THE DIFFRACTION OF A CIRCULARLY SYMMETRICAL ELECTROMAGNETIC WAVE BY A CO-AXIAL CIRCULAR DISC OF INFINITE CONDUCTIVITY

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Abstract

A disc of infinite conductivity, whose radius is a, and whose center is at the origin, lies in a plane which is perpendicular to the z-axis of cylindrical coordinates, r, z, ϕ. A circularly symmetrical electromagnetic wave of wave-length λ impinges on the disc, and the resultant field is required. The solution depends upon solving an integral equation of the first kind. When $2\pi a/\lambda<1$, this equation reduces to an integral equation similar to Abel's which may be solved explicitly. As an illustration the solution is obtained for the diffraction of a wave due to an oscillating electric dipole whose axis is the axis of z. It is mentioned that these equations have been used in determining the powerflow into the earth below a vertical antenna which is grounded by a circular disc lying on the surface.

1. Formulation of the Problem

Consider a disc of radius a whose center is at the origin and whose plane is perpendicular to the Z-axis of cylindrical coordinates r, z, ϕ. A circularly symmetrical electromagnetic wave impinges on the disc, and the resultant field is required. Heretofore in such problems the disc has been treated as the limiting case of an oblate ellipsoid of revolution.[1] A solution in the form of a definite integral may, however, be directly obtained without making use of this limiting process.

The electric intensity $\boldsymbol{E}'$ and the magnetic intensity $\boldsymbol{H}'$ of the incident wave are assumed to be of such nature that:

$$H_r' = H_z' = E_\phi' = 0$$

and that H_ϕ', E_r', and E_z' are independent of ϕ. The dependence of this wave on time is to be of the form $e^{-i\omega t}$.

In diffraction problems, the field due to the disc is considered separately from the incident wave. This field alone will be considered here, the incident wave being taken into account only through the boundary conditions. The electric intensity $\boldsymbol{E}$ and the magnetic intensity $\boldsymbol{H}$ of the field due to the disc must satisfy Maxwell's equations for free space, which in this case reduce to:

$$H_r = H_z = E_\phi = 0 \tag{1}$$

[1] Hertzfeld, Wiener Berichte (1911) p. 1587; Möglich, Ann. der Physik **83**, 609 (1927).

$$\left.\begin{aligned} i\omega\mu H_\phi &= \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \\ -\ i\omega p E_r &= -\frac{\partial H_\phi}{\partial z} \\ -\ i\omega p E_z &= \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) \\ \frac{1}{r}\frac{\partial}{\partial r}(rE_r) &+ \frac{\partial E_z}{\partial z} = 0 \end{aligned}\right\} \tag{2}$$

In these equations $\boldsymbol{E}$ and $\boldsymbol{H}$ are expressed in the rationalized practical system of units.[2]

The disc is assumed to be of infinite conductivity. Thus the radial component of the electric intensity of the total field must vanish on the surface of the disc. This requires that

$$(E_r)_{z=0} = f(r) \quad \text{for} \quad r < a \tag{3}$$

where $f(r)$ is the negative of the radial component of the electric intensity of the incident wave at $z=0$. From symmetry considerations it is seen that

$$(E_z)_{z=0} = 0 \quad \text{for} \quad r > a \tag{4}$$

and that

$$(H_\phi)_{z=0} = 0 \quad \text{for} \quad r > a. \tag{5}$$

It is required further that $\boldsymbol{E}$ and $\boldsymbol{H}$ be continuous functions of (r, z) throughout space except at the disc, and that they vanish to proper orders at infinity.

