THE EFFECT OF END LOSSES ON THE CHARACTERISTICS OF FILAMENTS OF TUNGSTEN AND OTHER MATERIALS

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Abstract

The leads of a tungsten filament in vacuum cool the ends of the filament and so affect the voltage, candle power, electron emission and other properties of the filament. For long filaments, where there is a central portion at a uniform temperature T_m, the temperature distribution near the lead is derived. A method for determining T_0, the temperature of the lead-filament junction, is given. Tables and formulas are presented which allow ready calculation of the effect of the leads on the properties of any long tungsten filament for which the current and diameter are known. From the more general results it has been found that the decrease in voltage due to the cooling of one lead may be represented by $\Delta V = 0.154\ (T_m/1000) - 0.081\ (T_o/1000) - 2.1 \cdot 10^{-8}\ T_o T_m - 0.056$. There is an extension of the theory to cover the cases of filaments in gases, filaments of other materials, etc.

Part II of the paper gives figures from which may be found the properties of filaments so short that the first theory does not apply. Some experimental checks of the theory are given.

In general the results and the methods of application have been placed first, and the mathematical derivations have been placed at the end of each part.

For a short filament with leads cooled in liquid air a negative slope of the volt-ampere characteristic when the central temperature is much smaller than T_m is observed.

Part I. The Long Filament

The extensive use of tungsten filaments in research and industry makes it important to consider how the cooling effects of the leads influence the characteristics of such filaments. For wide ranges of temperature the characteristics of hypothetical filaments which are not cooled by leads may be found from tables of the properties of tungsten[1–7] The magnitudes of the lead losses have been evaluated experimentally[1, 8, 9] and by theoretical methods.[10] It is felt that there is still a place for a systematic treatment

[1] W. E. Forsythe and A. G. Worthing, Astrophys J. **61,** 146 (1925).
[2] C. Zwikker, Royal Acad. Amsterdam **34,** No. 5 (1925).
[3] H. A. Jones and I. Langmuir, G. E. Review **30,** 310, 354, 408 (1927).
[4] H. A. Jones, Phys. Rev. **28,** 202 (1926).
[5] I. Langmuir, Phys. Rev. **7,** 154 (1916).
[6] I. Langmuir, Phys. Rev. **7,** 302 (1916).
[7] I. Langmuir, G. E. Review **19,** 208 (1916).
[8] A. G. Worthing, Journ. Frank. Inst. **194,** 597 (1922).
[9] T. H. Amrine, Trans. Ill. Eng. Soc. **8,** 385 (1913).
[10] A. G. Worthing, Phys. Rev. **4,** 524 (1914); R. Ribaud and S. Nikitine, Ann. de Physique **7,** 5 (1927); V. Bush and K. E. Gould, Phys. Rev. **29,** 337 (1927).

which may be conveniently applied to a filament operated under any ordinary conditions.[11]

Temperature distribution. For most tungsten filaments the cooling effect of the leads does not extend appreciably to the central portion of the filament. The absolute temperature, T_m, of this portion may be calculated from the diameter of the filament and the current through it.[12] The temperature of other parts of the filament is best expressed as a fraction, θ, of T_m. Thus $\theta = T/T_m$, where T represents the absolute temperature of any point of the filament.

We may consider the effect of each lead independently, since the two effects do not overlap. Of fundamental importance is the variation of θ with x, the distance from the lead. This is shown later to be governed by the equation,

$$\phi = a d\theta/dx \tag{1}$$

where ϕ is a function of θ, and is given by Eq. (30). a is a parameter of the dimension of length, and depends on T_m and the filament diameter D.

TABLE I. *Values of a_0 for various values of T_m. (For 'a' use Eq. (2).)*

T_m(°K)	a_0(cm)	T_m	a_0	T_m	a_0
600	5.84	1700	0.646	2800	0.275
700	4.08	1800	.582	2900	.261
800	3.01	1900	.527	3000	.247
900	2.33	2000	.481	3100	.235
1000	1.863	2100	.441	3200	.223
1100	1.524	2200	.406	3300	.213
1200	1.274	2300	.377	3400	.209
1300	1.084	2400	.351	3500	.195
1400	.936	2500	.329	3600	.187
1500	.821	2600	.309	3655	.183
1600	.724	2700	.291		

a_0, the value of a for $D = 0.01$ cm (4 mil approx.) is given in Table I. a for other values of D may be found from

$$a = a_0(D/0.01)^{1/2}. \tag{2}$$

Integration of (1) gives

$$(x/a)_0^\theta = \int_0^\theta d\theta/\phi. \tag{3}$$

The values of $(x/a)_0^\theta$ for various values of θ are tabulated in the second column of Table II. Note that a is effectively a unit of length. The symbol $(x/a)_{\theta_1}^{\theta_2}$ will in general represent the distance, expressed in a-units, from a point at temperature $\theta_1 T_m$, to a point at temperature $\theta_2 T_m$. To obtain dis-

[11] I. Langmuir, Trans. Faraday Soc. **17**, 634 (1922). See also references (7) p. 210, (6) p. 312, (3) p. 356. In these papers formulas for ΔV and ΔV_H are given. They were derived by methods similar to the ones used in this paper.

[12] Ref. (3), p. 312, Table I, column 4.

tance in a-units, divide the distance x in cm by the value of a, in cm, found from Table I.

TABLE II. *Values of* $(x/a)_0^\theta$ *and* $\beta(\theta_0)$.
For $n>4$: $\beta(\theta_0)=(x/a)_0^{\theta_0}$. For $n=1.2$, see columns 3 and 6.

θ	$(x/a)_0^\theta$	$\beta(\theta_0)$ $n=1.2$	θ	$(x/a)_0^\theta$	$\beta(\theta_0)$ $n=1.2$
0.0	0.000	0.000	0.7	0.7394	0.464
.1	.0419	.040	.8	.9766	.532
.2	.1110	.102	.85	1.1354	
.25	.1522	.137	.9	1.3592	.598
.3	.1974	.172	.95	1.7260	.630
.4	.2999	.245	.99	2.5535	.654
.5	.4200	.320	.999	3.7224	.659
.6	.5628	.392	1.000		.660

The temperature distribution near a cooling lead is represented by the curve farthest to the right (labeled 0.995) in Fig. 2 (Part II). The increment in the abscissa $(x/a)_{\theta}^{\theta c}$ from the ordinate θ_2 to ordinate θ_1 gives the distance along the filament from a point at temperature $\theta_1 T_m$ to a point at temperature $\theta_2 T_m$.

In practice the junction of lead and filament will be at a temperature $T_0=\theta_0 T_m$. The exact determination of this temperature is rather difficult. If the leads are short and fairly heavy we may assume $T_0=T_R$, where T_R is the room temperature. The error due to this assumption in the value of any filament-property for the whole filament[13] computed by the methods of this paper will be less than 1 percent when the length of the lead is less than a certain maximum length l_0. We find that l_0 is given approximately by

$$l_0 = 0.32(x/a)(D_L/0.1)^2(\lambda_L/0.586)/A. \tag{4a}$$

A is the filament current in amperes and (x/a) represents the half length of the filament. D_L is the lead diameter in cm. λ_L is the thermal conductivity of the lead in watts cm^{-1} deg^{-1}. For nickel leads $\lambda_L/0.586=1$, for tungsten leads $\lambda_L/0.586=2.73$, for molybdenum $\lambda_L/0.586=2.49$.

Thus with nickel leads for which $D_L=0.1$ cm and $l<1.6$ cm, used with a 20 cm filament of $D=0.02$ cm for which the highest operating temperature is $T_m=2400°$ we may assume $T_0=T_R$, since we find $A=4.02$,[12] $x/a=20.2$, and thence from Eq. (4a) $l_0=1.6$ cm.

For leads such as those used in incandescent lamps it is sufficiently accurate for many purposes to assume that $T_0=(1/4)T_m$.

If desired we may evaluate $\Delta T=T_0-T_R$ in terms of the lead length l. We find that

$$\Delta T = l \cdot A\,(0.586/\lambda_L)(0.1/D_L)^2\,(\Delta T_0). \tag{4b}$$

$(\Delta T)_0$ is the value of ΔT for a nickel lead for which $D_L=0.1$ cm, $l=1$ cm and $A=1$ amp. It is given in Table III as a function of the value of T_m for the filament. Note that these values are for a filament of constant current,

[13] Such as the voltage or the candle power of the whole filament.

and hence that the diameter of a filament for which $\Delta T = (\Delta T)_0$ is smaller for the higher temperatures in the table.

The data from Table III are not dependable above $T_0 = 1000°$, where radiation loss is appreciable, nor when the resistance loss in the lead is large.

TABLE III. $(\Delta T)_0 = T_0 - T_R$ *for nickel leads when* $A=1$, $D_L = 0.1$, $l = 1$.

T_m(°K)	$(\Delta T)_0$	T_m	$(\Delta T)_0$	T_m	$(\Delta T)_0$
1000°	21°	1800°	45°	2800°	80°
1200	26	2000	51	3000	87
1400	32	2200	58	3200	95
1600	38	2400	65	3400	102
		2600	72		

The actual temperature distribution of the filament is given by (cf. Eq. (3))

$$(x/a)_{\theta_0}^{\theta} = \int_0^{\theta} d\theta/\phi - \int_0^{\theta_0} d\theta/\phi. \tag{5}$$

To find the distance from the lead, x, of a point of known temperature θ, we may evaluate the two integrals above by means of Table II, and then obtain x from $(x/a)_{\theta_0}^{\theta}$ by using Table I. Conversely we may find the temperature θ of a point at any given distance, x.

