SHATTER OSCILLATIONS, THEIR NATURE AND THEORY*

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(Received December 30, 1929)

Abstract

Shatter oscillations are an apparently novel type in which the pressure sinks periodically to the minimum value which the liquid can sustain, the liquid mass becoming then porous or "shattered"; they tend to be much slower but more powerful than purely elastic vibrations and the wave-form is very different, pressure impulses alternating with long intervals of quiet. An experimental case is described and the general theory of such oscillations is developed. Further experiments are needed.

An Experiment

Several years ago the writer was privileged to witness some interesting experiments upon liquid oscillations in the plant of the Goulds Pumps Company at Seneca Falls, N. Y., which were of an astonishing and apparently little-known type. It is the purpose of this paper to describe them briefly, together with the theory proposed by the author, in the hope that some physicist may be induced to make a detailed study of the phenomena and so to find out whether the theory is correct and adquate.

In the most interesting experiment a pipe 31 m long and full of water was connected at one end to a tank which contained a considerable mass of water and over it air at a pressure of about 4 atmospheres (the pressure of the atmosphere included). At the other end of the pipe was a small pump with its

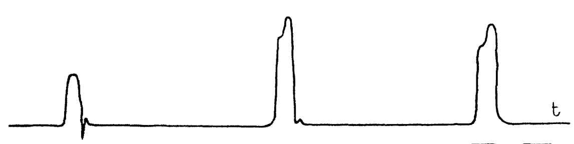

Fig. 1. Indicator record showing pressure distribution.

discharge valve removed and its suction valve blocked shut, so that it served merely to "fan" the water with an approximately simple-harmonic motion. A recording gauge recorded the pressure of the water in the pipe at a point near the pump.

Under the circumstances one might expect the water column to oscillate like a "closed organ pipe" (i.e. open only at the tank end, where the pressure must have been sensibly constant); the speed of sound in water in such a pipe being about 1360 m per sec, the fundamental period should be around 0.091 sec. Nothing of the sort was observed. At suitable pump speeds oscillations

*Part of this paper was read before the National Academy of Sciences; an abstract of this part appears in Science, **70**, 618 (1929).

did occur, and of enormous amplitude, but they were much slower than the slowest possible elastic oscillation, and the distribution of pressure was of the remarkable type shown in Fig. 1, which is copied from one of the actual indicator cards. Isolated "humps" of pressure appeared at intervals, in the figure one every 0.7 second, separated by long quiet intervals during which the pressure seemed to be about zero (certainly below atmospheric); the humps always occurred during discharge strokes of the pump (which are shown by horizontal lines in the diagram, four per oscillation in that case). The maximum pressure was usually around 25–30 atmospheres but in one case it rose to 75 atmospheres before the gaskets blew out of the pump and stopped the experiment.

Now the direction in which to look for an explanation of this phenomenon seems obvious enough. In an ordinary elastic oscillation a compression is followed by an equal rarefaction, but water cannot withstand a tension of 20–30 atmospheres; under special conditions tensions up to 5 atmospheres have been observed, but in practice, according to what appears to be the opinion of engineers, the pressure never sinks below zero (i.e. one atmosphere below atmospheric)—at all events no appreciably lower pressures were ever observed in these experiments, although the indicator was often capable of showing them.

We have, then, a case, apparently new in hydrodynamical theory, of large-amplitude oscillations in a liquid whose pressure cannot sink below a certain critical pressure, p_0. It was at first naively supposed that under these circumstances large gaps might form in the liquid mass; for instance, in the present case, that the column might break loose from the pump and retreat several feet down the pipe. But a little reflection shows that this cannot happen when the minimum pressure is really quite definite. For in order to form such a gap the layer of liquid on one side or the other of it must at some time experience an acceleration directed away from the incipient gap, and, since by hypothesis the pressure is p_0 in the gap and cannot become less than this in the liquid, no force producing such an acceleration can be developed by pressure differences in the liquid (although it could arise, of course, from gravity in a non-horizontal column, and large gaps could then form). For theoretical purposes the stituation might be idealized satisfactorily by assuming that when the pressure sinks to p_0 the elasticity drops discontinuously to zero. What happens in practice, however, must be that the liquid mass does break, at many closely spaced points, in consequence of local inhomogeneities of composition or of state and under the action of cohesive forces, so that the liquid becomes full of small crevices or, as we shall call it, "shattered." We shall study this process a little more closely.

