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A GENERAL THEORY OF THE PLASMA OF AN ARC

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Abstract

The conception of random positive ion velocities corresponding to ion temperatures in a plasma has serious theoretical difficulties and is lacking in direct experimental verification. It is more reasonable to assume that each ion starts from rest and subsequently possesses only the velocity which it acquires by falling through a static electric field which is itself maintained by the balance of electron and ion charges. This new viewpoint thus ascribes motions to the positive ions which, for long free paths, are ordered rather than chaotic, each negative body in contact with the discharge collecting ions from a definite region of the plasma and from it only. The resulting integral equatio[illegible] the plasma-sheath potential distribution have been set up for plane, cylindrical, and spherical plasmas, for long, short and intermediate length ion free paths, and for both constant rate of ionization throughout the plasma and rate proportional to electron density, and these equations have been solved for the potential distribution in the plasma in all important cases. The case of short ion free paths in a cylinder with ion generation proportional to electron density gives the same potential distribution as found for the positive column by Schottky using his ambipolar diffusion theory, with the advantages that ambipolarity and quasineutrality need not appear as postulates. The calculated potential distribution agrees with that found experimentally. The potential difference between center and edge of plasma approximates $T_e/11{,}600$ volts in all long ion free path cases. The theory yields two equations. One, the *ion current equation*, simply equates the total number of ions reaching the discharge tube wall to the total number of ions generated in the plasma, but it affords a new method of calculating the density of ionization. The second, the *plasma balance equation*, relates rate of ion generation, discharge tube diameter (in the cylindrical case), and electron temperature. It can be used to calculate the rate of ion generation, the resulting values checking (to order of magnitude) those calculated from one-stage ionization probabilities. The potential difference between the center of the plasma and a non-conducting bounding wall as calculated from the ion current equation agrees with that found experimentally.

The solution of the general plasma-sheath equation has been extended into the sheath surrounding the plasma to determine the first order correction which is to be subtracted from the discharge tube radius to obtain the plasma radius. The wall sheath in the positive column is several times the thickness given by the simple space charge equation.

Actually the ions do not start from rest when formed but have small random velocities corresponding to the gas temperature, T_g. In the long ion free path cases this leads to an error of the order of only T_g/T_e in the calculated potential distributions.

In the plasma surrounding a fine negatively charged probe wire the potential difference between plasma potential maximum and sheath edge may be so small that the ions generated within the plasma potential maximum are not trapped but can traverse the maximum by virtue of their finite initial velocities. This justifies the use of a sufficiently fine negatively charged wire in the usual way to measure positive ion concentrations, although certain difficulties appear which are thought to be connected with the collector theory rather than the present plasma theory.

A positively charged cylindrical probe collects electrons in the same manner as previously supposed, except that the sheath about it is considerably thickened by the presence of ions generated in the sheath.

The plasma balance equation completes the number of relations needed to determine completely the conditions in a positive column. Taking the arc current as the independent arc variable, the five dependent variables are axial electric field, density of ionization, electron temperature, positive ion current to the wall, and the rate of generation of positive ions. The five relations which determine them are the plasma balance equation, the ion current equation, an ion generation equation, a mobility equation, and an energy balance equation. The essential nature of these relations is recognized even though present knowledge is insufficient to complete all of them.

Stability in the positive column has not been considered. The possibility exists that instability of one type or another may lead to the oscillations which can occur in an arc.

List of Symbols

It has been thought advisable to give below a list of the symbols used in the present paper together with their meanings and a reference to the place of their first appearance in the body of the article. The occasional duplication of symbols will, it is thought, not lead to confusion.

A	Coefficient of $\Delta_s{}^2\eta$;	Eq. (58)
A	Area of probe electrode;	Eq. (87)
A'	Coefficient of $\Delta_q{}^2\eta$, (similar to A);	Eq. (79)
a	Discharge tube radius, maximum value of r in plasma.	
a_0, a_1, etc.	Numerical coefficients;	Eq. (17)
B	Numerical coefficient;	Eq. (80)
C	Numerical coefficient;	Eq. (80)
C	As in "Case $CL\lambda$" indicates cylindrical plasma;	Table II
D	Ion-into-gas diffusion constant;	Eq. (27)
e	Electronic charge;	Eq. (1)
f_λ	Defined by	Eq. (93)
H_1, H_2	Integrals;	following Eq. (80)
h_0	Numerical plasma constant;	Eqs. (51) and (53)
I_n	Random electron current density at tube wall;	following Eq. (55)
I_p	Positive ion current density, random;	Eqs. (1), (87), and (88)
	at discharge tube wall;	Eq. (49)
i_B	Arc current;	Eq. (97)
i_e, i_p	Electron, positive ion currents to probe;	Eq. (87)
J	N_z when N_z is constant;	Eq. (24)
J	As in "Case CLJ" denotes that N_z is constant;	Table II
J_0	Bessel Function of zero order;	Table IId
k	Boltzmann constant;	Eq. (1)
L	As in "Case $CL\lambda$" indicates long ion free paths;	Table II
L and M-S	Refers to I. Langmuir and H. M. Mott-Smith, Jr. articles;	Footnote (1)
l_e	Mean free path of electrons;	Eq. (96)
l_g	Mean free path of gas molecules.	
l_p	Mean free path of positive ions.	
M	As in "Case $CM\lambda$" indicates "medium" ion free paths;	Table II
M.D.	Maxwell Distribution.	
m_e	Mass of electron;	Eq. (1)
m_p	Mass of positive ion;	Eq. (1)
N	Number of electrons per cm length of arc;	Eq. (97)
N_z	Number of ions generated $\text{cm}^{-3}\cdot\text{sec}^{-1}$ at z;	Eq. (2)
n_e	Electron density;	Eq. (3)
n_0	Electron density at plasma potential maximum;	Eq. (3)