By the use of data on the characteristics of tungsten filaments as functions of temperature,[1, 3] and from the temperature distribution along a filament as found above, the properties of the filament at each point can be calculated. For example, we can determine the electron emission, the radiated energy, the luminous intensity, et cetera, at each point.

The effect of lead losses on filament characteristics. Many filament properties which are functions of the temperature would be strictly proportional to the length of the filament if the temperature were everywhere uniform. Let h be a quantity which measures some one of these properties *per unit length* at any given absolute temperature T. For example, h may represent the voltage drop per cm or the electron emission per cm of length. Let h_m be the value of h at the temperature T_m. Nearly all the properties of tungsten which we shall need to consider vary in proportion to some definite power of the temperature over rather wide ranges. Thus we may put

$$h = h_m \theta^n \tag{6}$$

$$n = (dh/h)(T/dT) \tag{7}$$

where n is approximately constant.[14]

If a filament of length $2x$ were all at its maximum temperature T_m the value H_m of any property for the whole filament would be

$$H_m = 2xh_m. \tag{8}$$

[14] For values see Ref. 3, p. 354 Table II or Ref. 1, p. 153 Table I-*B*.

The cooling effect of the leads makes H, the actual value of the property for the whole filament, less than H_m. ΔH, the amount of this decrease due to one lead, may be thought of as the total value of the property over a short length of uncooled filament. Designate by ΔV_H the voltage drop across this length. ΔV_H is then the volt-equivalent of ΔH, and the fractional decrease of H_m can thus be expressed as a fraction of the total voltage V_m

$$\Delta V_H/V_m = \Delta H/H_m. \tag{9}$$

The ratio H/H_m is a measure of the extent to which the cooling effect changes the property. We have

$$H/H_m = (H_m - 2\Delta H)/H_m = (V_m - 2\Delta V_H)/V_m. \tag{10}$$

The factor 2 accounts for two leads.

TABLE IV. *Values of* $B_1 = \int_0^1 (1-\theta^n)d\theta/\phi$.

n	B_1	n	B_1	n	B_1
1	0.583	9.0	1.626	20	2.032
1.2	0.660	10	1.682	22	2.079
2.0	0.882	11	1.728	24	2.124
3.0	1.076	12	1.772	25	2.145
4.0	1.217	13	1.813	26	2.165
5.0	1.329	14	1.850	28	2.203
5.1	1.339	15	1.885	30	2.238
6.0	1.421	16	1.918	35	2.315
7.0	1.500	17	1.949	40	2.384
8.0	1.566	18	1.978	50	2.497
		19	2.006	60	2.589

If V is the actual voltage drop (in volts) and ΔV the value which ΔH has when the property measured by H is voltage

$$H/H_m = (V + 2\Delta V - 2\Delta V_H)/(V + 2\Delta V). \tag{11}$$

It will be shown later that the value of ΔV_H is given by

$$\Delta V_H = 1.812 \cdot 10^{-5} T_m^{1.3}[B_1 - \beta(\theta_0)]. \tag{12}$$

B_1 is given in Table IV. It is a function of n, the temperature exponent for the property H in question. For $n > 5$, B_1 is given by the equation

$$B_1 = 0.5170 + 1.1660 \log_{10} n - 0.0591/n + 0.2224/n^2 + 0.0140/n^3 - 0.468/n^4 + \cdots. \tag{13}$$

The coefficient of Eq. (12), $1.812 \cdot 10^{-5} T_m^{1.3}$, is given in Table V.

$\beta(\theta_0)$ in Eq. (12) is a function of θ_0. It is independent of n if $n > 4$ and $\theta_0 < 0.5$, and is given by

$$n > 4, \theta_0 < 0.5 \quad \beta(\theta_0) = (x/a)_{0^\circ}^{\theta}. \tag{14}$$

This value of x/a is to be taken directly from Table II for $\theta = \theta_0$. For $n = 1.2$ (the exponent for resistance and the only important small value of n) $\beta(\theta_0)$ is given in the third column of Table II.

TABLE V. $1.812 \cdot 10^{-5} T_m^{1.3}$

T_m	$1.812 \cdot 10^{-5} T_m^{1.3}$	T_m	$1.812 \cdot 10^{-5} T_m^{1.3}$
1000°	0.1439	2300°	0.4250
1100	.1629	2400	.4493
1200	.1825	2500	.4737
1300	.2024	2600	.4985
1400	.2229	2700	.5236
1500	.2438	2800	.5490
1600	.2651	2900	.5744
1700	.2869	3000	.6003
1800	.3091	3100	.6266
1900	.3316	3200	.6528
2000	.3544	3300	.6797
2100	.3777	3400	.7066
2200	.4011	3500	.7337

Since the current is constant, the voltage has the same temperature exponent as the resistance. Hence ΔV may be found from Eq. (12) by setting $n=1.2$. In many cases to apply Eq. (11) it may be easier to find the theoretical voltage $V_m = V + 2\Delta V$ directly from the resistivity at T_m and the filament dimensions. The wattage *input* depends on the resistance of the filament and hence ΔV_H in this case is to be found for $n=1.2$. The wattage *radiated* on the other hand depends on $n=5.1$ or thereabouts.[15]

Method of application. In finding the value of B_1 for Eq. (12) from the value of n, we notice that for most properties of tungsten n is not constant as was assumed, but varies slightly with the temperature. We must take a mean value of n, that is, its value at some effective temperature T_E. This temperature is roughly that at which $h = h_m/2$. This temperature and the corresponding value of n may be found directly.[14]

As an alternate method to find T_E we note that for some properties h may be quite accurately expressed as

$$h = CT^k e^{-b/T}. \tag{15}$$

Thus for candle power Wiens' law (using a Crova wave-length) gives $k=0$, $b=25200°$. The Richardson-Dushman equation for the electron emission from pure tungsten has $k=2$, $b=52600°$. The rate of evaporation of a tungsten filament is expressed by Eq. (15) with $k=0$, $b=94100°$. Setting $h=h_m/2$ in Eq. (15) and using Eq. (6) to evaluate the term in k, we find approximately

$$b/T_E = b/T_m + [1 - k/n] \log_e 2. \tag{16}$$

Differentiating Eq. (15) and comparing with Eq. (7) we see that

$$n = k + b/T. \tag{17}$$

Hence from Eq. (16) and the values of the constants given above we obtain the following equations for effective values of n in terms of T_m

$$\text{candle power } n = 25200°/T_m + 0.7 \tag{18}$$

[15] Ref. 3, p. 312, Table I, column 5.

$$\text{electron emission } n = 52600°/T_m + 2.6 \tag{19}$$

$$\text{evaporation } n = 94100°/T_m + 0.7. \tag{20}$$

The decrease in the rate of evaporation near the leads is ordinarily not a matter of experimental interest, but under certain conditions its effects may be directly observed. Nitrogen or carbon monoxide in the presence of a tungsten filament at very high temperatures gradually disappears because every atom of tungsten which evaporates combines with a molecule of the gas to form a stable and non-volatile compound. Thus the rate of "clean-up" of the gas depends on the total amount of metal that evaporates.

The direct application of Eq. (12) as outlined above is the most accurate method for the evaluation of ΔV_H. In many cases where only approximate results are desired ΔV_H may be found from the following empirical equations, which were found to fit the data calculated from Eq. (12). The deviation from the results of Eq. (12) is less than the amount tabulated for the given range. The actual error of the results may in some cases be larger than this, due to approximations in the derivation of Eq. (12).

TABLE VI. $\Delta V_H = P(T_m/1000) - Q(T_0/1000) - R$ *volts.*

H is	P	Q	R	Range— T_m	Range— T_0	Range— θ_0	Max. error (volts)
Voltage*	0.154*	0.081*	0.056*	1000–2500°	any values	0.1–0.5	0.004
Candle Power	.338	.182	−.004	600–3500	300–1400°	.1– .5	.01
Electron emission	.440	.158	.072	1000–3500	300– 900	.07–.5	.009
Evaporation	.480	.160	.060	1500–3500	300– 900	.07–.5	.008
Watts radiated	.293	.160	.084	1100–3000	300– 900	.1– .4	.01

* For voltage (Watts input) a term $-2.1\cdot10^{-8}T_0T_m$ is to be added to the right hand side of Eq. (21).

In many cases with short, heavy leads $T_0 = 300°$. In these circumstances the following approximate equations hold.

TABLE VII. $\Delta V_H = P_0(T_m/1000) - S$ *volts* (22

H is	P_0	S	Range—T_m	Max. error (volts)
Voltage	0.148	0.080	1000–2500	0.004
Candle Power	.338	.051	600–3500	.01
Electron Emission	.439	.119	1000–3500	.007
Evaporation	.477	.103	1500–3500	.001
Watts radiated	.287	.121	1000–3100	.01

Computation of T_m. If we know the diameter of a filament and the current through it, T_m may be obtained directly from Tables which give temperature tabulated against current divided by $d^{3/2}$.[12]

If the diameter is not known, but if the length $2x$ is known, the voltage V and amperage A corresponding to the temperature we wish may be found. Assuming that the filament is all at the maximum temperature T_m, we compute $VA^{1/3}/(2x)$ and find a first approximation for T_m.[16] For

[16] Ref. 3, p. 312, Table I, column 6 gives $VA^{1/3}/(2x)$ as a function of T.

this value of T_m there is a certain voltage correction ΔV. This gives us a much better value, $V+2\Delta V$, for the voltage if the filament were all at T_m. $(V+2\Delta V)\ A^{1/3}/(2x)$ then gives us a second approximation for T_m. As many approximations as desired may be made.