Theory of Shattering in a Liquid Column

The point at which the pressure first sinks to the minimum value, p_0, must be one where, momentarily,

$$\frac{\partial p}{\partial t}<0, \quad \frac{\partial u}{\partial x}=-\frac{1}{\epsilon}\frac{\partial p}{\partial t}>0, \tag{1}$$

in which p = pressure, t = time, ϵ = elasticity, u = particle velocity, x = distance along the column. Continuity being assumed, p will then proceed to reach p_0 at neighboring points and a shattered region will form in which $p = p_0$ and is thus uniform, so that $\partial u/\partial t = 0$. In this region, according to (1), $\partial u/\partial x > 0$ (see also below, after Eq. (2)); the liquid will therefore expand at a constant rate through enlargement of the crevices so long as the shattered condition persists. The boundaries of the region, which we shall call "shatter-fronts," advance thru the liquid with a speed V which can easily be calculated. During an interval dt let the shatter-front advance positively a distance Vdt from a point P to P'; then at the beginning of dt the pressure was p_0 at P and $p_0 + (Vdt)\,\partial p/\partial x$ at P', whereas at the end of dt it has risen at P' by an amount, actually negative, $(\partial p/\partial t)\,dt$ and has now become p_0; thus $(Vdt)\,\partial p/\partial x + (\partial p/\partial t)dt = 0$ and

$$V = -\frac{\partial p}{\partial t}\bigg/\frac{\partial p}{\partial x} = \epsilon\frac{\partial u}{\partial x}\bigg/\frac{\partial p}{\partial x}\,. \tag{2}$$

A second and rather peculiar condition for the propagation of the shatter-front arises from the condition that the liquid must be left behind it in an expanding condition (or at least not in a contracting one, since then the pressure would immediately rise above p_0 again). In the same notation, at the beginning of dt let the particle velocities at P and P' resp. be u and $u + (Vdt)\,\partial u/\partial x$; then at the end of dt the velocity is still u at P (because of the uniformity of pressure), but at P' it has changed by $(\partial u/\partial t)dt$. The condition stated then requires that the final velocity at P' shall exceed that at P, or that $u + (Vdt)\,\partial u/\partial x + (\partial u/\partial t)\,dt \geqq u$; and, since $\partial u/\partial t = -(1/\rho)\,\partial p/\partial x$ where ρ is the density and the speed of sound is $c = (\epsilon/\rho)^{1/2}$, we have by (2):

$$k^2\left(\frac{\partial u}{\partial x}\right)^2 \geqq \left(\frac{\partial p}{\partial x}\right)^2, \quad V^2 \geqq c^2, \tag{3}$$

where $k = (\epsilon\rho)^{1/2}$. It follows that the shatter-front moves with a speed above that of sound; its progress therefore depends upon conditions in the liquid due to other causes than what may be going on in the shattered region.

In its advance the shatter-front may eventually come to a point where the first of Eqs. (3) is no longer satisfied. It will then turn into a "reconsolidating front" and immediately start back in the opposite direction, for the higher pressures in the adjacent unbroken region will accelerate the liquid toward the shattered region. As the reconsolidating front reaches each point in the shattered region, the crevices suddenly close up and the liquid already present undergoes an impulsive acceleration and compression which give to it both the particle velocity u and the pressure p that obtain on the unbroken side of the advancing front. For convenience let us resolve conditions in the unbroken column as usual into two wave-trains moving in opposite directions; let the unbroken column lie on the left, toward $-x$, and let u_1, p_1 be particle velocity and pressure in the wave-train moving positively and u_2, p_2 the corresponding quantities in the other wave-train. Then by the usual laws for elastic waves

$p_1 = ku_1$, $p_2 = -ku_2$, and at the front $p = p_1 + p_2$, $u = u_1 + u_2$. Here u_1, p_1, referring to the "incident" waves, are determined by conditions to the left and can be regarded as known, while u_2 and p_2 can be regarded as referring to waves "reflected" from the boundary of the shattered region and are to be found. Let U = velocity of advance of the reconsolidating front and in the shattered region let u_s = particle velocity and f = degree of shattering or fraction of the space that is empty of liquid, both taken at a point just ahead of the advancing front. Then in time dt the front sweeps over a section PP' of length Udt; during this time the liquid flowing in at P, of volume udt per unit of cross-section, must fill up not only the crevice-space $fUdt$ but also the space $(1-f)$ Udt $(p-p_0)/\epsilon$ emptied through compression of the liquid already in the section PP' and the space emptied by the outflow past P' of a volume $u_s dt$ of the shattered column. Hence

$$u = \left[f + (1-f)\frac{p-p_0}{\epsilon}\right]U + u_s. \tag{4}$$

(We assume u/U small so that terms in u^2 or u/U can be neglected, as in the ordinary theory of sound.) During the same interval dt the difference in pressure at the ends of the section PP' must impart to the liquid already in it the additional momentum acquired as its velocity changes impulsively from u_s to u; whence,

$$p - p_0 = \rho U(1-f)(u-u_s). \tag{5}$$

From these two equations we find:

$$(p-p_0)^2 + \frac{\epsilon f}{1-f}(p-p_0) = k^2(u-u_s)^2, \tag{6}$$

$$\frac{1}{U^2} = \frac{1-f}{c^2}\left[1 - f + \frac{\epsilon f}{p-p_0}\right] = \frac{1}{c^2}\left[1 + f^2 + f\left(\epsilon\frac{1-f}{p-p_0} - 2\right)\right]. \tag{7}$$

Equation (6) expresses the "boundary condition" that must hold under these conditions at the junction of shattered and unbroken regions. Eq. (7) gives U, and shows, among other things,that $U < c$ under the conditions assumed, since $\epsilon/(p-p_0)$ will be many times greater than 2.