n_p	Positive ion density;	Eq. (2)
n_z	Electron density at z;	Eq. (8)
P	Number of ions formed per electron per cm of path;	Eq. (91)
P	As in "Case PLJ" indicates plane plasma;	Table II
$p+1$	Term number in a power series;	following Eq. (17)
p_g	Gas pressure in baryes;	Eq. (91)
q	Ion mobility per unit charge;	Eq. (29)
q	Dimensionless length parameter;	Eq. (79)
q'	Ion mobility factor;	Eq. (35)
r	Distance in plasma;	Eq. (2)
S	Value of s at plasma potential maximum;	Eq. (82)
S	As in "Case $SL\lambda$" indicates spherical plasma;	Table II
S	As in "Case $PS\lambda$" indicates short ion free paths;	Table II
s	Dimensionless length parameter;	Eq. (10)
s'	Abbreviation for $ds/d\eta$.	
s''	Abbreviation for $d^2s/d\eta^2$.	
s_0	Value of s } for which $d\eta/ds=\infty$.	
s_0''	Value of $d^2s/d\eta^2$ } for which $d\eta/ds=\infty$.	
s_t	Value of s;	Fig. 7
s_z	Dimensionless length parameter;	Eq. (11)
s_ϕ	Value of s at sheath edge;	preceding Eq. (62)
T_e	Electron temperature;	Eq. (3)
T_g	Gas temperature;	Eq. (27)
T_p	Positive ion temperature;	Eq. (1)
V	Space potential at r relative to plasma potential maximum.	
V_i	Ionization potential of gas;	Eq. (91)
V_w	Wall potential with respect to plasma potential maximum.	
V_z	Space potential at z;	Eq. (7)
v	$=v_z$ when v_z is independent of z;	Eq. (27)
$\bar{v}$	Average thermal velocity of gas molecules.	
$\bar{v}_e$	Average thermal velocity of electrons;	Eq. (96)
v_f	Component of average final velocity of ion in free path;	preceding Eq. (35)
v_0	Component of average initial velocity of ion in free path;	preceding Eq. (35)
$\bar{v}_p$	Mean ion drift velocity;	preceding Eq. (57)
v_z	Velocity at r of ion generated at z;	Eq. (2)
W	Potential equivalent of electron velocity;	Eq. (91)
W_e	kT_e/e;	Eq. (71)
X	Potential gradient drawing ions toward sheath.	
x_p	Length unit;	defined by Eq. (71)
Z	Longitudinal potential gradient in positive column;	Eq. (97)
z	Distance in plasma;	Eq. (2)
α	Factor for transforming variables;	Eq. (10)
α_λ	Defined by;	Eq. (94)
β	Geometrical parameter;	Eq. (2)
β	Probability-of-ionization constant;	Eq. (91)
δs	$s-s_0$;	Eq. (63)
$\delta\eta$	$\eta-\eta_0$;	Eq. (63)
δs_ϕ	s_0-s_ϕ;	Eq. (66)
$\delta\eta_\phi$	$\eta_0-\eta_\phi$;	Eq. (66)
ϵ	Base of natural logarithms.	
ζ	Potential parameter, $\eta^{1/2}/S$;	Eq. (83)
η	$-eV/kT_e$.	
η_0	Value of η for which $d\eta/ds=\infty$.	
$\bar{\eta}_0$	Mean value of $\eta_0-\eta$;	Footnote (19)
η_s	$\eta-\eta_\phi+\bar{\eta}_\phi$.	

η_z	$-eV_z/kT_e$.	
η_w	$-eV_w/kT_e$;	Eq. (56)
η_ϕ	Value of η at sheath edge;	preceding Eq. (62)
$\bar{\eta}_\phi$	Mean value of $\eta_\phi-\eta$;	Footnote (19)
θ	Integration parameter;	Eq. (15)
λ	Number of ions generated per electron per sec;	Eq. (8)
λ	As in "Case $CL\lambda$" indicates generation of ions proportional to electron density;	Table II
μ	$-eV/kT_p$;	Eq. (79)
μ_e	Electron mobility;	Eq. (96)
μ_z	$-eV_z/kT_p$;	Eq. (79)
ξ	Distance parameter in sheath;	Eq. (70)
ρ	Fractional part of tube radius occupied by sheath;	Eq. (77)
ρ	Integration parameter;	Eq. (15)
ρ_e	Electron reflection coefficient;	Eq. (56)
ρ_z	Integration parameter;	Eq. (15)
Σ	Constant slope of i_p^2 vs. V or i_e^2 vs. V plot;	Eq. (88)
σ	Distance parameter, s/S;	Eq. (83)
τ	T_p/T_e (Section V only).	
τ	Mean free time of positive ions.	
ϕ	Arbitrary fraction;	Eq. (61)

I. Introduction

THERE is a large amount of evidence to show that the vast majority of free electrons in the plasma of a gaseous discharge possess velocities distributed according to the Maxwell Distribution Law (hereafter "M. D."). The temperatures to which these velocity distributions correspond lie roughly in the range between 5000 and 70,000°K. The method for measuring these temperatures need not be gone into[1] beyond mentioning that it depends on the Boltzmann density distribution which the electrons assume in the sheath about a negatively charged collector. The validity of the method hinges, first, on the confinement of the electrode potential changes to a sheath about the electrode, and second, to the smallness of changes in sheath cross-section or volume compared to the whole cross-section or volume of the plasma.

No such direct measurements on the positive ions are possible for various reasons. The most direct evidence that the ions possess considerable velocities lies in the saturation current to a collector at negative voltages. It has been reasoned that since the potential difference between plasma and electrode is confined to the sheath, only those ions will be collected which are headed for the sheath edge anyway. The random ion current density I_p in the discharge can then be found by dividing the observed saturation current by the sheath area. The ion density is equal to the electron density, n_e in the field-free plasma. On the assumption that the velocities are distributed according to the M. D. Law, but only over one hemisphere since no ions leave the collector, kinetic theory gives

$$T_p^{1/2}=(2\pi m_p/k)^{1/2}I_p/2en_e=2.02\times 10^{13}(I_p/n_e)(m_p/m_e)^{1/2} \qquad (1)$$

[1] I. Langmuir and H. M. Mott-Smith, Jr., General Electric Review **27**, 449, 538, 616, 762, 810 (1924). Hereafter these articles will be referred to as "L and $M-S$, Part I, etc., to Part V," respectively.

A saturation electron current at small positive voltages on the same collector can be observed if the collector area is not too great. This can be treated in the same way, giving[2]

$$T_e^{1/2}=4.03\times 10^{13} I_e/n_e.$$

Combining the two it is found that

$$T_p/T_e=(I_p/2I_e)^2 m_p/m_e$$

In a typical set of measurements made on a mercury arc[3] the average value of I_e/I_p was 405 ± 25. For mercury $m_p/m_e=3.678\times 10^5$, whence it would follow that $T_p/T_e=0.55$. It seems entirely unreasonable, however, that the ion energy should even approach the electron energy in view of the fact that it is the electrons primarily which supply energy to the rest of the plasma and the positive ions with their large relative mass and frequent impact with slow atoms are not adapted to acquiring large random kinetic energies.

But evidence in favor of a large random ion current is found in another experiment. When two equal plane electrodes, back to back and insulated from each other, are placed in the positive column of a discharge so that one electrode faces the anode, the other the cathode, the two electrodes receive comparable ion currents even though to reach one of the electrodes the ions must flow toward the anode. A similar conclusion offers itself in the related experiment in which the sheaths on a spherical electrode immersed in a positive column are seen to be of equal thickness on cathode and anode sides.[4]

There is other evidence also. Experiments in which a double electrode is used, the electrode facing the discharge being pierced with fine holes,[5] give ion temperatures of several thousand degrees. But how accurately Maxwellian these results show the velocity distribution to be is somewhat doubtful.