Shorter filaments. With shorter filaments the cooling effects of the two leads overlap, and the temperature at any point may be found approximately by adding the cooling effect of each lead at that point. With still shorter filaments these temperatures and the values of H/H_m found as above are in error. The amount of the error depends on n, the temperature exponent of the property in question. Part II, Table X gives in column 2 the maximum value of the half length $(x/a)_0^{\theta_c}$ for which the error in H/H_m is less than 1 percent. Column 3 gives similar information for 5 percent error. For details see Part II.

Derivation of the Equations

The fundamental differential equation giving the temperature distribution near a cooling lead is[8]

$$A^2r+[\lambda(d^2T/dx^2)+(d\lambda/dT)(dT/dx)^2]\pi D^2/4=w. \quad (23)$$

The symbolism is explained in Table VIII. The terms A^2r and w correspond respectively to the rate of production of energy and the rate of radiation

Table VIII. *Symbols.*

A	Filament current in amps	w	$=h$ for power radiated
D	Filament diameter (cm)	r	$=h$ for resistance
D_L	Lead diameter (cm)	v	$=h$ for voltage drop
l	Length of lead (cm)	H	Value of any property for the whole filament
x	Distance along filament (cm)		
a	Unit of length (cm) Table I	H_m	Value of H if the whole filament were at T_m
a_0	a for $D=0.01$ cm		
sub m	Value at the uncooled central portion of the filament	V	$=H$ for voltage drop
		H_c	Value of H if the whole filament were at T_c
sub c	Value at the center of the filament (Part II)	n	Temperature exponent for any property $=d \log h/d \log T$
T	Absolute temperature		
T_R	Room temperature	ρ	$=n$ for resistance
$T_0=\theta_0 T_m$	Lead-filament junction temperature	ω	$=n$ for radiation
ΔT	$=T_0-T_R$	k	$=n$ for thermal conductivity
θ	$=T/T_m$	ϕ	$=ad\theta/dx$
$(x/a)_{\theta_1}^{\theta_2}$	Distance in a-units from point at θ_1 to point at θ_2	ΔV_H	see Eq. (12)
		B_1	Table IV
λ	Thermal conductivity of filament	B_{15}	Value of B_1 for $\omega=5$
λ_L	Thermal conductivity of lead	$\beta(\theta_0)$	Table II
h	Value of any property per cm of filament length	τ_0	$=T_0/T_c$

of energy per unit length of filament, while the expression involving λ corresponds to the net rate of conduction of energy into an element of the filament.

An inspection of tables giving the characteristics of tungsten filaments as functions of the temperature shows that it is possible to express Eq. (23) in a much simpler form.[3]

The resistance of a tungsten filament can be expressed quite accurately over a wide range of temperatures by the equation $r=cT^{\rho}$, where c is a constant and $\rho=1.20$ (for the range between 600°–3,000°K). Similarly the radiated power may be expressed approximately by the relation $w=c'T^{\omega}$, where ω is fairly constant, having the values, 5.65 at 1,000°K, 5.12 at 1700°, 4.93 at 2,000°, 4.71 at 2,400° and 4.48 at 3,000°. In the majority of experiments in which it is desired to calculate the cooling effect of the leads the temperature of the hottest part of the filament will probably be below 2,400°. By averaging the values of this exponent from 2,000° to 400°, weighing each in proportion to the corresponding value of w, the effective exponent is found to be 5.1. We shall, therefore, take this to be the value of ω. Even at very high filament temperatures, where the effective exponent would be about 4.7, we shall see that the error made by using $\omega=5.1$ is practically negligible.

The heat conductivity of tungsten at temperatures from 1,300 to 2,500° has been given by Forsythe and Worthing.[1] It ranges from 0.93 watts $\text{cm}^{-1}\,\text{deg}^{-1}$ at 1,300° to 1.21 at 2,500°. We find that the empirical equation

$$\lambda=0.840(T/1000)^{0.4} \qquad (24)$$

expresses the values of λ at the 13 observed points given in their table (at 100° intervals) within an error of 0.0022 or about 0.2 percent. This equation is used throughout this paper.

In the central uncooled portion w_m, the power radiated per unit length, is equal to A^2r_m, where r_m is the resistance per unit length at this place. Since A is constant throughout the length of the filament, the temperature exponent of A^2r is the same as that of r, that is ρ. Hence we can replace the first term in Eq. (23) by $w_m\theta^{\rho}$. Similarly w may be replaced by $w_m\theta^{\omega}$. From Eq. (24) we obtain the relation $\lambda=\lambda_m\theta^{0.4}$, where λ_m is the thermal conductivity at temperature T_m. Using the values of λ and $d\lambda/dT$ from this relation, we obtain from Eq. (23)

$$d^2\theta/dx^2+0.4(d\theta/dx)^2/\theta=(\theta^{\omega-0.4}-\theta^{\rho-0.4})/a^2 \qquad (25)$$

where a is a parameter defined by

$$a^2 = \pi D^2\lambda_m T_m/4w_m. \qquad (26)$$

We can replace w_m by its value v_m^2/r_m where v_m is the voltage drop per cm at temperature T_m. The factor r_mD^2 which then occurs in the equation is independent of D and varies as $T_m^{1.2}$ (it is in fact the function R' given by Jones and Langmuir).[3] Thus

$$r_mD^2=7.89\cdot 10^{-9}T_m^{1.2}. \qquad (27)$$

From Eqs. (24), (26), (27)

$$a=1.812\cdot 10^{-5}T_m^{1.3}/v_m. \qquad (28)$$

In computing a from this equation v_m was found from Tables of characteristics.[13] Since $v_m \propto D^{-1/2}$, we obtain Eq. (2).

In Eq. (25) set $\omega=5.1$, $\rho=1.2$ and substitute ϕ as defined by Eq. (1) for x

$$\phi d\phi/d\theta+0.4\phi^2/\theta=\theta^{4.7}-\theta^{0.8}. \tag{29}$$

By taking as new variables ϕ^2 and $\log\theta$ this equation becomes linear and may be solved in the usual way to give

$$\phi^2=(3-5\theta^{2.6}+2\theta^{6.5})/6.5\theta^{0.8}. \tag{30}$$

The constant of integration, 3, is fixed by the condition that $d\theta/dx=0$, and therefore $\phi=0$ at $\theta=1$, the center of the filament.

For values of θ close to unity (30) may be expanded in terms of $Z=1-\theta$

$$\phi^2=3.9Z^2-4.81Z^3+3.715Z^4-0.8886Z^5+0.4068Z^6+0.27Z^7+\cdots. \tag{31}$$

Temperature distribution. When $\theta\leqq 0.6$ the integral in Eq. (2) may be found by expanding $1/\phi$ in the series

$$1/\phi=1.472\theta^{0.4}[1+(5/6)\theta^{2.6}+(25/24)\theta^{5.2} -(1/3)\theta^{6.5}+1.447\theta^{7.8}-(5/6)\theta^{9.1}+\cdots]. \tag{32}$$

Integration gives

$$(x/a)_0^\theta=1.0514\theta^{1.4}+0.3067\theta^4+0.2323\theta^{6.6} -0.0621\theta^{7.9}+0.231\theta^{9.2}-0.117\theta^{10.5}+\cdots. \tag{33}$$

When $\theta\geqq 0.6$ Eq. (31) gives

$$1/\phi=0.5064[1+0.6167Z+0.0941Z^2-0.1809Z^3 -0.227Z^4-0.172Z^5+\cdots]/Z. \tag{34}$$

Integrating we obtain the indefinite integral, and find the integration constant by comparison with Eq. (33) at $\theta=0.6$, where both series are sufficiently convergent to give results accurate to 1 part in 1,000

$$(x/a)_0^{1-z}=0.2247-1.1660\log_{10}Z-0.3123Z-0.0238Z^2 +0.0305Z^3+0.0287Z^4+0.0174Z^5+\cdots. \tag{35}$$

Column 2, Table II was calculated from Eqs. (33) and (35).

Lead-filament junction temperature. The rate of flow of heat, Q, in watts, past any point of the filament is equal to the integral, from that part to the center, of the difference between the heat generated by resistance and that lost by radiation.

$$Q=w_m a\int_\theta^1(\theta^{1.2}-\theta^{5.1})d\theta/\phi. \tag{36}$$

Setting $w_m=Av_m$, substituting for a from Eq. (28), and integrating

$$Q=(1.812\cdot 10^{-5}T_m{}^{1.3})6.5^{-0.5}(3-5\theta^{2.6}+2\theta^{6.5})^{0.5}A \tag{37}$$

where θ corresponds to the temperature of the point in question.

