A NEW RELATIVITY THEORY OF THE UNIFIED PHYSICAL FIELD

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Abstract

This theory abandons the attempt to geometrise physics. Its aim is to give invariant differential equations of motion for mass-charge particles in regions where the indeterminacy of position and momentum is not significant.

Eddington's displacement rule is used to define an indeterminate vector field, and a simple generalisation of it is used to define an indeterminate tensor field. The vector field gives the possible velocity of a mass-charge particle at any point, and vector lines in the field are necessarily the tracks of the particle. The tensor field defines an invariant element of arc by means of which the orbit equations are given in the familiar parametric form.

Physical or actual fields are necessarily determinate functions and are defined by the variations of the indeterminate fields round closed loops in the usual way; but an alternative definition is suggested which does not depend upon such abstract processes. The usual identifications of the gravitation and electric field tensors give the classical laws for small fields. In the pure gravitation field the vector lines become Einstein's "geodesics", and for negligible gravitation field they become the empirical equations of motion of a charge in regions remote from atomic nuclei.

The theory is purely a descriptive apparatus, and its usefulness is definitely limited by the principle of indeterminacy. This admitted limitation enables the theory to avoid the inconsistency between field and atomic theory, an inconsistency which appears as merely the result of ignoring the limitation.

Introduction

All unified field theories have hitherto been attempts to geometrise completely the theory of the physical field: and such is the loyalty to the geometrical ideal that quite unintelligible properties are introduced such as non-integrable length and generalised parallel displacement. Such geometrical analogies are entirely useless as aids to imaginative conception[1] and therefore, in the writer's opinion, a hindrance to sound mathematics and to further progress. In the present theory geometrical interpretation is dispensed with. Stated briefly the aim of the theory is to derive a spacetime description of the motion of a material particle in regions where the indeterminacy in the particle's position and momentum is not significant; the motion being given by differential equations of invariant form.

Apart from Whitehead's theory[2] which does not appear to have attracted much attention, and which was based on the reactionary principle of absolute

[1] For a statement of the opposite view see Eddington's "The Mathematical Theory of Relativity" Art. 83.

[2] Dr. A. N. Whitehead, "The Principle of Relativity" Chaps. IV, V.

acceleration, the theory of O. Klein[3] appears to be the only theory which has succeeded in so deriving the equations of motion of charged particles in electric fields. Since the latter theory introduces an unknowable fifth dimension, it would seem less satisfactory than the present one which requires no such *ad hoc* hypothesis.

The writer has tried so to state the theory that it shall be obviously nothing more than a descriptive apparatus: not its validity, but only its usefulness as an apparatus depends upon the existence of the entities whose motions it proposes to describe. The limits to the usefulness of the method come out quite clearly when the fact of indeterminacy is taken into account. Thus the present theory will lead to the classical result that an accelerated electron must radiate energy, but it also contains the admission that such a result is meaningless when the acceleration is large and taking place in fields close to atomic nuclei. This seems to be a distinct advance on the classical theory, at least as usually stated; for no such admission is contained in the latter theory.

Thus the present theory suggests that the apparent inconsistency between the field theory and atomic theory is nothing more, in all probability, than a limitation in the usefulness of the field theory due to the fact of indeterminacy. Once the exact nature of this limited usefulness is defined the inconsistency evaporates.

I. The Vector Field of Possible Velocity

Begin with space time coordinates x_i, $i=1, 2, 3, 4$, and the fundamental array Γ_{jk}^{i} $i, j, k=1, 2, 3, 4$. The latter is composed of sixty four independent continuous single-valued functions of the coordinates.

Select arbitarily a 4-vector A^r at a definite point x_i and proceed by the rule

$$dA^r = -\Gamma_{si}{}^r A^s dx_i \tag{1}$$

to construct a vector field, by defining

$$A'^r = A^r + dA^r$$

at

$$x_i' = x_i + dx_i.$$

The field at any point will depend upon the track along which the rule (1) has been applied. We shall call this field the field of possible velocity, and use it to describe the motions of particles.

A vector line in this field will be a line, every element of which has coordinate components proportional to the components of the vector at the point; and so defined, will be the track of a particle with the corresponding initial conditions. The equations of a vector line follow immediately from Eq. (1) as

$$d^2x_i = -\Gamma_{jk}{}^i dx_j dx_k. \tag{2}$$

This is a set of four differential equations of the second order, their solutions containing eight arbitrary constants, the four components of initial velocity and the coordinates of the starting point.

[3] O. Klein Zeits. f. Physik **46**, 188 (1927).