Although all this evidence points to the possession of considerable velocities by the positive ions, and the concept of a random velocity distribution among the positive ions has been a generally useful one in explaining a multitude of observations, there has been no convincing determination of the velocity distribution, and the hypothesis of large random velocities has grave theoretical difficulties to overcome.[6] A further difficulty crops up as soon as the attempt is made to join the sheath to the plasma, that is, to investigate theoretically the nature of the sheath edge.[7] In quite general terms the perplexity is this. Consider two nearby points in the discharge, one just inside the sheath about a negatively charged electrode and one just outside

[2] There appears to be some justification for using 4.03 in the equation pertaining to electrons as compared to half of this for the ions. See I. Langmuir, Phys. Rev. **33**, 964–5 (1929).

[3] L and $M-S$, Part II, Table III.

[4] L and $M-S$, Part V, p. 812.

[5] L. Tonks, H. Mott-Smith, Jr., and I. Langmuir, Phys. Rev. **28**, 104 (1926).

[6] In applying his quasi-neutral diffusion theory to the positive column of an arc W. Schottky, Phys. Zeits. **25**, 346 (1924), abandons the idea that the positive ions have a temperature comparable with the electron temperature.

[7] I. Langmuir, Phys. Rev. **33**, 976 (1929).

the sheath. At the outside point suppose that there is a one-sided M. D. of velocity among the ions—one-sided since no ions come out through the sheath. At this point the electron and ion densities are equal. The potential at the inside point is slightly less than in the plasma, with the result that both electron and ion densities are less there, the electrons according to the Boltzmann Law and the ions because of their greater average velocity. This can readily be seen by plotting the theoretical densities against the potential decrease. But these curves show an astonishing relation—for small negative potentials the electron density exceeds that of the positive ions because the electron density curve approaches the plasma potential with a finite, the ion curve with an infinite slope. By Poisson's Equation any such predominance of negative charge at the sheath edge requires positive curvature in the potential distribution curve there, thus making it impossible to merge the sheath into the plasma.

We have now reached a new point of view which seems in every way to be more satisfactory. We suppose as before that the electrons possess a M.D. of velocity and such a high mobility that they obey the Boltzmann Law irrespective of any drift in the plasma away from their points of origin. (This is, of course, not true of the longitudinal gradient in a positive column, but the arc current is so very much greater than the drift currents necessitated by the generation of electrons, that this gradient by its smallness justifies rather than invalidates our assumption.) But the positive ions are supposed to have negligible velocity when formed and to acquire only such velocities as correspond to the electric fields through which they pass. In the case of long mean free paths each ion will thus fall freely under the influence of the small plasma fields set up by the electrons and ions themselves until it strikes the tube wall or an electrode. For short free paths the ion will be impeded in its motion by collisions with atoms but still will be mainly guided by the electric field in which it finds itself.

This point of view, it will be seen, ascribes a less chaotic motion to the ions than they possessed according to the old concept. Thus in a sphere or cylinder at very low pressure the ions all move radially outward, each with a velocity corresponding to its point of origin, and for any geometrical configuration it becomes theoretically possible to associate each element of wall or electrode area with a tubular region of the plasma which alone contributes ions to that area.[8] Accompanying this picture is the idea that it is the presence of an electrode (or tube wall) in contact with the discharge which is responsible for the ion current flowing to that electrode by reason of its influence, as a boundary condition, on the potential distribution in the plasma.[9]

[8] This is, of course, an idealized representation. In any case the ions possess the random motions which they had as atoms just before ionization occurred, but this is usually small compared to the potentials through which the ion falls in the plasma. Cases in which these small velocities are not negligible appear later. At the higher pressures collisions with atoms introduce additional randomness.

[9] Later, in Section VI, it will be found that very small electrodes have no effect on the potential distribution through the body of the plasma and can, therefore, be used as true probes.

In order to handle this theory mathematically it is necessary to know the space distribution of the ion generation. Two cases will be considered below. If the ions are generated wholly by electrons which have acquired their velocity in the sheath surrounding an electron source, and the mean free path is long compared to the tube dimensions, the generation will, in many cases, be essentially uniform throughout the plasma. If, on the other hand, the ion generation is caused by the "ultimate" electrons themselves which are constantly renewing their energy, as in the positive column of a discharge, the rate of generation of ions will be proportional to the electron density. Other cases may suggest themselves, as for instance, generation by fast electrons at higher pressures where the mean free path of the electrons is small, and these cases can also be handled by the methods developed here.

For all geometrical configurations except those possessing the simplest symmetry, the mathematical difficulties become very great, and for this reason the quantitative treatment is confined in the following section to plasmas bounded externally by two infinite parallel planes, or by an infinite circular cylinder, or by a sphere, and in Sections VI and VII to plasmas bounded internally by a cylinder.

II. Potential Distribution in the Plasma. Part I

It is evident that if ions are to flow to electrodes and walls under the influence of the electric fields hypothecated, there must be a potential maximum in the plasma, and, in the simple cases to be discussed here, symmetry considerations place it at the center of the boundary structure and the ions thus move outward in straight lines. It is advantageous to select the origin of coordinates at the center (median plane, cylinder axis, or sphere-center) of the structure and to denote distance in cm from this center by r. If the potential at $r=0$ be taken as zero, it is negative elsewhere and the ions generated at any point z acquire a certain velocity v_z by the time they pass some further point r. In general v_z may be a complicated function of the potential distribution, and the high and low pressure cases are distinguished by the type of function assumed for v_z. If the number of ions generated per second per unit volume at z is N_z, their density when they pass r is, in each case, respectively,

Plane: $N_z dz/v_z$

Cylinder: $N_z z dz/r v_z$

Sphere: $N_z z^2 dz/r^2 v_z$

Since ions generated at every value of z which is less than r contribute to n_p, the ion density at r, this density is given by

$$n_p = r^{-\beta} \int_0^r N_z z^\beta dz/v_z \tag{2}$$

where β assumes the values, 0, 1, 2, for plane, cylindrical, and spherical cases respectively.

The electron density at any point is given by

$$n_e = n_0 \exp\,(eV/kT_e) \tag{3}$$

where n_o is the electron density at the origin, T_e is the electron temperature, and V is the space potential. Poisson's Equation may be written

$$\nabla^2 V = -4\pi e(n_p - n_e).$$

Substituting Eqs. (3) and (2) in this equation we have

$$\nabla^2 V - 4\pi e n_0 \exp(eV/kT_e) + 4\pi e r^{-\beta} \int_0^r N_z z^\beta dz/v_z = 0. \tag{4}$$

This is the general integral equation for the potential distribution throughout plasma and sheath. In the form of Eq. (4) and in subsequent forms to which this equation may be transformed in various cases it will be known as the *complete plasma-sheath equation*. Throughout its various metamorphoses it will continue to consist of three terms, namely, one term corresponding to the Poisson differential coefficient, another corresponding to the electron density, and a third corresponding to the positive ion density. In its complete form this equation is far too complicated to handle, but it can, fortunately, be simplified in two important regions. First, in the plasma it will appear that the Poisson term is negligible. Dropping it from the complete equation leaves the *plasma equation*, the equation with which the present section deals. Second, in the sheath bounding the plasma other simplifications can be made leading to the *sheath solution* which will be discussed in Section IV.