For short fairly heavy leads at temperatures below 1,000°K the resistance loss and radiation loss in the lead are negligible. All the heat that flows into the lead from the filament must flow out at the cold end, which may be assumed to be at room temperature. Taking the heat conductivity of nickel leads as constant at 0.586 watts cm^{-1} deg^{-1}, the leads have a constant temperature gradient given by

$$\lambda_L(dT/dl)\pi D_L^2/4 = Q. \tag{38}$$

Consequently for the total temperature difference $\Delta T = T_0 - T_R$ for the leads we obtain

$$\Delta T = 4lQ/(\pi D_L^2 \lambda_L). \tag{39}$$

Eqs. (37) and (39) give Eq. (4b) for ΔT in terms of $(\Delta T)_0$. To find the latter we notice that the lead temperature θ_0 has little effect on Q. Setting $\theta_0 = 0.24$, $A=1, l=1$, $D_L = 0.1$, $\lambda_L = 0.586$, we obtain from Eqs. (37) and (39)

$$(\Delta T)_0 = 145[1.812 \cdot 10^{-5} T_m^{1.3}]. \tag{40}$$

Eq. (40) was used in conjunction with Table V to compute Table III.

Filament characteristics. We have

$$2\Delta H = H_m - H = 2a \int_{\theta_0}^{1} (h_m - h) d\theta/\phi. \tag{41}$$

Applying Eqs. (6) and (28) and substituting for h_m/v_m the equal ratio H_m/V_m, where V_m is the voltage drop that would exist between the ends of the filament if it were all at the temperature T_m

$$\Delta V_H = \Delta H(V_m/H_m) = 1.812 \cdot 10^{-5} T_m^{1.3} \int_{\theta_0}^{1} (1-\theta^n) d\theta/\phi. \tag{42}$$

ΔV_H is a convenient symbol for the expression $\Delta H(V_m/H_m)$. It represents the voltage across a section of uncooled filament of such length that H_m for this section would equal the decrease caused by the cooling effect of the lead. The advantage of this nomenclature is that it requires no knowledge of filament diameter or length.

By breaking up the integral of Eq. (42) into

$$B_1 = \int_0^1 (1-\theta^n) d\theta/\phi \tag{43}$$

and

$$\beta(\theta_0) = \int_0^{\theta_0} (1-\theta^n) d\theta/\phi \tag{44}$$

we obtain Eq. (12).

In evaluating B_1, we meet the difficulty that we must use two different series for $1/\phi$, one for small and the other for large values of θ. Let $1/\phi_1$

be the value of $1/\phi$ given by the first six terms only of the series in Eq. (32) and $1/\phi_2$ by the first six terms only of the series in Eq.(34). We have

$$
\begin{array}{ll}
0 \leqq \theta \leqq 0.5 & \phi = \phi_1 \\
0.5 < \theta < 0.65 & \phi_1 = \phi = \phi_2 \\
0.65 \leqq \theta \leqq 1 & \phi = \phi_2 .
\end{array}
$$

Since the series are equivalent between 0.5 and 0.65 we may put

$$B_1 = \int_0^t (1-\theta^n)d\theta/\phi_1 + \int_t^1 (1-\theta^n)d\theta/\phi_2$$

where $0.5 < t < 0.65$. A simple transformation gives

$$B_1 = \int_0^t (1-\theta^n)(1/\phi_1 - 1/\phi_2)d\theta + \int_0^1 (1-\theta^n)d\theta/\phi_2 . \tag{45}$$

The value of $1/\phi_1 - 1/\phi_2$ is practically zero for values of θ between 0.5 and 0.65, and therefore the value of the first integral is not dependent on the actual value of t. Even at $\theta = 0.1$ the value of $1/\phi_1 - 1/\phi_2$ is only 0.117, but at $\theta = 0$ it becomes 0.573. Thus the larger part of the first integral is for values of $\theta < 0.1$. In this range θ^n may be neglected in comparison with unity for all large values of n. Even if $n = 1$ the error in neglecting θ^n will be small.

Calling the first integral of Eq. (45) F_1, we have

$$
\begin{aligned}
F_1 &= \int_0^t d\theta/\phi_1 - \int_0^t d\theta/\phi_2 \\
&= (x/a)_0^t - \int_0^t d\theta/\phi_2 .
\end{aligned} \tag{46}
$$

Designate the right hand side of Eq. (35) by F_2. From the derivation of this equation

$$F_2 + C = \int d\theta/\phi_2$$

C being a constant of integration. Hence

$$\int_0^t d\theta/\phi_2 = [F_2 + C]_{\theta=0}^{\theta=t} = [F_2 + C]_{z=1}^{z=1-t} = F_2(1-t) - F_2(1).$$

But from Eq. (35) $F_2\,(1-t) = (x/a)_0^t$. Hence from Eq. (46)

$$F_1 = F_2(1) = -0.0348$$

the numerical value being found by setting $z = 1$ in the expression for F_2 in Eq. (35). Due to the definition of ϕ_2 this value is not affected by the missing terms of the series.

Putting the value of $1/\phi_2$ from Eq. (34) in the second integral of Eq. (45), we find

$$B_1 = -0.0348 + 0.5064 \sum_{p=1}^{6} A_p \int_0^1 (1-\theta^n)(1-\theta)^{p-2} d\theta$$

where A_1, A_2, A_3 are the coefficients, 1, 0.6167, etc., in the series in Eq. (34). This reduces to

$$B_1 = 0.2247 + 0.5064 \int_0^1 (1-\theta^n) d\theta/(1-\theta) - 0.3123/(n+1)$$
$$-0.0477[(n+1)(n+2)]^{-1} + 0.1833[(n+1)\cdots(n+3)]^{-1}\cdots \quad (47)$$

the coefficients of the next two terms being 0.6894 and 2.092.

If n is an integer we have

$$\int_0^1 (1-\theta^n) d\theta/(1-\theta) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + 1/n \quad (48)$$

and for other values of n the integral can be expressed in terms of gamma functions. For $n = 1.2$, its value is 1.1216. For large values of n the integral is given by the series

$$\int_0^1 (1-\theta^n) d\theta/(1-\theta) = 0.5772 + \log_e n + 1/2n$$
$$-[12n(n+1)]^{-1} - [12n(n+1)(n+2)]^{-1}. \quad (49)$$

Inserting this value in Eq. (47) and expressing as a series in reciprocal powers of n we obtain Eq. (13).

In computing Table IV for values of n less than 5, Eq. (47) was used but Eq. (13) was found more convenient for the larger values.

To evaluate $\beta(\theta_0)$ we note that in general $\theta_0 < 0.5$ and hence that for fairly large n the term θ^n is negligible, and Eq. (14) holds. For ΔV, when $n = 1.2$, $\beta(\theta_0)$ for the third column of Table II was obtained by using the series expansions of $1/\phi$ given by Eqs. (32) and (34).

Application under other conditions. For filaments in the presence of gas, or for filaments of materials other than tungsten there will be changes in the values used in Eq. (29) for ω, ρ and k, where k is the temperature exponent of the thermal conductivity. By methods similar to the derivation of Eq. (30) we find that the general expression for ϕ is

$$\phi^2 = 2\theta^{-2k}[(1-\theta^{\rho+k+1})/(\rho+k+1) - (1-\theta^{\omega+k+1})/(\omega+k+1)]. \quad (30a)$$

The integral B_1 in Eq. (12) may be found by numerical integration or by direct integration in some cases. Thus, when $\omega + k + 1 = 2(\rho + k + 1)$, ϕ^2 is a perfect square, and

$$B_1 = (\omega-\rho)^{-1/2}\{\psi[(n+k+1)/(\rho+k+1)] - \psi[(k+1)/(\rho+k+1)]\} \quad (50)$$

where $\psi(x) = d\ln\Gamma x/dx$ is the logarithmic derivative of the gamma function. Table IVa gives some values of B_1 obtained in some of these ways.

TABLE IVa.
$B_1 = \int_0^1 (1-\theta^n) d\theta/\phi$ for various exponents ω, ρ, k, n

ω	ρ	k	n	B_1
5.1	0.0	1.0	5.1	1.695
4.3	1.85	−.4	20.	4.079
6.0	1.0	1.0	1.0	0.371
4.0	1.0	1.0	1.0	0.428
4.0	1.0	1.0	5.0	1.118
4.0	1.0	1.0	10.0	1.486
4.0	1.0	1.0	20.0	1.871
4.0	1.0	1.0	40.0	2.264
3.8	1.2	0.4	27.2	2.522

Data such as those in Table IVa show that the variation of B_1 with ω is small and may be represented thus

For $n=\rho$ $$B_1 = B_{15}[1+0.077(5.0-\omega)] \tag{51a}$$

For $n=20$ $$B_1 = B_{15}[1+0.100(5.0-\omega)] \tag{51b}$$

where B_{15} is the value of B_1 for $\omega=5.0$. B_{15} may be found by these equations from tables of B_1 such as Table IVa. Fig. 1 is a plot of B_{15} as abscissa against ρ as ordinate for constant values of k. The full lines are for voltage correction, $n=\rho$. The dotted lines are for $n=20$.