II. The Fundamental Tensor Field

Erect, again at any initial point, the (symmetrical tensor) components a_{ik} and just as in §1, define a tensor field by

$$\begin{aligned} da_{ik} &= (a_{rk}\Gamma_{is}{}^{r} + a_{ir}\Gamma_{ks}{}^{r})dx_s \\ -\, da^{ik} &= (a^{rk}\Gamma_{rs}{}^{i} + a^{ir}\Gamma_{rs}{}^{k})dx_s \end{aligned} \tag{3}$$

where a^{ik} is the usual normalised subdeterminant of a_{ik}.

An associated covariant vector

$$A_i = a_{ik}A^k$$

may be defined at any point, and combining Eqs. (1) and (3) shows

$$dA_i = \Gamma_{is}{}^{k}A_k dx_s. \tag{1'}$$

Provided we remember that Eqs. (3) are not integrable, and that therefore the double partial differentiation of the a_{ik}'s is non-commutative, they can be expressed in the form

$$\partial a_{ik}/\partial x_s = a_{rk}\Gamma_{is}{}^{r} + a_{ir}\Gamma_{ks}{}^{r} \text{ etc.} \tag{3'}$$

Insert these equations in the usual definition of the Christoffel three-index symbol of the second kind, we deduce that, if we put

$$\Delta_{jk}{}^{i} = \tfrac{1}{2}(\Gamma_{jk}{}^{i} + \Gamma_{kj}{}^{i}) \tag{4}$$

$$\nabla_{jk}{}^{i} = \tfrac{1}{2}(\Gamma_{jk}{}^{i} - \Gamma_{kj}{}^{i}) \tag{4'}$$

then

$$\{ij,\, k\} = \Delta_{ij}{}^{k} + a_{ri}a^{ks}\nabla_{sj}{}^{r} + a_{rj}a^{ks}\nabla_{si}{}^{r}. \tag{5}$$

Using Eqs. (4), (4'), and (5), Eq. (2) becomes

$$d^2x_i + \{jk,\, i\}dx_j dx_k = 2a_{rj}a^{is}\nabla_{sk}{}^{r}dx_j dx_k. \tag{2'}$$

III. Fundamental Invariants

If we define invariant

$$A^2 = A_s A^s$$

we can show by Eqs. (1) and (1') that the value of A is a constant independent of position, depending only on the initial values of A_s, A^s. Similarly if we define the invariant

$$ds^2 = a_{ij}dx_i dx_j \tag{7}$$

and apply the rule of association

$$d\cdot dx_k = -\,\Gamma_{ij}{}^{k}dx_i dx_j \tag{1''}$$

the value of ds will also be a constant depending only on the initial values of the displacement ds_i and the tensor a_{ij}. Comparison of Eqs. (1'') and (2) shows that we may therefore use ds as an invariant differential to give the Eqs. (2) the form

$$d^2x_i/ds^2 + \Gamma_{jk}{}^i dx_j/ds dx_k/ds = 0. \tag{8}$$

Since ds is an invariant for coordinate transformations, we may use the variation equation

$$\delta \int ds = 0$$

to deduce[4] the tensor character of the expression

$$d^2x_i + \{jk, i\} dx_j dx_k = t^i \tag{10}$$

and as usual also the tensor

$$D^{ij}{}_{,k} = \partial D^{ij}/\partial x_k + \{sk, i\} D^{sj} + \{sk, j\} D^{si}. \tag{11}$$

and the similar derivatives of tensors of different order.

Note that although (10) is a tensor at any point, it is not a determinate function of position; solutions of (9) are not fixed by the initial conditions, as are those of Eq. (2).

IV. Actual Physical Fields

We have introduced the idea of a possible field, indeterminate at any point, but actual fields will be determinate at every point. Such fields may be obtained from the fundamental array in the usual way.

Thus the total change in the possible vector field on going round a closed space-time track is[5]

$$\Delta A^i = -\tfrac{1}{2} \iint {}^*B^i{}_{jrs} A^j dS^{rs} \tag{12'}$$

and similarly the change in the tensor field is[6]

$$\Delta a_{ij} = -\tfrac{1}{2} \iint (a_{kj} {}^*B^k{}_{irs} + a_{ik} {}^*B^k{}_{jrs}) dS^{rs} \tag{12''}$$

where

$${}^*B^r{}_{ijk} = -\partial\Gamma_{ij}{}^r/\partial x_k + \partial\Gamma_{ik}{}^r/\partial x_j + \Gamma_{ik}{}^s\Gamma_{sj}{}^r - \Gamma_{ij}{}^s\Gamma_{sk}{}^r \tag{12}$$

is a tensor determined by the array.