Eq. (4) can be simplified immediately by the substitution of a new dimensionless variable η for V,

$$\eta = -eV/kT_e. \tag{5}$$

The equation then becomes

$$(kT_e/4\pi e^2 n_0)\nabla^2\eta + \epsilon^{-\eta} - n_0^{-1} r^{-\beta} \int_0^r N_z z^\beta dz/v_z = 0. \tag{6}$$

This equation must now be adapted to the various particular cases.

Case 1. Long mean free path and ion generation proportional to electron density. In this case each ion falls freely from the point at which it is generated. Accordingly,

$$v_z = [2e(V_z - V)/m_p]^{1/2} = [2kT_e(\eta - \eta_z)/m_p]^{1/2} \tag{7}$$

where V_z and V are the potentials at z and r respectively. Here, also

$$N_z = \lambda n_z = \lambda n_0 \epsilon^{-\eta_z} \tag{8}$$

where λ is the number of ions generated by an electron in one second. Substituting these expressions in Eq. (6) we have

$$(kT_e/4\pi e^2 n_0)\nabla^2\eta + \epsilon^{-\eta} - \lambda(m_p/2kT_e)^{1/2} r^{-\beta} \int_0^r z^\beta \epsilon^{-\eta_z}(\eta - \eta_z)^{-1/2} dz = 0. \tag{9}$$

In order to solve this equation the substitution

$$s=\alpha r \tag{10}$$

is necessary where s is the new variable and α is an adjustable constant. Remembering that $\Delta^2\eta$ has the dimensions of ηr^{-2}, it is seen that Eq.(9) takes on the form

$$(\alpha^2 kT_e/4\pi e^2 n_0)\nabla_s^2\eta+\epsilon^{-\eta}-\lambda(m_p/2kT_e)^{1/2}s^{-\beta}\alpha^{-1}\int_0^s s_z^{\beta}\epsilon^{-\eta_z}(\eta-\eta_z)^{-1/2}ds_z=0 \tag{11}$$

where ∇_s^2 indicates that derivatives are taken with respect to s, and η_z is the same function of s_z that η is of s. At this point α is so chosen that the coefficients of the second and third terms are equal, that is

$$\alpha=\lambda(m_p/2kT_e)^{1/2} \tag{12}$$

and as α has the dimensions of a reciprocal length, s as well as η is dimensionless. This substitution puts the differential equation in the form

$$(m_p\lambda^2/8\pi e^2 n_0)\nabla_s^2\eta+\epsilon^{-\eta}-s^{-\beta}\int_0^s s_z^{\beta}\epsilon^{-\eta_z}(\eta-\eta_z)^{-1/2}ds_z=0. \tag{13}$$

Later it will be shown that the introduction of typical values for the constants in the coefficient of $\nabla_s^2\eta$ renders that term negligible over practically the whole range of s in most cases. Thus the definition of the plasma as the region where ion and electron space charges are essentially equal receives additional justification. If now the sheath edge is defined as the surface at which this essential equality fails, it is left somewhat indefinite, thus requiring a detailed treatment of the plasma-sheath transition, which will be touched on later.

Dropping the first term, then, we have to solve the integral equation

$$\epsilon^{-\eta}-s^{-\beta}\int_0^s s_z^{\beta}\epsilon^{-\eta_z}(\eta-\eta_z)^{-1/2}ds_z=0. \tag{14}$$

The fact that the plasma equation can be reduced to this dimensionless form immediately enables us to draw the important conclusion that the potential distribution curve will be of the same shape irrespective of the particular values which the constants which originally entered the equation may have. To make the general curve fit any particular case it will simply be necessary to change the ordinate (η) scale according to the value of T_e and the abscissa (s) scale according to the tube dimensions and the thickness of the sheath on the wall when that is appreciable.

The solution of Eq. (14) is made possible by regarding η rather than s as the independent variable and putting successively

$$\eta=\rho^2,\qquad \eta_z=\rho_z^2,\qquad \rho_z=\rho\sin\theta. \tag{15}$$

This converts the equation to the form

$$s^{\beta}\epsilon^{-\rho^2}-\int_0^{\pi/2} s_z^{\beta}\epsilon^{-\rho_z^2}(ds_z/d\rho_z)d\theta=0. \tag{16}$$

If now s be expressed as a power series in ρ

$$s = a_0 + a_1\rho + a_2\rho^2 + \cdots \tag{17}$$

with s_z being expressed as the same function of ρ_z, it is obvious that the integrand itself can be expressed in a series

$$b_0 + b_1\rho_z + b_2\rho_z^2 + \cdots$$

in which each b is a function of certain of the a's. The substitutions indicated in Eq. (15) give the $(p+1)$term of this series the form $b_p\rho^p \sin^p \theta$. The integration limits in terms of θ are 0 and $\pi/2$, and this integral of $\sin^p \theta d\theta$ being given in the tables, the integral term of Eq. (16) comes out as a power series in ρ. The coefficients of this series can then be equated term by term with those in the expansion of the first term in Eq. (16) in order to evaluate a_0, a_1, a_2, etc. The boundary condition that $s = 0$, at $\rho = 0$ requires that $a_0 = 0$. In the final solutions it also turns out that $0 = a_2 = a_4 = a_8$, etc., so, for simplicity, these values will be assumed immediately in obtaining the actual solutions. We shall, therefore, use,

$$s = a_1\rho + a_3\rho^3 + a_5\rho^5 + \cdots . \tag{18}$$

The second boundary condition that the electric field at the origin be zero requires that $d\eta/ds = 0$, that is that $ds/d(\rho^2) = \infty$ there, a condition which the above solution is seen to satisfy also.

The cylindrical case may be used to illustrate the evaluation of the coefficients in detail. For this case $\beta = 1$ so that Eq. (16) becomes

$$s\epsilon^{-\rho^2} - \int_0^{\pi/2} s_z(ds_z/d\rho_z)\epsilon^{-\rho_z^2} d\theta = 0. \tag{19}$$

From Eq. (18)

$$s_z ds_z/d\rho_z = a_1^2\rho_z + 2(2a_1a_3)\rho_z^3 + 3(a_3^2 + 2a_1a_5)\rho_z^5 + \cdots .$$

Also

$$\epsilon^{-\rho_z^2} = 1 - \rho_z^2 + \rho_z^4/2! - \rho_z^6/3! + \cdots .$$

Replacing ρ_z^{2m+1} in the product by $\rho^{2m+1} \sin^{2m+1}\theta$, the coefficient of ρ^{2m+1} is seen to be

$$\begin{aligned}(-1)^m \sin^{2m+1}\theta \{ & a_1/m! - 2(2a_1a_3)/(m-1)! + 3(a_3^2 + 2a_1a_s)/(m-2)! - \cdots \\ & \pm k[a_k^2 + 2(a_1a_{k-1} + a_3a_{2k-3} + \cdots)]/(m-k+1)! + \cdots \text{to } k = m+1\}\end{aligned} \tag{20}$$