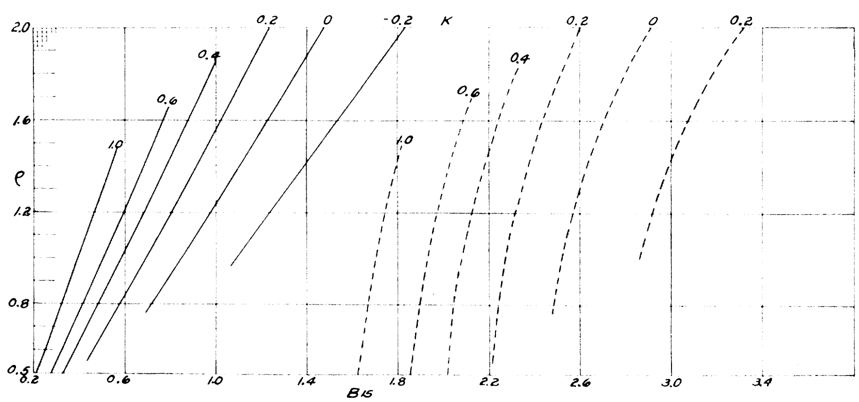

Fig. 1. Plot of B_{15} ($\omega=5$) against ρ for constant k.
Full lines $n=\rho$. Dotted lines $n=20$.

The deviations of B_1 found from Fig. 1 from the value of B_1, for the same n, found in Table IV may be expressed as a fraction, N, of the latter value. The variation of N with n, for constant values of ω, ρ and k may be approximately expressed by

$$N(n) = N(20) - \alpha(20-n)[N(20)-N(\rho)]/(20-\rho) \tag{51c}$$

where

n	ρ	5	10	30	40
α	1	0.42	0.27	0.15	0.11

We may find B_1 for any exponents n, ω, k and ρ by finding $N(20)$ and $N(\rho)$ for the appropriate values of ω, k and ρ from Fig. 1 and Eqs. (51a) and (51b). Eq. (51c) and Table IV then give the desired B_1.

Other metals. From Eq. (26) and the derivation of Eq. (28) we see that

$$a = (\lambda_m T_m R_m / v_m)^{1/2} \tag{28a}$$

where R_m is the resistivity of the metal in question at the temperature T_m. We then have

$$\Delta V_H = (\lambda_m T_m R_m)^{1/2} [B_1 - \beta(\theta_0)]. \tag{12a}$$

The Wiedemann-Franz law states that, at a given temperature, λR, and hence the coefficient in Eq. (12a), is approximately the same for most metals and alloys. This coefficient is thus given by $1.812 \cdot 10^{-5} T_m{}^{1.3}$ in Table V. Consequently the magnitude of ΔV_H may be found for any metal which obeys this law with the help of Fig. 1 and the assumption that the change of $\beta(\theta_0)$ is similar to that of B_1.

End losses from filaments in the presence of gas. Gas around a filament causes a loss of heat by conduction and increases the voltage required to reach a given T_m. Eq (12a) shows that for a fixed T_m, ΔV_H is the same for filaments in gas and vacuum except for the variation in the values of B_1 and $\beta(\theta_0)$. The latter may be evaluated by considering the conduction loss to be part of the radiation loss w in Eq. (23). Thus if at T_m the conduction loss is 1/4 of the radiation loss, and if the conduction loss varies as $T^{1.7}$,[17] the effect on B_1 may be represented as a change in the effective value of ω from 4.6 to 3.8, which by Table IVa means an increase of about 15 percent in the values of B_1 given in Table IV. In general the values of ΔV_H for vacuum hold with fair accuracy for small gas pressures, but the temperature distribution is altered.

For new filament materials or other new conditions there will be no accurate knowledge of the filament characteristics and the temperature exponents for the application of the above method, which method nevertheless will, we hope, still be capable of indicating whether or not lead losses are important in any given case.

PART II. LEAD LOSSES IN SHORT FILAMENTS

Temperature distribution. Shorter filaments do not admit the assumption, made for most of the results of Part I, that the central portion of the filament is not cooled by the leads. Even the calculation of temperature distribution by adding the effects of the two leads is not very good in these cases. Thus when x/a, the filament half length, is 1.53, for which the cooling effect of one lead at the other gives $\theta = 0.994$, the long filament case gives a central temperature corresponding to $\theta = 0.85$, while the true value is $\theta = 0.8$. Theoretically the maximum temperature T_m can only be attained in an infinitely

[17] These conditions are approximately those for a filament of 0.007 cm diameter at $T_m = 2900°$ in 10 mm of N_2. See I. Langmuir and G. M. J. Mackay, J. Amer. Chem. Soc. **36**, 1717 (1914). For larger filaments the conduction losses are relatively smaller.

long filament (cf. Eq. (35)). A "short filament" is one for which this fact invalidates the conclusions of Part I.

Let T_c be the actual temperature at the center of the filament, and let the larger value T_m still indicate the temperature of a hypothetical portion uncooled by leads as calculated from the current and diameter of the filament. Let $\theta_c = T_c/T_m$.

Eq. (1) holds as before with a more complicated value for ϕ. If $(x)_{\theta_0}^{\theta_c}$ is the distance from the center to a lead at temperature $T_0 = \theta_0 T_m$, we have, as before

$$(x/a)_{\theta_0}^{\theta_c} = \int_{\theta_0}^{\theta_c} d\theta/\phi \tag{52}$$

a has the same values as before, and is given in Table I.

Table IX gives the values of $(x/a)_0^{\theta_c}$ for various values of θ_c. Fig. 2 gives the plot of $(x/a)_\theta^{\theta_c}$ as abscissa against θ as ordinate, for constant values of θ_c. The value of θ_c for any curve is of course the intercept on the θ axis.

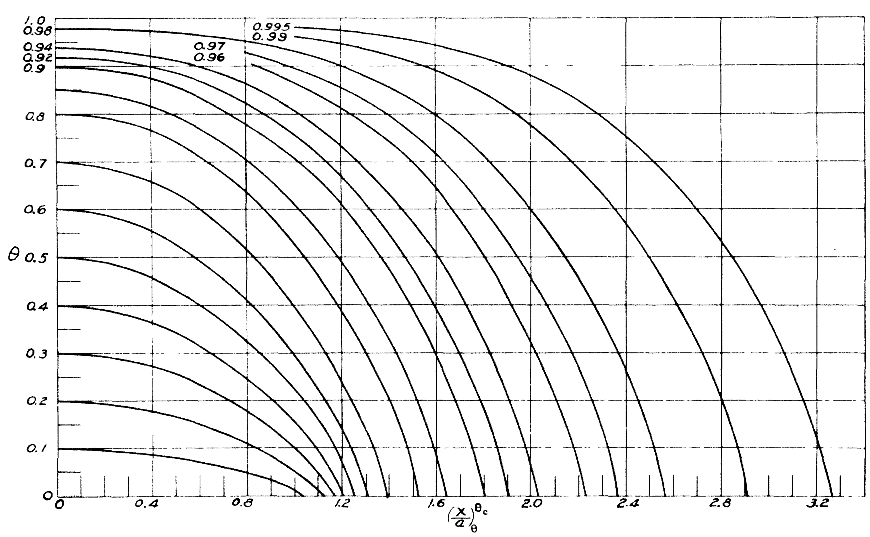

Fig. 2. Plot of $(x/a)_\theta^{\theta_c}$ against θ for constant values of θ_c.

These curves are temperature distribution curves. Thus if $\theta_c = 0.8$ the curve with that intercept gives us the temperature (θ) at any distance, $(x/a)_\theta^{\theta_c}$, from the center. To find for any filament the value of θ_c, we need know only one point on the temperature distribution curve. Thus for a filament of known length and of known lead temperature, a point having the coordinates $(x/a)_{\theta_0}^{\theta_c}$, θ_0 is determined on the plot. This point lies on some temperature distribution (constant θ_c) curve—probably not one of these drawn. The value of θ_c may readily be found, however, by interpolating between the two nearest curves. By continuing this interpolation down to the x/a scale, the value $(x/a)_0^{\theta_c}$ is found. This is the value that would obtain if the given filament were prolonged until its leads were at 0°K, and were left otherwise unchanged. If desired, $(x/a)_0^{\theta_c}$ may be found from θ_c by interpolating in Table IX.

Of course the value of any property h at any point of the filament may be found by first finding the temperature at that point, then applying published data on filament characteristics.[1,3]

TABLE IX. *Relation between the temperature $T_c=\theta_c T_m$ at the center of a short filament with leads at °K and the length (2x) of the filament.*

θ_c	$(x/a)_0^{\theta_c}$	θ_c	$(x/a)_0^{\theta_c}$	θ_c	$(x/a)_0^{\theta_c}$
0.005	0.7325	0.4	1.2065	0.94	2.031
.01	.8262	.5	1.2510	.95	2.118
.03	.9221	.6	1.3077	.96	2.225
.05	.9704	.7	1.3905	.97	2.365
.1	1.0401	.8	1.5280	.98	2.565
.2	1.1154	.85	1.6399	.99	2.912
.3	1.1645	.9	1.802	.995	3.261
		.92	1.899	.999	4.074

The effect of lead losses on characteristics. If all the filament were at the actual maximum temperature, T_c, the value H_c of any property for the whole filament would be

$$H_c = 2h_c x = 2ah_c(x/a)_{\theta_0}^{\theta_c}. \tag{53}$$

h_c is the value of the property in question for 1 cm of filament at temperature T_c. The ratio of the actual value for the whole filament, H, to the hypothetical value is H/H_c. The value of this ratio when $\theta_0=0$ we designate as $(H/H_c)_0$. If $n>4$, H is independent of θ_0 for constant θ_c for all the practical range. Hence, applying Eq. (53)

$$H = 2ah_c(x/a)_0^{\theta_c}(H/H_c)_0. \tag{54}$$

H, and hence H/H_c, is a function of n, the temperature exponent of the property in question. In Fig. 3, ordinates $(H/H_c)_0$ are plotted against abscissae $(x/a)_0^{\theta_c}$ for constant values of n (the full lines). The scale for $(x/a)_0^{\theta_c}$ at the top, and the scale for θ_c on the bottom may be used interchangeably. They correspond as in Table IX.