2. Solution of the Problem

An appropriate solution of Eq. (2) is:

$$E_r = \int_0^\infty J_1(\lambda r)\phi(\lambda)(\lambda^2 - k^2)^{1/2} e^{\mp z(\lambda^2-k^2)^{1/2}} d\lambda \tag{6}$$

$$E_z = \pm \int_0^\infty J_0(\lambda r)\phi(\lambda)\lambda e^{\mp z(\lambda^2-k^2)^{1/2}} d\lambda \tag{7}$$

$$H_\phi = \mp i\omega p \int_0^\infty J_1(\lambda r)\phi(\lambda) e^{\mp z(\lambda^2-k^2)^{1/2}} d\lambda^{1/2} \tag{8}$$

The upper signs are used for positive z, while $k=\omega/c$, where c is the velocity of light. The symbols J_0 and J_1 represent the ordinary Bessel's functions. The quantity $\phi(\lambda)$ is an arbitrary function of λ, independent of the coordinates.

[2] In this system: Magnetic intensity, $\boldsymbol{H}$, is expressed in ampere-turns per cm. Magnetic flux density, $\boldsymbol{B}$, is expressed in webers per sq. cm. ($=10^8$ maxwells per sq. cm.). Permeability, $\mu=B/H=4\pi/10^{-9}$ for free space. Electric intensity, $\boldsymbol{E}$, is expressed in volts per cm. Displacement, $\boldsymbol{D}$, is expressed in coulombs per sq. cm. Permittivity, $p=D/E=8.85/10^{-14}$ for free space.

It may be chosen in such a manner as to satisfy the boundary conditions ((3), (4), and (5)) and is, of course, subject to the restriction that it must produce proper convergence of the integrals.

The method of solution is somewhat analogous to Beltrami's theory of symmetrical potentials.[3] Choose:

$$\phi(\lambda) = \int_0^a F(s) \cos \left[s(\lambda^2 - k^2)^{1/2}\right] ds \tag{9}$$

where $F(s)$ is arbitrary. From Eq. (7)

$$r(E_z)_{z=0} = \frac{\partial}{\partial r} r \int_0^\infty J_1(\lambda r)\phi(\lambda) d\lambda \tag{10}$$

Substituting the value of $\phi(\lambda)$ given by Eq. (9), and changing the order of integration, there results:

$$r(E_z)_{z=0} = \pm \frac{\partial}{\partial r} r \int_0^a F(s) ds \int_0^\infty J_1(\lambda r) \cos \left[s(\lambda^2 - k^2)^{1/2}\right] d\lambda \tag{11}$$

For $r>a$

$$r(E_z)_{z=0} = \pm \frac{\partial}{\partial r} r \int_0^a \frac{F(s) \cosh (ks) ds}{r} = 0. \tag{12}$$

Thus Eq. (4) is satisfied.

Next $F(a)$ will be chosen so that Eq. (5) is satisfied. From Eq. (8)

$$(H_\phi)_{z=0} = \mp i\omega p \int_0^\infty J_1(\lambda r)\phi(\lambda) d\lambda. \tag{13}$$

Now from Eq. (9)

$$\phi(\lambda) = F(a) \frac{\sin \left[a(\lambda^2 - k^2)^{1/2}\right]}{(\lambda^2 - k^2)^{1/2}} - \int_0^a F'(s) \frac{\sin \left[s(\lambda^2 - k^2)^{1/2}\right]}{(\lambda^2 - k^2)^{1/2}} ds. \tag{14}$$

Thus

$$(H_\phi)_{z=0} = \mp i\omega p \left[F(a) \int_0^\infty J_1(\lambda r) \frac{\sin \left[a(\lambda^2 - k^2)^{1/2}\right]}{(\lambda^2 - k^2)^{1/2}} d\lambda \right.$$
$$\left. - \int_0^a F'(s) ds \int_0^\infty J_1(\lambda r) \frac{\sin \left[s(\lambda^2 - k^2)^{1/2}\right]}{(\lambda^2 - k^2)^{1/2}} d\lambda . \right] \tag{15}$$

For $r>a$

$$(H_\phi)_{z=0} = \mp i\omega p \left[\frac{F(a) \sinh (ka)}{kr} - \int_0^a \frac{F'(s) \sinh (ks) ds}{kr} \right] = 0 \tag{16}$$

This equation gives $F(a)$ in terms of $F'(s)$:

$$F(a) = \int_0^a F'(s) \frac{\sinh (ks)}{\sinh (ka)} ds. \tag{17}$$

[3] Webster: Partial Differential Equations of Mathematical Physics. p. 368 (1927).