Now

$$\int_0^{\pi/2} \sin^{2m+1}\theta d\theta = \frac{1\cdot 2\cdot 4\cdot 6\cdots(2m)}{1\cdot 3\cdot 5\cdot 7\cdots(2m+1)}$$

so that, denoting the bracketed expression in (20) by σ_m the coefficient of ρ^{2m+1} in the integral is

$$(-1)^m\sigma_m \frac{1\cdot 2\cdot 4\cdot 6\cdots(2m)}{1\cdot 3\cdot 5\cdot 7\cdots(2m+1)}. \tag{21}$$

For the first term of Eq. (19) we have to form the product

$$(a_1\rho+a_3\rho^3+\cdots)(1-\rho^2+\rho^4/2!-\cdots).$$

The coefficient of ρ^{2m+1} here is

$$-(-1)^m[a_1/m!-a_3/(m-1)!+a_5/(m-2)!-\cdots\pm a_{2m+1}].$$

Setting the sum of this expression and (21) equal to zero we find that the condition to be satisfied by the a's is

$$a_1/m!-a_3/(m-1)!+a_5/(m-2)!-\cdots=\frac{1\cdot 2\cdot 4\cdot 6\cdots(2m)}{1\cdot 3\cdot 5\cdot 7\cdots(2m+1)}\times[a_1^2/m!-2(2a_1a_3)/(m-1)!+3(a_3^2+2a_1a_5)/(m-2)!-\cdots] \quad (22)$$

To evaluate the a's, m is set equal successively to 0, 1, 2, etc.
For $m=0$, $a_1=a_1^2$ whence $a_1=1$
For $m=1$, $1-a_3=(2/3)\ (1-4a_3)$, whence $a_3=-0.2$
For $m=2$, $1/2!+0.2+a_5=(8/15)[1/2!+0.8+3(0.04+2a_5)]$ whence $a_5=-0.026061$ and so on, giving for the solution

$$s=\eta^{1/2}(1-0.2\eta-0.026061\eta^2-0.0064894\eta^3-0.0019840\eta^4-0.00067937\eta^5-\mathit{0.000253}\eta^6-\mathit{0.000101}\eta^7-\cdots) \quad (23)$$

The coefficients of the last two terms are estimated from the trend of the preceding ones.

If this series be used to calculate the potential distribution, it is found that at a certain value of η, say η_0, $d\eta/ds$ passes through infinity and that thereafter s decreases with increasing η. We recognize in η_0 and in the corresponding value s_0 of s the extreme limit for the validity of the plasma solution since $d^2\eta/ds^2$ is infinite at s_0 and the $\nabla_s{}^2\eta$ term of Eq. (13) ceases to be negligible at some lesser value of s. By setting the derivative of Eq. (23) equal to zero, it is found by successive approximations that $\eta_0=1.155$, $s_0=0.7722$.

The results of a similar calculation for the plane case are given in Table IIa at the conclusion of this section.

Case 2—Long mean free path and ion generation constant throughout the plasma. This case differs from the previous one in that here

$$N_z=J \quad (24)$$

J being the number of ions generated per cm^3 per sec. throughout the plasma. Examination of the previous treatment will show that all the conversion factors and integral equations there used can be adapted to the new case by omitting the ϵ^{η_z} under the integral sign and substituting J/n_0 for λ everywhere. Thus, instead of Eqs. (12) and (13) we have

$$\alpha=J(m_p/2kT_e)^{1/2}/n_0 \quad (25)$$

$$(m_pJ^2/8\pi e^2n_0^3)\nabla_s{}^2\eta+\epsilon^{-\eta}-s^{-\beta}\int_0^s s_z{}^\beta(\eta-\eta_z)^{-1/2}ds_z=0. \quad (26)$$

This equation can be solved by the identical method used before and the resulting solutions having the same general character as the earlier ones are given in Table IIb.

Case 3. Short mean free paths and uniform ion generation. A. If the mean free paths are short compared to the dimensions of the plasma and the field strength is so small that the energy which an ion picks up between impacts with atoms is small compared to the thermal energy of the atoms, the ion drift velocity at r is made up of two components, one arising from diffusion, the other from the electric field,

$$v_z = -(D/n_p)dn_p/dr - (eD/kT_g)dV/dr$$

where T_g is the temperature of the gas and D is the diffusion constant for the ions into the gas. The fact that v_z has ceased to be a function of z will be indicated henceforward by dropping the subscript z. Since ion and electron densities are equal in the plasma, n_p can be substituted for N_e in Eq. (3), thus making it possible to eliminate n_p and dn_p and giving

$$v = -(eD/k)(T_e^{-1} + T_g^{-1})dV/dr$$

and since $T_g << T_e$, the diffusion component is negligible and

$$v = -(eD/kT_g)dV/dr. \tag{27}$$

Assuming D to be the same as the interdiffusion constant for the gas, we have [10] $D = 0.561 l_g \bar{v}$ where l_g is the mean free path and $\bar{v}$ the average speed of the atoms in their random motion. Expressing $\bar{v}$ in terms of T_g we have

$$v = 2(2kT_g/\pi m_p)^{1/2} = 1.597(kT_g/m_p)^{1/2}$$

Thus Eq. (27) can be put in the form

$$v = -0.895 l_g (kT_g m_p)^{-1/2} e dV/dr$$
$$= -qedV/dr \tag{28}$$

where

$$q = 0.895 l_g/(kT_g m_p)^{1/2}. \tag{29}$$

for convenience. Now $N_z = J$ can be brought from under the integral sign in Eq. (6) and since $v_z = v$ of Eq. (28) is independent of z this, too, comes out, giving

$$(kT_e/4\pi e^2 n_0)\nabla^2\eta + \epsilon^{-\eta} - (J/n_0 qkT_e)r^{-\beta}(d\eta/dr)^{-1}\int_0^r s_z^\beta ds_z = 0. \tag{30}$$

The substitutions used in Case 1 here lead to

$$\alpha^2 = J/n_0 qkT_e \tag{31}$$

and

$$(J/4\pi e^2 n_0^2 q)\nabla_s^2\eta + \epsilon^{-\eta} - s^{-\beta}(ds/d\eta)\int_0^s s_z^\beta ds_z = 0. \tag{32}$$

[10] Jeans, Dynamical Theory of Gases, 2nd Edition, §440.

As before the first term can be neglected for the present and the solution comes out very simply as

$$\epsilon^{-\eta}=1-s^2/2(\beta+1). \tag{33}$$

A marked difference between this solution and those previously obtained is that here $\eta_0=\infty$ although s_0 is finite, having the values

$$s_0=1.414, \quad 2.000, \quad 2.449\,; \quad \beta=0,1,2. \tag{34}$$

In this case then, the potential at which the present solution fails to be a good approximation must be very far indeed from η_0 although the corresponding value of s may not be very different from s_0.