For a given filament, knowing x, a, and θ_0, $(x/a)_0^{\theta_c}$ is determined from Fig. 2. Fig. 3 then yields $(H/H_c)_0$ for the value of n corresponding to the property in question. H may then be found from Eq. (54).

For $n \leqq 4$, H is not independent of θ_0. For $n=1.2$, the resistance-exponent, the dotted lines at the top of Fig. 3 give the value of H/H_c [*not* $(H/H_c)_0$] for constant values of $\tau_0=\theta_0/\theta_c$. Note however that this is still given in terms of the abscissa $(x/a)_0^{\theta_c}$.

For values of x/a less than 1.0, and hence not in Fig. 2, H/H_c is constant at the value for $x/a=1.0$.

It is to be remarked that for $T_c<1000$°K none of the given data apply accurately, nor is the temperature distribution accurate. This is because of the uncertainty in the value of λ here.

Limits of long filament case. Table X gives the smallest value of $(x/a)_0^{\theta_c}$ for which the errors in H/H_c made in applying the long filament instead of the short filament case are less than the percentage at the head of the column. The errors of course depend somewhat on the value of n for the property under consideration. A (+) after the value of n indicates that the long filament case gives too large a value of H/H_c, or of H; that is, too small a value of B_1. In the neighborhood of $n=5$ the sign of the error changes. For $n=1$ (temperature distribution) the long filament case gives too high temperatures, even when the cooling effects of the two leads are added.

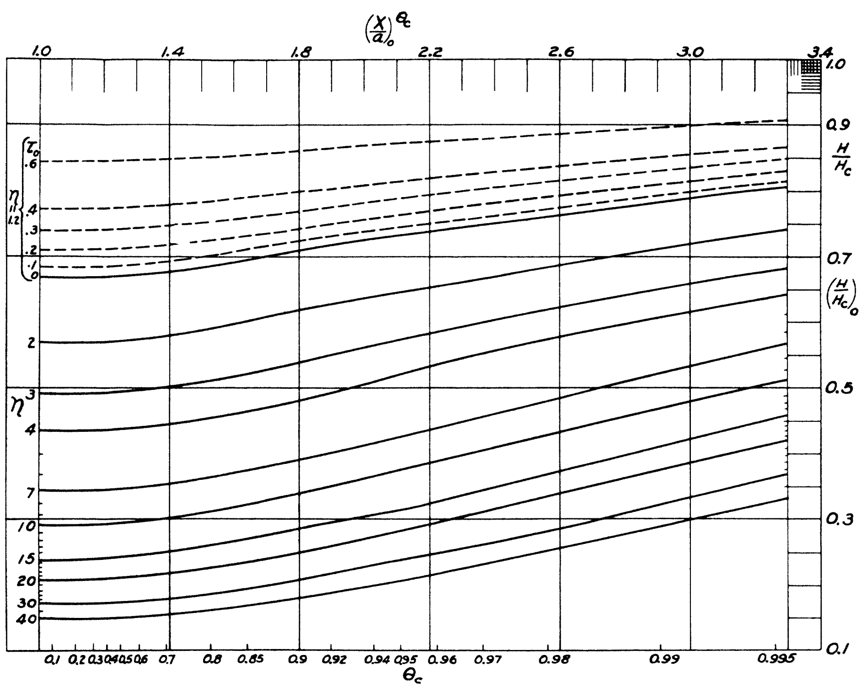

Fig. 3. Full lines $(H/H_c)_o$ for constant n against $(x/a)_0^{\theta_c}$ and θ_c. Dotted lines H/H_c for $n=1.2$ constant τ_o against $(x/a)_0^{\theta_c}$ and θ_c.

Lead temperatures may be found with sufficient accuracy for most cases from Table III of Part I. For high accuracy or for very short filaments Eqs. (36) and (39) may be applied directly, using Eq. (55).

TABLE X. *Errors in H/H_c made in assuming a filament to come under Part I. This able gives approximately the smallest value of $(x/a)_0^{\theta_c}$ for which the error made by that assumption is less than the percentage given at the head of the column.*

n	1 percent	5 percent	n	1 percent	5 percent
1.0(+)	1.9	1.6	10(−)	2.8	2.3
1.2(+)	1.9	1.7	15(−)	2.9	2.6
4(+)	1.9	1.8	20(−)	3.0	2.7
5(±)	1.6	1.6	30(−)	3.2	3.0
7(−)	2.3	2.0	40(−)	3.3	3.0

Computation of temperature distribution. Eq. (30) holds with a new constant of integration determined by the condition $\phi = ad\theta/dx = 0$ when $\theta = \theta_c$ (the maximum temperature):

$$1.3\phi^2 = \theta^{-0.8}(\theta_c^{2.6} - 0.4\theta_c^{6.5} - \theta^{2.6} + 0.4\theta^{6.5}). \tag{55}$$

The integration in Eq. (52) is best carried out by series approximations.
Case I. $\tau = \theta/\theta_c$ is small

$$(x/a)_0^\theta = \theta_c \int_0^t d\tau/\phi. \tag{56}$$

Expressing Eq. (55) in terms of τ, then expanding $1/\phi$ by the binomial theorem and integrating the resulting series term by term

$$(x/a)_0^\theta = 0.8144A^{0.5}\theta_c^{0.1}\tau^{1.4}[1 + 0.175A\tau^{2.6} + 0.0795A^2\tau^{5.2} - 0.0354A\theta_c^{3.9}\tau^{6.5} + \cdots] \tag{57}$$

where $A = 1/(1 - 0.4\,\theta_c^{3.9})$.

In Eq. (57) $\tau^{9.2}$ has been neglected. If $\theta \leqq 0.6\,\theta_c$ the error is less than 1 percent. If in this series we set $\theta_c = 1$, we get the series of Eq. (33) of Part I.
Case II. $\sigma = 1 - \tau = (\theta_c - \theta)/\theta_c$ is small and θ_c is also small.

Eq. (55) may be expanded in terms of σ to give

$$\phi^2 = 2\theta_c^{1.8}\sigma(1-\theta_c^{3.9})(1-\sigma)^{-0.8}[1 - 0.8\sigma(1 - 2.438\theta_c^{3.9}) + 0.16\sigma^2(1 - 24.78\theta_c^{3.9}) + 0.016\sigma^3(1 + 229\theta_c^{3.9}) + \cdots].$$

Terms of the form $(1 - b\theta_c^{3.9})/(1 - \theta_c^{3.9})$ were expanded by the binomial theorem, and $\theta_c^{7.8}$ and higher powers neglected. Expanding $1/\phi$ and integrating

$$(x/a)_\theta^{\theta_c} = \theta_c \int_0^\sigma d\sigma/\phi = \theta_c^{0.1}(1 + \tfrac{1}{2}\theta_c^{3.9})(2\sigma)^{0.5}[1 - 0.33\theta_c^{3.9}\sigma - 0.024\sigma^2(1 - 10\theta_c^{3.9}) - 0.0034\sigma^3(1 + 5.38\theta_c^{3.9})]. \tag{58}$$

This is accurate when σ^4 and $\theta_c^{7.8}$ may be neglected in comparison with unity. At $\theta_c = 0.56$ this error is about 1 percent.
Case III. σ is small and θ_c large.

Let $z = 1 - \theta$ and $z_c = 1 - \theta_c$. z and z_c are both small. Expanding (55) by the binomial theorem

$$1/\phi = 0.5064(1 - 0.4z - 0.12z^2 + \cdots)(z^2 - z_c^2)^{-0.5}[1 + 1.0167C_1 + C_2 + \cdots]. \tag{59}$$

where

$$C_1 = (z^3 - z_c^3)/(z^2 - z_c^2) \doteq z[1 + (1/2)(z_c/z)^2]$$
$$C_2 = 1.5504C_1^2 - 0.930(z^4 - z_c^4)/(z^2 - z_c^2)$$
$$\doteq [0.621 + 1.008(z_c/z)^2]z^2$$

substituting, multiplying Eq. (59) out, collecting like terms and integrating

$$(x/a)_{\theta}^{\theta_c} = \int_{z_c}^{z} dz/\phi = 0.5064(1+0.852z_c^2)\ \text{arc cosh}\ (z/z_c) + 0.3123(z^2-z_c^2)^{0.5}$$
$$+0.2574z_c\ \text{arc cos}\ (z_c/z) + 0.0239z(z^2-z_c^2)^{0.5} \quad (60)$$

This series is good if $\theta_c \geqq -0.7$; $\theta \geqq 0.5$.

Case IV. θ_c is nearly unity.

We may choose z small enough for series (60) to converge, but at the same time much larger than z_c. The small quantity z_c then has little effect on the integral from 0 to $1-z$. To evaluate this integral we may neglect z_c entirely and use Eq. (35) of the long filament case, which, by the choice of the integration constant, gives the value of $(x/a)_0^{\theta}$. z as chosen above is sufficiently small for this series also to converge. Adding Eqs. (35) and (60), expressing arc cosh (z/z_c) as a logarithm and arc cos (z_c/z) as a series, and remembering that z is much larger than z_c, we obtain

$$(x/a)_0^{\theta_c} = 0.5757 - 1.16596 \log_{10} z_c + 0.4043z_c + 1.9z_c^2. \quad (61)$$

The coefficient of the last term was chosen empirically so as to compensate for the missing terms. Note that the result is independent of the specific value of z, as long as it has such a value as to make the derivation valid.

