that the right hand halves of the curves, Figs. 4 to 6 inclusive, will remain practically unchanged and that the left hand halves will diminish monotonically as we pass to the left, vanishing almost completely at their left extremities.

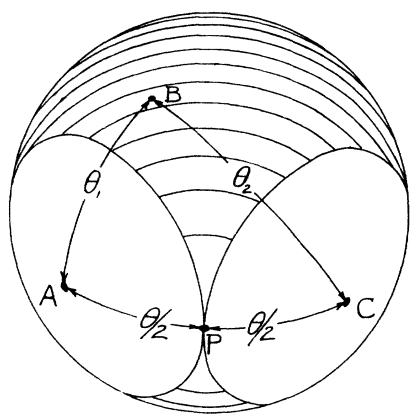

Fig. 13. Coordinate system for doubly modified double scattering taking into account the absence of modified scattering at small angles.

For scattering angles θ greater than 90° as the circles of exclusion move out of the low shift hemisphere into the high shift hemisphere, it is evident that energy will reappear in the left extremity of the spectral distributions and the suppressed portion of our curves will displace toward the right. The left hand regions or regions of low shift will however always be more depressed than the right hand regions of great shift. Fig. 14 shows the spectral distributions to be expected from the preceeding discussion.

Summary of Conclusions

1. The ratio of double to single scattering for a spherical scatterer is proportional to the radius of the scatterer (neglecting absorption) and is given by the formula

$$\frac{\text{Doubly scattered energy}}{\text{Singly scattered energy}} = \frac{9}{32}\sigma r R(\theta)$$

in which the $R(\theta)$ is greatest for about $\theta = 90°$ and has minima at $\theta = 0$ and $\theta = 180°$, the latter being the smaller (Fig. 3). R never differs greatly from 2.5.

2. For MoK radiation scattered by a sphere of graphite 1 cm in diameter double scattering is about 14% of single scattering (neglecting absorption).

3. From the last it seems probable that in most experiments the effect of triple and higher multiplicities of scattering can be completely neglected.

4. Once modified doubly scattered radiation may account for a slight broadening of the Compton line except in regions near $\theta = 0$ and 180°.

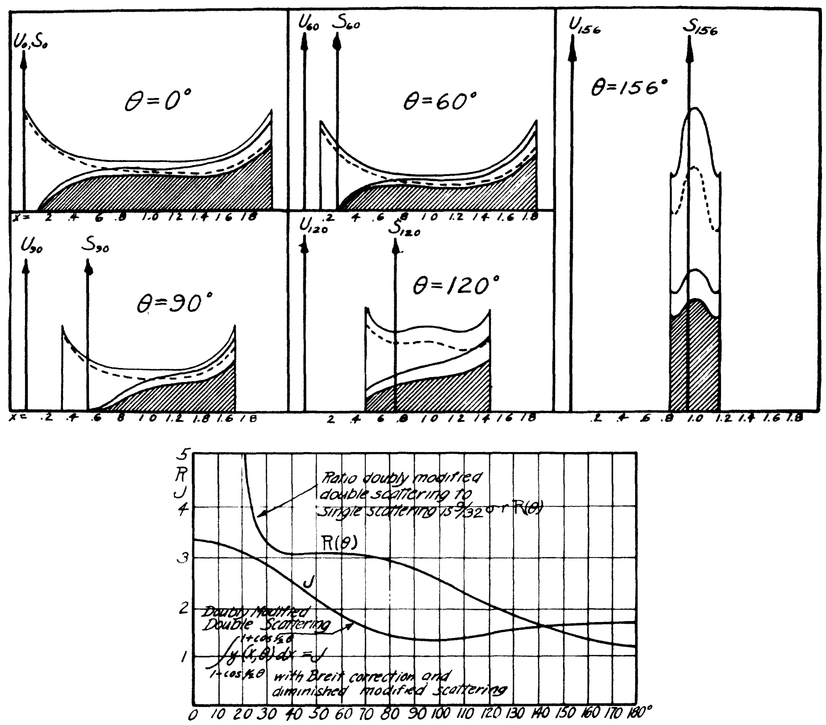

Fig. 14. Spectral distributions due to doubly modified double scattering corrected for diminished modified scattering at small angles (shaded curves). The two curves $R(\theta)$ and J are the same quantities as those shown in Fig. 3 after application of the Breit correction and correction for diminished modified scattering at small angles.

5. Twice modified doubly scattered radiation when corrected qualitatively for diminished modified scattering at small angles may contribute a faint asymmetric line or edge in the position

$$\Delta\lambda = 2\frac{h}{mc}(1 + \cos \tfrac{1}{2}\theta)$$

6. For hard radiation where the Thompson formula with the Breit correction applies with fair accuracy to modified scattering alone, twice modified doubly scattered radiation contributes a spectral band of breadth

$$4\frac{h}{mc}\cos \tfrac{1}{2}\theta$$

List of Symbols

$\Delta\Omega$, $\delta\Omega$—Solid angle measured in steradians.
θ—The angle of single scattering.
θ_1—The first angle of double scattering.
θ_2—The second angle of double scattering.
M—The multiplicity of scattering.
λ_0—The wave-length of the primary radiation.
λ_1, λ_2—The wave-lengths after first and second scatterings.
h—Planck's constant.
m—The mass of the electron.
c—The velocity of light.
$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$—Unit vectors representing the directions of the entry, intermediate, and exit rays in double scattering. (Fig. 2).
α, ϕ—The colatitude and longitude angles of the spherical polar coordinates of reference in double scattering. (Fig. 2.)
$x=(\Delta\lambda)/(2h/mc)$—wave-length variable used for convenience in describing spectral distributions due to double scattering.
$E_{n1}E_{a1}E_{n2}E_{a2}E_{n3}E_{a3}E_{n4}E_{a4}$—components of the electric intensity in the radiation. The subscript, n, indicates the component normal to the plane of one scattering angle while the subscript, a, indicates the component parallel to that plane. The subscript 1 refers to the initial beam and the first scattering angle, subscript 2 refers to the intermediate beam and the first scattering angle, 3 to the intermediate beam and the second scattering angle, 4 to the final or exit beam and the second scattering angle. See Fig. 2.
B—The dihedral angle between the planes of θ_1 and θ_2. (Fig. 2.)
r_1—The distance from the point of first scattering to the point of second scattering.
r_2—The distance from the point of second scattering to the point of observation.
r—The radius of the spherical scattering body.
σ—The scattering coefficient for the material of the scatterer and the radiation used.