Next $F'(s)$ is determined in such a manner that Eq. (3) is satisfied. Substituting Eq. (14) in Eq. (3):

$$(E_r)_{z=0} = \int_0^\infty J_1(\lambda r)\Big[F(a)\sin\left[a(\lambda^2-k^2)^{1/2}\right] - \int_0^a F'(s)\sin\left[s(\lambda^2-k^2)^{1/2}\right]ds\Big]d\lambda. \tag{18}$$

The value of $F(a)$ from Eq. (17) is now substituted in Eq. (18). Changing the order of integration:

$$(E_r)_{z=0} = \int_0^a F'(s)\frac{\sinh(ks)}{\sinh(ka)}ds\int_0^\infty J_1(\lambda r)\sin\left[a(\lambda^2-k^2)^{1/2}\right]d\lambda - \int_0^a F'(s)ds\int_0^\infty J_1(\lambda r)\sin\left[s(\lambda^2-k^2)^{1/2}\right]d\lambda. \tag{19}$$

For $r<a$

$$r(E_r)_{z=0} = f(r) = -\int_0^r sF'(s)\frac{\cos\left[k(r^2-s^2)^{1/2}\right]}{(r^2-s^2)^{1/2}}ds + i\int_0^a F'(s)\left[\frac{a\sinh\left[k(a^2-r^2)^{1/2}\right]\sinh(ks)}{(a^2-r^2)^{1/2}\sinh(ka)} - \frac{s\sinh\left[k(s^2-r^2)^{1/2}\right]}{(s^2-r^2)^{1/2}}\right]ds \tag{20}$$

This is an integral equation of the first kind for the determination of $F'(s)$. It probably possesses no solution of a simple form. If, however, the second integrand be expanded in a power series in ka, we find that the leading term is

$$\frac{iF'(s)ka}{\sinh(ka)}\left[\frac{4}{3}\,\frac{sr^2(a^2-s^2)}{a^5}\,\frac{(ka)^5}{5!} + \text{higher order terms}\right]$$

Let us now assume that $ka<1$, the final result being, of course, subject to this restriction. Remembering that r in Eq. (20) is less than a, it appears that the second integral of Eq. (20) will be much smaller than the first unless abnormally great values of $F'(s)$ occur between $s=r$ and $s=a$. We will therefore assume that the second integral of Eq. (20) is negligible as compared to the first. The validity of this assumption for any definite problem must be checked after $F'(s)$ is calculated, by direct test of the final result in the original conditions of the problem. In the special cases for which calculation has been completed, the error involved has been found to be less than one percent.

We thus have left an integral equation similar to Abel's equation,[4] which may be solved in much the same manner. This equation is:

$$rf(r) = -\int_0^r \frac{F'(s)s\cos\left[k(r^2-s^2)^{1/2}\right]ds}{(r^2-s^2)^{1/2}}. \tag{21}$$

[4] Bocher, An Introduction to the Study of Integral Equations, p. 8, (1914).

It is interesting to note that this equation for the determination of $F'(s)$ no longer contains the parameter a. Thus we will proceed formally as though the functions $f(r)$ were known[5] for all values of r. In the final equations $F'(s)$ will involve a knowledge of $f(r)$ only from $r=0$ to $r=a$. Let

$$-F'(s) = \int_0^k \beta g[(k^2-\beta^2)^{1/2}]I_0(\beta s)d\beta + \int_0^\infty \lambda g[(\lambda^2+k^2)^{1/2}]J_0(\lambda s)d\lambda \quad (22)$$

where $g[\rho]$ is a function to be determined. Substituting this value for $F'(s)$ in Eq. (21) and changing the order of integration,