B.[11] At pressures intermediate between those just discussed and those contemplated in Cases 1 and 2 the ion temperature will be determined less by the gas temperature and more by the energy acquired in a free path. Thus T_g in Eq. (29) must be replaced by T_p, the ion temperature, and this becomes proportional to dV/dr so that the drift velocity is proportional to the square root of the field strength.[12] A rough value for the drift velocity can be calculated as follows. Let the ions be accelerated by a uniform field X and let the average components of velocity in the direction of the field at the beginning and the end of a mean free path be v_0 and v_f respectively. The gas atoms are moving so slowly compared to the ions that they can be assumed to be at rest. The persistence of velocity for the resulting type of collision[13] is 0.5 so that

$$v_f=2v_0.$$

Denoting the mean free time of the ion between collisions by τ there are the additional relations

$$v_f=v_0+(eX/m_p)\tau$$
$$l_p=v_0+(eX/m_p)\tau^2$$

where l_p is the positive ion mean free path which is $2^{1/2}$ times the atom free path on account of the higher velocity. The last equation neglects the existence of a velocity component perpendicular to the field, but the effect of such a component in decreasing the progress made between collisions is small enough to be neglected in this rough calculation. Eliminating τ and

[11] Here, as later in Case 4B, the significance of the analysis is somewhat doubtful, first because of the small radial field strength near the potential maximum as mentioned below, and second, because in the very important class of cases pertaining to the positive column of an arc there is a uniform longitudinal field. Both in the free fall and the short path cases the ion motions can be simply resolved into components, leaving the theoretical results unaffected by this uniform field, but this resolution fails when, as here, the drift velocity is proportional to the square root of the field.

[12] P. M. Morse, Phys. Rev. **31**, 1003 (1928) uses a formula for ion mobility which is an adaptation of an expression for electron mobilities derived by K. T. Compton, Phys. Rev. **22**, 333 (1923). Morse does not, however, take into account the finite persistence of velocity of the ions, with the result that his consolidated numerical coefficient is 0.858 compared to 1.2 derived here, Eq. (35).

[13] The Dynamical Theory of Gases, J. H. Jeans, p. 279, 2nd edition.

v_f from these three equations, solving for v_0, and then finding $v=(v_0+v_f)/2$, the drift velocity, it is found that

$$v=1.2(eXl_p/m_p)^{1/2}$$

If in the present case we put

$$q'=1.2(l_p/m_p)^{1/2} \tag{35}$$

then

$$v=-q'(edV/dr)^{1/2}. \tag{36}$$

Substituting this expression for v and J for N_z in Eq. (6), and at the same time introducing s we have

$$\alpha=(J^2/n_0^2q'^2kT_e)^{1/3} \tag{37}$$

$$(1/4\pi e^2)(kTJ^4/n_0^7q'^4)^{1/3}\nabla_s^2\eta+\epsilon^{-\eta}-s^{\beta}(ds/d\eta)^{1/2}\int_0^s s_z^{\beta}ds_z=0 \tag{38}$$

the solution for which, neglecting the $\nabla_s^2\eta$ term as usual, is

$$\epsilon^{-\eta}=[1-2s^3/3(1+\beta)^2]^{1/2}. \tag{39}$$

In this case, too, $\eta_0=\infty$ while

$$s_0=1.145,\quad 1.816,\quad 2.378;\quad \text{for}\quad \beta=0,1,2. \tag{40}$$

A unique feature of this potential distribution is that η approaches zero at $s=0$ along a cubic rather than along a parabola, as in all the previous cases. It is probable, however, that this is of little significance because the approach of the electric field to zero in the neighborhood of the origin makes Eq. (28) rather than Eq. (36) the more applicable there.

Case 4. Short mean free paths and ion generation proportional to electron density. Here again two cases must be distinguished according to the length of the ion free paths relative to the electric field and gas temperature.

A. In the range of shorter free paths Eq. (30) is readily adapted by substituting $\lambda n_0\epsilon^{-\eta_z}$ for J, leading to the equations

$$(\lambda/4\pi e^2n_0q)\nabla_s^2\eta+\epsilon^{-\eta}-s^{-\beta}(ds/d\eta)\int_0^s \epsilon^{-\eta_z}s_z^{\beta}ds_z=0 \tag{42}$$

$$\alpha=(\lambda/qkT_e)^{1/2}. \tag{41}$$

The solution, when the first term is neglected as usual, can be easily transformed into the Bessel Equation of order $(1-\beta)/2$

$$d^2w/ds^2+(1/s)dw/ds+[1-(1-\beta)^2/4s^2)]w=0$$

by putting $w=s^{(\beta-1)/2}\epsilon^{-\eta}$. The solutions are given in Table IId.

The solution for the cylindrical case,

$$\epsilon^{-\eta}=J_0([\lambda/qkT_e]^{1/2}r),$$

is identical with Schottky's solution which he gives[14] as

$$n=n_0J_0([a/D_a]^{1/2}r)$$

[14] W. Schottky, Phys. Zeits. **25**, 635 (1924).

(the a being the present λ) if one assumes the ion temperature and mobility entering the expression for the ambipolar diffusion constant D_a to be small compared to the electron temperature and mobility, respectively. The new treatment is thought to have two advantages over the earlier treatment, first in that the present method, by including the Poisson term in the fundamental equation, will tell us when this term can no longer be neglected and how the problem can be handled beyond this point (see Section IV), and second, in that the radial ambipolarity of the diffusion is shown not to be essential to the solution. Schottky recognizes that his solution is, in fact, inconsistent with this idea, for its assumption that n is zero at the tube wall is equivalent to having the wall at an infinite negative potential, in which case the electron current would be zero. The essential requirement appears to be that the loss of electrons to the walls shall not be sufficient to disturb materially the M. D. of electrons in the plasma.

B. In the range of longer free paths Eqs. (37) and (38) can be readily adapted by writing λn_0 for J and putting $\epsilon^{-\eta_z}$ under the integral sign, giving

$$\alpha=(\lambda^2/q'^2kT)^{1/3} \tag{43}$$

$$(1/4\pi n_0e^2)(kT\lambda^4/q'^4)^{1/3}\nabla_s^2\eta+\epsilon^{-\eta}-s^\beta(ds/d\eta)^{1/2}\int_0^s\epsilon^{-\eta_z}s_zds_z=0. \tag{44}$$

With the omission of the first term the equation can be integrated once to assume the form

$$\epsilon^{-\eta}=\left\{1-2\int_0^s y^{-2\beta}\left[\int_0^y\epsilon^{-\eta_z}s_z^\beta ds_z\right]^2dy\right\}^{1/2}.$$

Solutions of the form

$$\epsilon^{-\eta}=1+a_1s+a_2s^2+\cdots$$

can then be obtained by substitution (see Table IId).

The theoretical potential distribution curves for a number of the cases so far discussed are shown in Figs. 1 and 2. Fig. 1 applies to cases where the ionization rate is proportional to electron density and thus covers Cases 1 and 4. Fig. 2, applying to a uniform ionization rate covers Cases 2 and 3. The abscissae are values of s/s_0 so that except for the small sheath thickness on the wall the radius of the tube is the unit of distance. The ordinates are values of η which can be converted readily to voltage by multiplication by $T_e/11{,}600$.