The reconsolidating front in its turn will diasppear upon meeting either a similar front moving in the opposite direction or an obstruction, with resulting final elimination of the shattered region (as a third alternative, of course, it might also be overtaken and destroyed by a fresh shatter-front). When two reconsolidating fronts meet, with pressure and particle velocity p, u in the one moving positively and approaching from the left and p', u' in the other one, then at the point of meeting these quantities will change impulsively to certain values p_m, u_m. Let us resolve the motion on each side into positive and negative wave-trains, with respective pressures $p_1 = (p+ku)/2$, $p_2 = (p-ku)/2$ on the left, and $p_1' = (p'+ku')/2$, $p_2' = (p'-ku')/2$ on the right. Then, after the two fronts meet, these wave-trains advance as usual and by their superposition determine subsequent conditions in the liquid; after the lapse of a very short time the trains will overlap a little at the point of meeting and will produce there the following values:

$$p_m = p_1 + p_2' = \frac{1}{2}(p+p') + \frac{k}{2}(u-u'),\ u_m = \frac{p_1}{k} - \frac{p_2'}{k} = \frac{1}{2}(u+u') + \frac{1}{2k}(p-p') \quad (8)$$

The case in which the reconsolidation front is stopped by an obstruction, such as the closed end of the pipe, can be handled by putting $u_m = 0$ in the last equations and eliminating p' and u'; this gives the usual impact formula:

$$p_m = p + ku. \quad (9)$$

The extension of the theory just developed to the three-dimensional case is easy, but for completeness we shall sketch it. The boundary of the shattered region is in this case a closed surface which may move either as a shatter-front or as a reconsolidating front, or in part as one and in part as the other; the theory developed above will apply provided we take the x-axis along the normal to the surface, with the single exception that we must also take account of expansion due to the component of particle velocity tangential to the surface. The tangential acceleration vanishes because over the surface $p = p_0$ and is constant. One finds thus as the conditions for the advance of a shatter-front, in the unbroken liquid just ahead of it:

$$\text{div } u > 0,\ k^2(\text{div } u)^2 = \left(\frac{\partial p}{\partial n}\right)^2, \quad (1'),\ (3')$$

n denoting distance along the normal to the front (which is also the direction of ∇p), and for V, the speed of advance of the front along its normal:

$$V = -\frac{\partial p}{\partial t}\bigg/\frac{\partial p}{\partial n} = \epsilon(\text{div } u)\bigg/\frac{\partial p}{\partial n};\quad V^2 \geqq c^2. \quad (2'),\ (3'')$$

When the boundary starts back through the shattered region as a reconsolidating front, we have as boundary conditions for the determination of p and u in the unbroken liquid,

$$(p-p_0)^2 + \frac{\epsilon f}{1-f}(p-p_0) = k^2(u_n - u_{sn})^2, \quad (6')$$

in exact analogy with (6), n denoting the component normal to the front, and also, owing to the absence of pressure gradient over the front,

$$u_t = u_{st}, \quad (6'')$$

the t denoting the vector component tangential to the front. These equations are equivalent to three scalar ones and suffice to determine the reflected wave in terms of the variables describing the incident waves and the conditions in the shattered region. The value of U, the speed of advance of the front along its normal, is given as before by Eq. (7).

Explanation of the Experiment

In the light of these theoretical developments the course of the oscillation described at the beginning of the article is believed to be as follows. The enormous gradient implied by the momentary high pressure at the pump sets the water into rapid motion toward the tank; this phase should last about $0.091/4=0.023$ sec. Then as the pressure sinks to p_0 a shatter-front sweeps quickly along the pipe and a considerable section of the column becomes shattered and remains so, the crevices widening steadily, for a considerable time. Some photographs of the column taken thru a short glass section of the pipe seemed to confirm the prediction of the theory as to its condition during this phase. Finally, as the reconsolidating front arrives at the pump and the now rapidly moving column is brought to rest, a "water-hammer" results and the high hump of pressure is thereby restored.

Unfortunately the mathematical discontinuity in the equations seems to preclude obtaining any simple solution as an illustrative case. The best method of attack seems to be to start with an assumed hump of pressure and determine the motion of the liquid by approximate methods of calculation. It seems most profitable, however, for such a calculation, rather laborious at best, to be undertaken in conjunction with a repetition of the experiment in which simultaneous pressure records are obtained at several points along the pipe, in order to obtain a complete check on the theory. No projected experiment of this sort is known to the writer at the present moment.