$$\begin{aligned} rf(r) &= \int_0^k \beta g[(k^2-\beta^2)^{1/2}]d\beta \int_0^r sI_0(\beta s)\frac{\cos[k(r^2-s^2)^{1/2}]}{(r^2-s^2)^{1/2}}ds \\ &\quad + \int_0^\infty \lambda g[(\lambda^2+k^2)^{1/2}]d\lambda \int_0^r sJ_0(\lambda s)\frac{\cos[k(r^2-s^2)^{1/2}]}{(r^2-s^2)^{1/2}}ds \quad (23) \\ &= \int_0^k \beta g[(k^2-\beta^2)^{1/2}]\frac{\sin[r(k^2-\beta^2)^{1/2}]}{(k^2-\beta^2)^{1/2}}d\beta \\ &\quad + \int_0^\infty \lambda g[(\lambda^2+k^2)^{1/2}]\frac{\sin[r(\lambda^2+k^2)^{1/2}]}{(\lambda^2+k^2)^{1/2}}d\lambda \quad (24) \end{aligned}$$

or, changing the variable of integration,

$$rf(r) = \int_0^\infty g(\rho)\sin(r\rho)d\rho. \quad (25)$$

This is a Fourier integral equation, the solution of which is

$$\begin{aligned} g(\rho) &= \frac{2}{\pi}\int_0^\infty \alpha f(\alpha)\sin(\alpha\rho)d\alpha \quad (26) \\ &= -2\alpha f(\alpha)\frac{\cos(\alpha\rho)}{\pi\rho}\bigg|_0^\infty + \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial\alpha}(\alpha f(\alpha))\frac{\cos(\alpha\rho)}{\rho}d\alpha. \quad (27) \end{aligned}$$

Since $f(\alpha)$ vanishes to a higher order than $1/\alpha$ at $\alpha=\infty$, the first term vanishes. Substituting the remainder in Eq. (22)

$$\begin{aligned} -F'(s) &= \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial\alpha}(\alpha f(\alpha))d\alpha \int_0^k \beta I_0(\beta s)\frac{\cos[\alpha(k^2-\beta^2)^{1/2}]}{(k^2-\beta^2)^{1/2}}d\beta \\ &\quad + \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial\alpha}(\alpha f(\alpha))d\alpha \int_0^\infty \lambda J_0(\lambda s)\frac{\cos[\alpha(\lambda^2+k^2)^{1/2}]}{(\lambda^2+k^2)^{1/2}}d\lambda \quad (28) \end{aligned}$$

$$F'(s) = -\frac{2}{\pi}\int_0^s \frac{1}{\alpha}\frac{\partial}{\partial\alpha}(\alpha f(\alpha))\frac{\alpha\cosh[k(s^2-\alpha^2)^{1/2}]}{(s^2-\alpha^2)^{1/2}}d\alpha \quad (29)$$

The solution of the problem is now complete, since we may write from Eq. (14) and Eq. (17),

[5] Since $f(r)$ is essentially a component of the incident field, it vanishes at infinity as $1/r^2$.

$$\phi(\lambda) = \int_0^a F'(s)\left[\frac{\sin\,[a(\lambda^2 - k^2)^{1/2}]\,\sinh\,(ks)}{(\lambda^2 - k^2)^{1/2}\sinh\,(ka)} - \frac{\sin\,[s(\lambda^2 - k^2)^{1/2}]}{(\lambda^2 - k^2)^{1/2}}\right]ds \tag{30}$$

and the components of the field are then obtained directly from Eq. (6), (7), and (8).