Thus we can obtain $(x/a)_0^{\theta}$ by Case I and $(x/a)_{\theta}^{\theta_c}$ by Case II or III. These two methods overlap in the admissible values of θ except for intermediate values of θ_c, when neither Case II nor Case III is very good. Numerical integration was used for accurate results in this region. The value of $(x/a)_0^{\theta_c}$ may be obtained by Case IV or by the formula

$$(x/a)_0^{\theta_c} = (x/a)_0^{\theta} + (x/a)_{\theta}^{\theta_c}.$$

Table VII and Fig. 2 were obtained by the above methods.

Filament characteristics. Assuming as in Part I that the value of the property in question varies as the nth power of the temperature, we have

$$H = 2h_c \int_0^x (T/T_c)^n dx = 2ah_cJ \quad (62)$$

where

$$J = (1/\theta_c^n) \int_0^{\theta_c} \theta^n d\theta/\phi - (1/\theta_c^n) \int_0^{\theta_0} \theta^n d\theta/\phi. \quad (63)$$

If n > 4, θ^n becomes rapidly smaller with decreasing θ. Hence we may neglect the second integral to obtain

$$J = (1/\theta_c^n) \int_0^{\theta_c} \theta^n d\theta/\phi. \quad (64)$$

From Eqs. (53) and (62)

$$(H/H_c)_0 = J/(x/a)_0^{\theta_c}. \quad (65)$$

There are two limiting cases to be considered.

Case I. θ_c is small.

We may neglect $\theta_c^{6.5}$ in Eq. (55). Introducing $\tau=\theta/\theta_c$ we obtain from Eqs. (63), (53), (62), (52), and (55)

$$H/H_c=\frac{\int_{\tau_0}^{1}\tau^{n+0.4}d\tau(1-\tau^{2.6})^{-0.5}}{\int_{\tau_0}^{1}\tau^{0.4}d\tau(1-\tau^{2.6})^{-0.5}}\cdot$$

The value of θ_c has cancelled out. If $\tau_0=\theta_0/\theta_c$ is held fixed, then H/H_c is independent of θ_c. Hence in Fig. 3 the values for $x/a<1.0$ are the same as those given for $x/a=1.0$.

Case II. θ_c is large.

As θ_c approaches unity we get the transition from the short filament to the long filament case. When $z_c=1-\theta_c$ can be neglected entirely, Eqs. (64) and (43) give

$$(x/a)_0^{\theta_c}-\theta_c^{\,n}J=B_1. \tag{66}$$

From Eq. (62)

$$H=2ah_m(\theta_c^{\,n}J).$$

Thus when (66) is satisfied the value of H increases in direct proportion to the increase of length. This is because the central portion, to which the increased length is added, is at practically constant temperature.

The degree to which Eq. (66) is satisfied is a measure of the approximation involved in assuming that the filament is long. To construct Table VIII the true values of J as found below were compared with the value calculated from Eq. (66). For $n=0$ the short filament temperature distributions were compared with those obtained from the long filament case by adding the cooling effect of the two leads.

Evaluation of J. If θ_c and hence $D=0.4\,\theta_c^{3.9}\,(1-0.4\,\theta_c^{3.9})^{-1}$ is small, we set $y=(\theta/\theta_c)^{2.6}$ in Eq. (64) and obtain, using Eq. (55)

$$J=\theta_c^{0.1}M(1/5.2)^{1/2}(1-0.4\theta_c^{3.9})^{-1/2} \tag{67}$$

where

$$M=\int_0^1 y^p dy(1-y)^{-1/2}[1-Dy(1-y^{3/2})(1-y)^{-1}]^{-1/2} \tag{68}$$

and $n=2.6p+1.2$. M may be expanded as a power series in D

$$M=G_0+(1/2)G_1D+(3/8)G_2D^2+(5/16)G_3D^3+\cdots \tag{69}$$

where

$$G_k=\int_0^1 y^{p+k}(1-y)^{-1/2}[(1-y^{3/2})(1-y)^{-1}]^k dy. \tag{70}$$

Each of these integrals may be expanded in terms of $u=1-y$, and then integrated by means of gamma functions.

$$G_0=\pi(-1/2)\pi(p)/\pi(p+1/2) \tag{71}$$

$$G_1=\frac{3\pi(-1/2)\pi(p+1)}{2\pi(p+3/2)}[1-(1/8)(p+5/2)^{-1}$$
$$-(1/32)(p+5/2)^{-1}(p+7/2)^{-1}+\cdots]$$

$$G_2=\frac{9\pi(-1/2)\pi(p+2)}{4\pi(p+5/2)}[1-(1/4)(p+7/2)^{-1}$$
$$-(1/64)(p+7/2)^{-1}(p+9/2)^{-1}+\cdots]$$

$$G_3=\frac{27\pi(-1/2)\pi(p+3)}{8\pi(p+7/2)}[1-(3/8)(p+9/2)^{-1}$$
$$+(3/64)(p+9/2)^{-1}(p+11/2)^{-1}+\cdots].$$

As θ_c and D become larger the series of Eq. (69) does not converge rapidly. A better series may be obtained by setting

$$M=G_0(1-g_1D-g_2D^2-\cdots)^{-1/2} \tag{72}$$

and determining the value of each g_i in terms of the G_k's by expanding Eq. (72) and equating coefficients with Eq. (69).

If θ_c is very small, D may be neglected entirely. For the case $n=0$, Eqs. (69), (71), (67) give

$$J=(x/a)_0^{\theta_c}=1.309(\theta_c)^{0.1}. \tag{73}$$

This is useful in determining the smaller values of Table VII.

When $0.9\leqq\theta_c<1$ even series (72) does not give accurate results. The values of J here were determined by means of Simpson's rule. The difficulties due to the infinite integrand at $\theta=\theta_c$ were avoided by an integration by parts.

If n is small, θ_o may not be neglected as in Eq. (64). The most important case is $n=1.2$ (the exponent for resistance). The values of J may be found from Eq. (63), the first integral being evaluated by the methods above, and the second integral by a series similar to that of Eq. (57). This holds over the useful range $\theta_0\leqq 0.6\ \theta_c$.

A similar method might be used for other small values of n. The calculations for the second integral may be simplified by neglecting $\theta^{6.5}$ in Eq. (55). At $\theta_0=0.6\theta_c$ the error made thus is less than 3 percent for any short filament ($\theta_c\leqq 0.94$). Since this integral is a correction term, the approximation is justified. By substituting $u=1-0.4\theta_c^{3.9}-(\theta/\theta_c)^{2.6}$, the integral is reduced to a form which may be evaluated in terms of elementary functions for $n=1.2$, 2.5, 3.8, 5.1 and 6.4. Interpolation may be used for intermediate values of n.

For $n\geqq 4$ Eq. (64) is valid, unless high accuracy is desired. When $n=10$, $\theta_c=0.7$, $\theta_0=0.5$, the error is less than 1 percent.

Experimental checks. Several lamps were made up to test the short filament theory by experiment. Filament L was a straight filament 1.6 cm long, $D=0.0256$ cm, cut from a length of wire that had been aged 24 hours at 2400°. It was welded to nickel leads 5 cm long, $D_L=0.254$ cm. The temperature distribution along this filament was measured by means of an optical pyrometer mounted on a carriage that could be moved along a horizontal scale which gave the position of the pyrometer accurately to 0.001 inch. The pyrometer had previously been calibrated against a standard lamp.

The curves in Fig. 4 show the temperature distribution for three different currents. The solid lines indicate the observed values, and a comparison of T_c in each case with the corresponding value of T_m listed in the corner of

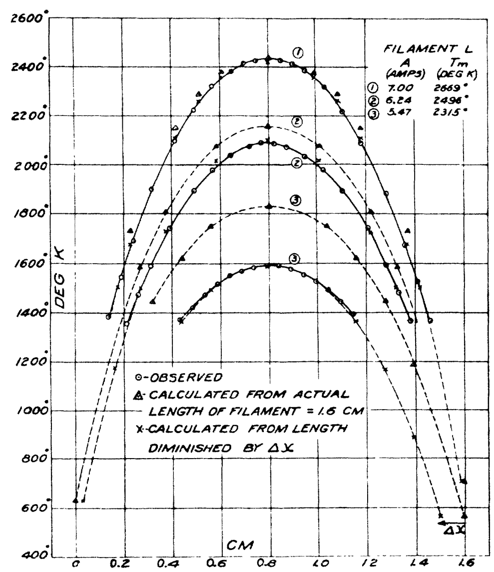

Fig. 4. Temperature distribution for filament L, $D=0.0256$ cm, $2x=1.6$ cm, wire aged throughout its length. Experimental and theoretical curves.

the graph, shows that the center of the filament is greatly cooled by the leads; in other words, filament L is a "short filament."