Define as usual

$${}^*R_{ij} = {}^*B^r{}_{ijr} \tag{13}$$

$${}^*G_{ij} = \tfrac{1}{2}({}^*R_{ij} + {}^*R_{ji}) \tag{13'}$$

[4] For an alternative deduction of the tensor character of (11) see the writer's "Mathematical Properties of a Continuum with Indeterminate Metric" shortly appearing in the P. R.S.

[5] See Eddington, reference 1, Art. 92.

[6] The proof of this is analogous to that of the preceeding equation.

and

$$*F_{ij} = *B^r_{,rij} \tag{13''}$$

Then by Eq. (12) we can show that

$$*F_{ij} = \partial\Gamma_{si}^s/\partial x_j - \partial\Gamma_{sj}^s/\partial x_i \tag{14}$$

or that

$$*F_{ij} = *K_{i,j} - *K_{j,i} \tag{14'}$$

where

$$*K_i = \Gamma_{si}^s + \partial f/\partial x_i \tag{14''}$$

in the original coordinates, but transforms as a vector; f being any function of position, invariant.

An alternative antisymmetrical tensor[7] may also be defined by

$$*H_{ij} = *K'_{i,j} - *K'_{j,i} \tag{15}$$

where

$$*K_i' = \nabla_{is}^s. \tag{15'}$$

In what follows no difference results from using $*H_{ij}$ instead of $*F_{ij}$.

(a). Pure gravitation field.

Let us limit our choice of the 64 array-components at any point by the twenty-four independent equations

$$\nabla_{jk}^i = 0. \tag{16}$$

Then Eqs. (4′), (4) and (5) show that Eq. (12) reduces to the classical Riemann-Christoffel tensor, both (15) and (13″) vanish, and (13′) becomes Einstein's symmetrical field tensor. By virtue of (12″) the components a_{ij} become determinate functions of position, Eqs. (3) becoming mere identities. In this case therefore our theory is simply a restatement of Einstein's from a new point of view; the vector lines or particle orbits, becoming Einstein's so-called geodesics.

(b). Negligible gravitation field.

We wish to study the field of a small charged particle, in particular, of an electron. The gravitation field of an electron may be neglected[8] to a first approximation, and if zero external gravitation field is taken then we may take the three index symbols as zero both in the orbit Eqs. (2′) and in the derivatives (11).

Maxwell's first set of laws

$$\partial F_{ij}/\partial x_k + \partial F_{jk}/\partial x_i + \partial F_{ki}/\partial x_j = 0 \tag{17}$$

are identically satisfied if either (14) or (15) are used as the electric field tensor; and his second set

$$J^i = \partial F^{ij}/\partial x_j \tag{17'}$$

[7] For proof (of tensor nature) refer to the paper mentioned in reference (4) above.

[8] See Eddington, reference 1, Arts. 78, 80.

are satisfied to the degree of approximation assumed, if we define

$$J^i = {}^*F^{ij}{}_{,j}. \tag{18}$$

Again we have approximately

$$\partial^2 D_i/\partial x_j \partial x_k = D_{i,jk} = D_{i,kj}$$

so that by (14′) and (18)

$$J_{i,j} = a^{rs}({}^*K_{i,r} - {}^*K_{r,i})_{,js} = a^{rs}({}^*K_{i,jrs} - {}^*K_{r,sij})$$

or

$$J_{i,j} - J_{j,i} = a^{rs}({}^*K_{i,j} - {}^*K_{j,i})_{,rs}$$

giving

$$a^{rs*}F_{ij,rs} = J_{i,j} - J_{j,i} \tag{19}$$

A solution of this[9] is

$${}^*F_{ij} = (1/4\pi)\iiint (1/r)(J_{i,j} - J_{j,i})dv \tag{20}$$

which gives the field at distance r from the volume dv containing the current density J_i. Suppose the current $J_i dv$ is due to a charge e moving with a velocity A_i, then $J_i dv = eA_i$, so that Eq. (20) becomes

$${}^*F_{ij} = (1/4\pi)(e/r)(A_{i,j} - A_{j,i})$$

or, by (1′) and (4′)

$${}^*F_{ij} = 2\nabla_{ij}{}^k A_k e/4\pi r. \tag{21}$$

If the charge e is not considered a point singularity, it must be supposed to be such that each element has the same velocity, and Eq. (21) to have been integrated over the volume dv containing e.