Thus

$$\begin{aligned} H_\phi &= \mp i\omega p \int_0^a F'(s)\frac{\sinh\,(ks)}{\sinh\,(ka)}ds \int_0^\infty J_1(\lambda r)e^{\mp z(\lambda^2-k^2)^{1/2}}\frac{\sin\,[a(\lambda^2 - k^2)^{1/2}]}{(\lambda^2 - k^2)^{1/2}}d\lambda \\ &\quad \pm i\omega p \int_0^a F'(s)ds \int_0^\infty J_1(\lambda r)e^{\mp z(\lambda^2-k^2)^{1/2}}\frac{\sin\,[s(\lambda^2 - k^2)^{1/2}]}{(\lambda^2 - k^2)^{1/2}}d\lambda \\ &= \pm\frac{i\omega p}{kr}\int_0^a F'(s)\left[e^{ikR_1}\frac{\sinh\,(kI_1)\sinh\,(ks)}{\sinh\,(ka)} - e^{ikR_2}\sinh\,(kI_2)\right]ds \end{aligned} \tag{31}$$

where

$$R_1 + iI_1 = [r^2 + (ia \pm z)^2]^{1/2}$$

and

$$R_2 + iI_2 = [r^2 + (is \pm z)^2]^{1/2}.$$

The upper signs are used for positive z and the lower for negative z. Derivatives of H_ϕ give E_r and E_z.

$$\left.\begin{aligned} E_r &= -\frac{i}{\omega p}\frac{\partial H_\phi}{\partial z} \\ E_z &= \frac{i}{\omega p r}\frac{\partial}{\partial r}(rH_\phi) \end{aligned}\right\} \tag{32}$$

In many important cases $f(\alpha)$ is given in the form of an integral.

$$f(\alpha) = \int_0^\infty h(\rho)J_1(\alpha\rho)d\rho. \tag{33}$$

Then

$$\frac{1}{\alpha}\frac{\partial}{\partial\alpha}(\alpha f(\alpha)) = \int_0^\infty \rho h(\rho)J_0(\alpha\rho)d\rho \tag{34}$$

and

$$\begin{aligned} F'(s) &= -\frac{2}{\pi}\int_0^\infty \rho h(\rho)d\rho \int_0^s \alpha J_0(\lambda\alpha)\frac{\cosh\,[k(s^2 - \alpha^2)^{1/2}]}{(s^2 - \alpha^2)^{1/2}}d\alpha \\ &= -\frac{2}{\pi}\int_0^\infty \rho h(\rho)\frac{\sin\,[s(\rho^2 - k^2)^{1/2}]}{(\rho^2 - k^2)^{1/2}}d\rho \end{aligned} \tag{35}$$

3. Illustrative Example

As an example, $F'(s)$ will be calculated for the diffraction of a wave due to an oscillating electric dipole. The axis of the dipole is the Z-axis and it is located a distance H above the disc. Then

$$E_r' = -\int_0^\infty \rho^2 e^{-(H-z)(\rho^2-k^2)^{1/2}} J_1(\rho r)\,d\rho \tag{36}$$

so that

$$f(\alpha) = \int_0^\infty \rho^2 e^{-H(\rho^2-k^2)^{1/2}} J_1(\rho\alpha)\,d\rho \tag{37}$$

and

$$h(\rho) = \rho^2 e^{-H(\rho^2-k^2)^{1/2}}. \tag{38}$$

Thus from Eq. (35)

$$F'(s) = -\frac{2}{\pi}\int_0^\infty \rho^3 e^{-H(\rho^2-k^2)^{1/2}} \frac{\sin\left[s(p^2-k^2)^{1/2}\right]}{(\rho^2-k^2)^{1/2}}\,d\rho \tag{39}$$

$$= \frac{2}{\pi}\int e^{ikH}\left[\frac{k(H^2-s^2)\sinh(ks)+2ikHs\cosh(ks)}{(H^2+s^2)^2}\right] - e^{ikH}\left[\frac{(3H^2s-s^3)\cosh(ks)+i(3Hs^2-H^3)\sinh(ks)}{(H^2+s^2)^3}\right]. \tag{40}$$

This value of $F'(s)$ is then substituted in Eq. (31) to obtain the components of the field. The resulting integrals are rather complicated, but may be numerically evaluated by mechanical means.

These equations have been used in determining the power flow into the earth below a vertical antenna which is grounded by a circular disc lying on the surface. (The disc is an idealized case. Actually a number of radial wires would be buried in the earth.) The results of this investigation will be published elsewhere.