Corresponding to the transition from long to short ionic free paths the respective potential distribution curves should show a progressive change. Curves for cylinder-long path, cylinder-medium and cylinder-short path in both Figs. 1 and 2 show this transition at larger values of s/s_0 but the cylinder-medium-path curve falls out of line at the lower values of s/s_0. This undoubtedly arises from the fact mentioned before that the mobility law underlying the medium-path curves is not valid in small fields, leaving the cases involving the long and the very short ionic free paths as those capable of the most rigorous treatment.

Comparison with experiment. Later, in Sections VI and VII, the orthodox use of a fine wire probe will be in the main justified on the basis of the new theory. Anticipating these conclusions, use can be made here of some measurements made by Mr. T. J. Killian[15] on the potential distribution along the diameter of the anode arm of a mercury arc for two different pressures of mercury vapor, corresponding respectively to 1.4° and 38.6°C. Killian measured also the electron temperature in each case, which gives all the informa-

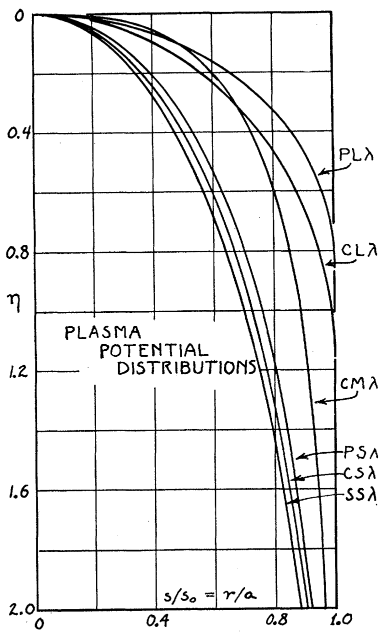

Fig. 1. Theoretical curves for rate of ionization proportional to electron density. For the meaning of curve designations see Table II.

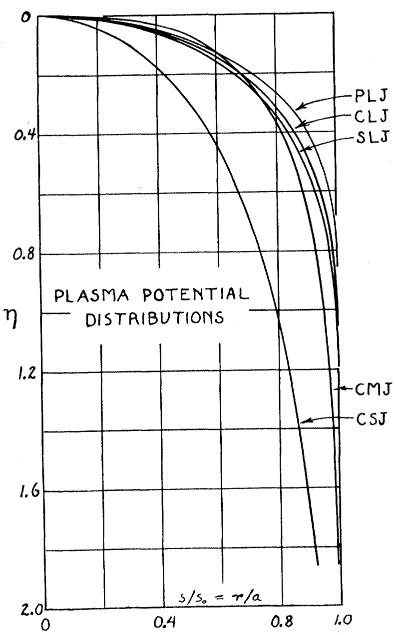

Fig. 2. Theoretical curves for rate of ionization constant. For the meaning of curve designations see Table II.

tion needed to check the observations against the theory. Actually, potentials were determined on both sides of the tube axis—but in view of the tube symmetry the points in Figs. 3 and 4 are plotted to one side only, those which have been transposed being indicated by crosses. The ordinates are, in each case, $11{,}600\ V/T_e = \eta$, V being the space potential in volts and 11,600 being e/k in degrees per volt. Along with the experimental points in each case is plotted one or two of the theoretical curves.

For the lowest gas pressure, Fig. 3, Curve $CL\lambda$ of Fig. 1 should give good agreement with the data, as it does. At some higher pressure Curve $CS\lambda$ may be more nearly approached, as appears in Fig. 4. In judging of the agree-

[15] T. J. Killian, forthcoming paper in The Physical Review.

ment between the theoretical and experimental results both the uncertainty of the potential measurements and also the absence of any adjustable constant in the theory must be considered. Since the transition from the $CL\lambda$ curve to the $CS\lambda$ curve should occur in the range where the free path length l_p is comparable with the tube radius a, the ratio l_p/a has been given on each figure. The value of l_p was calculated by multiplying Langmuir and Jones' value of 70 cm[16] for the electronic mean free path at 1 barye and 650°K by 1/4 and correcting to 300°K and the appropriate pressure. When the potential distribution curve assumes the short-free-path form as in Fig. 4 one should expect l_p/a to be rather less than 0.37, the value found. On

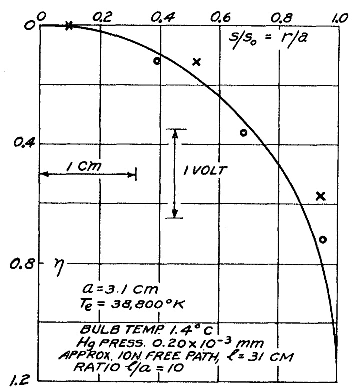

Fig. 3. Comparison of experimental and theoretical plasma potential distributions for long ion free paths.

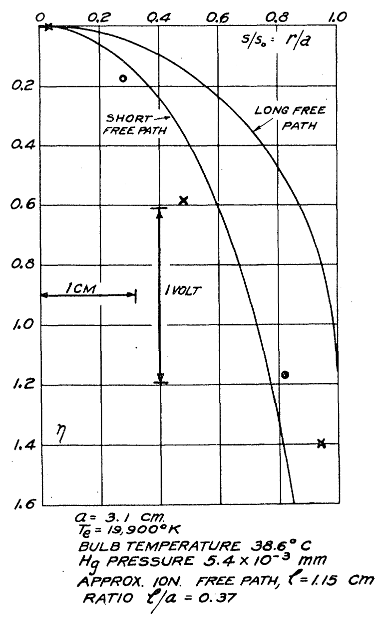

Fig. 4. Comparison of experimental and theoretical plasma potential distributions for short free paths.

the other hand free-path values—and even the concept itself—are somewhat uncertain. Probably the fact that the transition occurs in the general neighborhood expected is all that can be asked.

The plasma balance equation. A digression may be permissable at this point to inquire into another consequence of the theory so far developed. In every case there is a fixed numerical value s_0 which is the upper limit of the values which s can assume. Now s_0 has already been identified with the tube radius in Figs. 3 and 4 and this is justifiable to the extent that the sheath thickness is negligible. Hence, using Eq. (10), we can write approximately for any case

[16] I. Langmuir and H. A. Jones, Phys. Rev. **31**, 357 (1928).

$$a/s_0 = r/s = 1/\alpha. \tag{45}$$

To fix our ideas let us confine our attention to the low pressure λ cases covered by Table IIa. After introducing the value of α given by Eq. (12) this becomes

$$a\lambda = s_0(2k/m_p)^{1/2}T_e^{1/2} = 0.5522\times 10^6 s_0(T_e m_e/m_p)^{1/2}(\text{cm}\cdot\text{sec}^{-1}). \tag{46}$$

For a cylinder, $s_0 = 0.7722$ whence

$$a\lambda = 0.4264\times 10^6(T_e m_e/m_p)^{1/2}\ \text{cm}\cdot\text{sec}^{-1} \tag{47}$$

and in Hg $(m_p/m_e)^{1/2}$ is 605.5 so that for a cylindrical mercury discharge at low pressure

$$a\lambda = 703.1 T_e^{1/2}. \tag{48}$$

We are thus enabled to calculate most easily the average rate at which each electron in the positive column of an arc ionizes atoms. Using some of the results given by Langmuir and Mott-Smith for a tube of 1.6 cm radius with bulb at 15.5°C corresponding to a vapor pressure of 1.05 baryes we find the values given in the 5th column of Table I.