Construct any sphere of radius a completely enclosing the charge e; then it is easy to show that the average of the values of de/r at all points over the surface of the sphere is de/a, so that the average of e/r is also e/a, and the average of the field taken over the surface of the sphere will therefore be

$$F_{ij}' = \frac{e}{4\pi a} 2\nabla_{ij}{}^k A_k. \tag{21′}$$

Suppose now that the external field varies only so slowly that its variation over the sphere of radius a can be neglected, then we may assume that it is possible to choose the sphere so that the external field F_{ij}'' is equal and opposite to the average F_{ij}'; this sphere will be defined as the boundary of the charged particle, or even as the particle itself. Actually the sphere is only an approximation: a more accurate definition would be the surface, assumed to exist, over which the external field just balances the corpuscle's field at every point.

[9] See Eddington, reference 1, Arts. 72, 74.

Granting this assumption,[10] that the particle is bounded by a surface of zero field, the equations of motion of the charge e are given by putting in (2′)

$$a_{rj}dx_j/ds = A_r,\ dx_k/ds = A^k,$$

and

$$2\nabla_{sk}{}^r A_r = (4\pi a/e)F_{sk}{}' = -\ (4\pi a/e)F_{sk}{}''$$

giving, of course with the approximation mentioned before,

$$m = e^2/4\pi a,\ md^2x_i/ds^2 = -\ eA^rF_r{}''^i \tag{22}$$

which, if m be interpreted as the mass of the particle, are the empirical equations of motion of charged particles, in regions where the field $F_{ij}{}''$ is not sensibly varied over the volume occupied by the charge.

V. The Principle of Indeterminacy

Heisenberg's principle of indeterminacy shows that the vector line method cannot give anything more than an approximate description of phenomena, for to do so it requires an exact knowledge of the initial conditions. The degree of approximation can be roughly estimated, however.

Let q, p be a typical pair of coordinates and momentum components actually measured as the initial conditions of the particle, and let the estimated uncertainties be Δq, Δp; definite values satisfying

$$\Delta q \cdot \Delta p = 0(h). \tag{23}$$

Describe a sphere of diameter D and centre q, where

$$D > \Delta q, \text{ and } D - \Delta q = 0(\Delta q). \tag{24}$$

If this sphere is moved so that its centre follows the track found by putting the measured initial conditions in (22), then the particle will certainly remain within the sphere for a finite time T. The greater T or the smaller Δq and D, the more useful the vector line description, and vice versa. We must be careful not to assert that if exact initial conditions could be found, then Eqs. (22) would give an exact description of the motion; for such an assertion can never be experimentally verified. Essentially, therefore the time T must be determined experimentally in every case; it is not possible to deduce from the purely abstract descriptive apparatus any binding limits to its applicability; an appeal to experiment must be made.

Nevertheless the following argument which, for the above reasons, cannot claim to be rigorous, may not be without interest.

In zero external field, (that is zero after the initial measurements have been made) an electron obeying Eq. (22) will move with its initial velocity unchanged. Any initial discrepancy between its measured and its "actual"

[10] Eddington's assumption (reference 1, Art. 80) that the two fields neutralise each other throughout the volume occupied by the charge could be used here just as well, but the one given in the text seems to be a useful alternative.

velocity will be transmitted as it were along its motion unchanged. Hence the time T in which the electron may drift out of the sphere of diameter D mentioned above will be

$$T = (D - \Delta q)/\Delta v$$

or by Eq. (23)

$$= (m/h)(D - \Delta q)(\Delta q) \tag{25}$$

For the electron $(m/h) = 0(1)$ so that by Eq. (24)

$$T = 0(\Delta q^2).$$

Suppose the initial velocity is measured as 10^8 cm/sec; and allow an uncertainty of no more than 10^3 cm/sec. Then by Eq. (23) the uncertainty in the initial position will be of the order of 10^{-3}cm. Thus T is of the order of 10^{-6} sec. and the length of track traversed in that time of the order of 100 cm.

When the field is not zero the initial discrepancy is not transmitted unchanged and the value of T becomes correspondingly uncertain, in general less. When the field varies so rapidly with position that it is sensibly different for different points within the sphere D, the various possible tracks starting from the points of Δq will in general diverge so rapidly that the sphere D will contain them only for a uselessly short time.

This non-rigorous argument thus leads us to suspect that in regions near atomic nuclei, where the space-variation of the field is great, the vector line method becomes useless; a result which is in full agreement with experimental data.

Independent of this argument is the further possibility that in the non-uniform fields near atomic nuclei, the boundary of the electron as defined in §4 can no longer exist. In this case the electron would cease to function as a particle and the process of deriving Eqs. (22) would no longer be possible.