The temperature distribution was calculated from the short filament theory and the results shown in the curves marked by triangles. In Curve 1 where $\theta_c=0.912$ and $T_m-T_c=234°$, the agreement with experiment is fairly good. In the more nearly extreme cases of Curve 2 where $\theta_c=0.836$, $T_m-T_c=408°$, and Curve 3 where $\theta_c=0.687$, $T_m-T_c=725°$, the temperatures given by the theory are too high. We attribute this departure from the observed values to an error in the value of λ at the cool ends of the filament. The heat conductivity of tungsten is not accurately known at temperatures below incandescence, and if we have used values of λ in this range that are too low, so that the calculated temperature gradient near the leads is too steep, the resulting T_c will be too high. An error from this source becomes important only when the short filament theory is put to the severe test of predicting T_c in cases where T_c is much less than T_m.

Since T_c must be known before either the voltage or the candle-power of a filament can be calculated, the following empirical addition to the theory has been devised, to be used in those cases in which T_c can be determined only by calculation. To compensate for the error that results from using values of λ that are too low, we proceed as if the length of the filament were shortened by an amount Δx such that the heat loss by conduction will be increased above Q_λ, the value corresponding to λ, by an amount $\psi = 4Q_\lambda \Delta x/(\pi D^2)$. Q_λ is given by Eq. (37) which for $\theta_0 = 0.24$ becomes approximately

$$Q_\lambda = 0.6654\, A\, \theta_c(1.812 \cdot 10^{-5} T_m{}^{1.3}) \tag{74}$$

It has already been pointed out, in connection with the derivation of Eq. (40), that changes in θ_0 have little effect on the factor 0.6654. θ_c supplies approximately the factor by which Q_λ must be reduced when the integration in Eq. (36) is carried only to θ_c instead of to 1. The term in brackets is given in Table V. Values of ψ derived from the data of Fig. 4 are given in Table XI. They were calculated using those values of Δx which made the theoretical and observed curves coincide at 1500°; the theoretical curves are indicated by crosses. ψ is tabulated as a function of T_0, since the error is greater at low lead temperatures.

TABLE XI.

$T_0 =$	300°	400°	500°	600°
$\psi =$	471	367	263	159

From Table V and Table XI one can calculate

$$\Delta x = \pi D^2 \psi/(4Q_\lambda) \tag{75}$$

and subtract Δx from the actual half length x of the filament before calculating T_c. This correction applies to calculations of T_c *only*, and the maximum values to be used are $0.15x$ for leads in air, and $0.22x$ for leads in liquid air.

Fig. 5 is a plot of the volt-ampere characteristics of filament G, 1.928 cm in length, $D = 0.0103$ cm. The curves labelled air, using the bottom scale for voltage are for the bulb at room temperature, 300°K. The curves labelled liquid air, using the top scale for voltage, are for the bulb immersed in liquid air.

The course of the liquid air observations for low voltages is interesting. For $450° < T_c < 1200°$ and $T_m \doteq 2{,}000°$, the current decreases with increasing voltage. This phenomenon has been obtained with all short filaments which have been tried in liquid air.

Thus for one value of the current there are in some cases three possible values of the central temperature of the filament. With a low temperature the heat generated is small and so the small temperature gradient is sufficient to carry away the heat and maintain equilibrium. Likewise, with a higher central temperature the heat generated is greater and the larger temperature gradient is necessary to preserve stability. Hence it is possible that all three central temperatures may be stable. The phenomenon may be explained in more detail by assuming that below 1,200° λ is larger than the value given by Eq. (24). Analysis shows that the general form of Eq. (73), which gives $(x/a)_0^{\theta_c}$ for small θ_c, is

$$(x/a)_0^{\theta_c} \propto \theta_c{}^{(1+k-p)/2}. \qquad (73a)$$

If k, the thermal conductivity exponent, is less than 0.2, then by Eq. (73a) the temperature distribution curves of Fig. 2 will cross near $\theta = 0$ and $x/a = 1$. Thus $(x/a)_0^{\theta_c}$ and hence also T_m and A decrease with increasing θ_c for small values of the latter. This **is** essentially the phenomenon observed above.

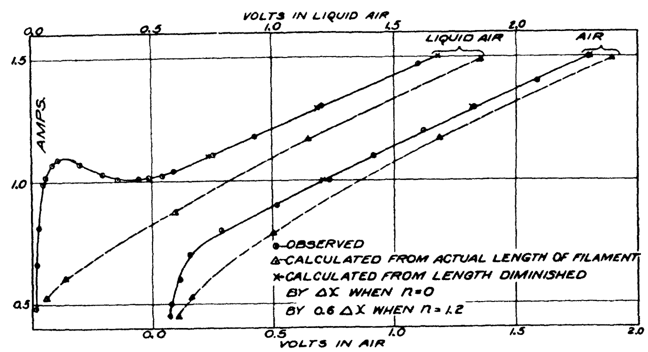

Fig. 5. Volt-ampere characteristics of filament G, $D = 0.0103$ cm, $2x = 1.928$ cm, wire aged throughout its length. Experimental and theoretical curves.

To obtain the calculated voltages, marked by crosses on the curves in Fig. 5, a correction of $0.6\Delta x$ subtracted from x was found sufficient to compensate for the decreased voltage drop along the ends of the filament due to the lower temperature which we assume to exist in this region. Thus for $n = 0$, (temperature distribution) the correction is Δx; for $n = 1.2$ it is $0.6\Delta x$; and for higher values of n no correction is needed.

Example of the calculations. To illustrate the method of calculating lead losses, consider filament G running at a current of 1.295 amps. From $D = 0.01030$ cm we find $D^{3/2} = 0.001045$, $A/D^{3/2} = 1239$, thence $T_m = 2222°$(12). Table I gives by interpolation $a_0 = 0.400$, whence, as $D^{1/2} = 0.1015$ we find from Eq. (2) that $a = 0.406$. To determine the lead junction temperature, we find in Table III that $(\Delta T)_0 = 59°$ and hence with $l = 5$ cm and $D_L = 0.254$ cm Eq. (4b) gives $\Delta T = 59°$. Thus $T_0 = 359°$, and by interpolation in Table XI $\psi = 410$. Table IV gives $(1.812 \cdot 10^{-5} T_m{}^{1.3}) = 0.4064$, so that from Eq. (75) we have $\Delta x = 0.100$ cm if we put $\theta_c = 1$. The half-length $x = 0.964$, therefore $x' = x - \Delta x = 0.864$. Dividing by $a = 0.406$ we have $(x'/a)_{\theta_0}^{\theta_c} = 2.128$. Since $T_0 = 359^0$ we find θ_0 by dividing by T_m, and have $\theta_0 = 0.1616$. We then find the point of coordinates 2.128, 0.1616 on Fig. 2, and drawing through it a curve parallel to the given curves we obtain the intercepts $\theta_c = 0.959$[18] and $(x'/a)_0^{\theta_c} = 2.220$. Multiplying θ_c by T_m we find $T_c = 2131°$.

Eq. (54) shows that the voltage of a short filament is given by

$$V = A(H/H_c)8x\rho_c/(\pi D^2) \tag{76}$$

ρ_c is the resistivity at T_c and is $61.12 \cdot 10^{-6}$ ohm cm[18]. To find H/H_c we must first obtain τ_0 by dividing 359° by T_c to obtain 0.168. From Fig. 3 for abscissa $(x'/a)_0^{\theta_c} = 2.220$ we find $H/H_c = 0.766$. We substitute for x in Eq. (76) the corrected value $x - 0.6\Delta x = 0.904$ and obtain $V = 1.315$ volts. The experimental value was $V = 1.330$.

To find the candle power we use Eq. (54) to give

$$\text{Candle power} = H_c = 2aDC_c'(H/H_c)_0(x/a)_0^{\theta_c} \tag{77}$$

C_c', the specific candle power of tungsten at T_c, is found to be 47.3 international candles cm^{-2}(18). From Eq. (18) using T_c instead of T_m we find the effective value of n to be 12.53. From Fig. 3 for abscissa $(x'/a)_0^{\theta_c} = 2.220$ we then find $(H/H_c)_0 = 0.362$. $(x/a)_0^{\theta_c}$ is obtained from $(x'/a)_0^{\theta_c}$ by multiplying by $x/(x - \Delta x)$ to give 2.477. We thus obtain $H_c = 0.355$ international candles. We did not measure the candle power in this instance, but in other instances similar calculations of candle power gave results that were in good agreement with experimental values.

Discussion of experimental checks. In ordinary practice the ends of a filament are "unaged," that is, they can never be heated to the temperature which a filament must once have before the properties become those of "aged" tungsten. It is known[6] that heating tungsten wire for one minute to a temperature of about 1500° causes the cold resistance to be lowered 15 to 20 percent, and presumably the heat conductivity would be increased at the same time by about the same amount. The filaments used for experimental checks were made from wire that had been previously aged throughout its length, in order to obtain uniform results, for the properties of unaged wire vary from sample to sample and depend on the history of the filament. It has been shown that in the case of aged tungsten the heat conductivity at the cool ends of the filament is higher than the values used in the theory. But where the filament is unaged in this region, the heat conductivity is lower than in the case of the aged filament which tends to restore λ to the values used in the theory so that in general the short filament theory may be used without the correction Δx to calculate lead losses from filaments that are made in the ordinary way.

[18] Where θ_c comes out to be less than 0.95 it is well to substitute θ_c equal to the calculated value instead of unity in Eq. (74), and thus arrive at a second approximation for Δx from Eq. (75).