TABLE I.[17]

Run No.	Arc Current, (amps.)	T_e	I_p (ma · cm^{-2}	$\lambda\times 10^{-4}$ calc. by plasma balance equation	$\lambda\times 10^{-4}$ calc. by ion generation equation (See Section VIII)
34b	0.5	27,500	0.17	7.29	1.6
35a	1.0	29,000	0.25	7.48	2.1
35b	2.0	26,600	0.44	7.15	1.4
37a	8.0	19,500	2.29	6.13	0.21

Eq. (46) will be called the *plasma balance equation* for the low-pressure proportional-ionization cases because it states the adjustment of electron temperature to ion generation which just fits the plasma into the space available for it. It will be noted that Eq. (46) can be derived directly from the s vs. r equation appearing in Table IIa by inserting the limiting values s_0 and a. This same substitution when made in each of the other s vs. r equations of Table II yields the plasma balance equation appropriate to the particular case. Because of the finite thickness of the positive ion sheath on tube wall, a does not correspond exactly to s_0, and the necessary correction will be derived in Section IV.

Hitherto, the number of known relations in a positive column has been one less than the number of variables to be fixed. The plasma balance equation is important because it is the missing relation. This phase of the theory will be discussed in Section VIII.

[17] Langmuir and Mott-Smith, II, Table III. Examination of the original data reveals that the electron temperature in Run 34b is somewhat uncertain with the consequence that the value 1.6 for $\lambda\times 10^{-4}$ may, possibly, be as much as 30 percent too low.

The plasma balance equation can, obviously, say nothing concerning the actual causal dependence of λ on T_e and other quantities. This causal relationship might conceivably be incompatible with the plasma balance equation and in that case the formation of a plasma would be impossible. This relationship will be further discussed in Section VIII.

The first order correction to the plasma equation. The total unimportance of the first order correction to the plasma solution throughout the greater part of the plasma becomes evident in Section IV but it will enter into any exact plasma-sheath transition calculation. The evaluation of this correction can be carried out by a method similar to that employed later in Section V. In long free path cases it will probably be advantageous to put $s_1+\delta s_1$ for s in the complete equation, s_1 being the solution already obtained and δs_1 being the correction desired. In short free path cases, on the other hand it appears that the useful substitution will be $\epsilon^{-\eta}=\epsilon^{-\eta_1}+\delta(\epsilon^{-\eta_1})$.

III. Positive Ion Currents in the Plasma

Pursuing the views advanced in the preceding sections one readily concludes that the positive ion current at the center of a discharge tube is zero, that at any other point it is radial (except for a longitudinal component in the positive column arising from the constant longitudinal gradient), and that it increases continually up to the sheath edge. If we abandon the approximation that each newly-created ion starts from rest, and make the more reasonable assumption that the newly-created ions have the same temperature as the gas—namely T_g, it is seen that there is a real, if small random ion current even at the center of the tube, but that does not concern us at present. The important thing to note here is that the ion currents en route to the walls are almost certain to remain unobservable by any direct measurements, for the introduction of any electrode which is apparently suitable for the purpose will itself so distort the plasma fields and ion motions as to destroy completely the effect sought. The result is that in seeking experimental agreement with the theory, we are limited to the observation of the ion current density at the tube wall. This current can be readily expressed in terms of the variables which have already been introduced. In all of the cases already discussed the number of ions reaching each unit of wall area in one second is the number generated per second in the volume subtended by that wall area. Thus

$$a^\beta I_p=\int_0^a eN_r r^\beta dr$$

where I_p is the positive ion current density at the wall. Introducing s here as before we have

$$I_p=aes_0^{-\beta-1}\int_0^{s_0} N_s s^\beta ds \tag{49}$$

or where $\rho(=\eta^{1/2})$ is the independent variable

$$I_p=aes_0^{-\beta-1}\int_0^{\eta_0^{1/2}} N_\rho s^\beta (ds/d\rho)d\rho. \tag{50}$$

Comparison of this with the integrand appearing in the integral equations of Case 1 above shows an identity of form. The only difference lies in the variable of integration. Whereas for each series term a $\sin^m\theta d\theta$ was integrated before, here a $\rho^m d\rho$ is to be integrated. Thus with ionization proportional to electron density

$$I_p = aes_0^{-\beta-1}\lambda n_0 \sum_0^\infty (-1)^m \sigma_m \eta_0^{m+(\beta+1)/2}/(2m+\beta+1)$$

TABLE II. *Plasma solutions and values of s_0, η_0, h_0. Designation of cases.*

Form of plasma	Ion free paths	Ion generation
P plane	L long	J constant throughout ($N_z = J$)
C cylindrical	M medium	λ proportional to electron density ($N_z = \lambda n_z$)
S spherical	S short	

TABLE IIa.[18] *Plasma solutions for Case 1 in text.*
Plasma equation: $s = G\eta^{1/2}(1+g_1\eta+g_2\eta^2+\cdots)$, $\alpha = s/r = \lambda(m_p/2kT_e)^{1/2}$

Constant	Case: $PL\lambda$	$CL\lambda$[19]	$SL\lambda$
G	$2/\pi$	1	$4/\pi$
g_1	−0.333333	−0.200000	−0.142857
g_2	−0.0333333	−0.0260260	−0.017722
g_3	−0.00476190	−0.00648941	—
g_4	$-0.0_3661376$	−0.0019840	—
g_5	-0.0_484181	-0.0_36794	—
g_6	-0.0_59715	*-0.0_3253*	—
g_7	—	*-0.0_310*	—
s_0	0.4046	0.7722	—
η_0	0.8540	1.155	—
h_0	0.8513	0.3500	—

TABLE IIb.[18] *Plasma solutions for Case 2 in text.*
Plasma equation: $s = G\eta^{1/2}(1+g_1\eta+g_2\eta^2+\cdots)$, $\alpha = s/r = (J/n_0)\ (m_p/2kT_e)^{1/2}$

Constant	Case: PLJ	CLJ	SLJ
G	$2/\pi$	1	$4/\pi$
g_1	−2/3	−0.600000	−0.571429
g_2	+4/15	+0.238182	+0.227712
g_3	−8/105	−0.068573	−0.0661527
g_4	—	+0.015303	+0.0147939
g_5	—	−0.0027721	−0.00265902
g_6	$[\pm 2^p/1\cdot 3\cdot 5\cdot\cdots\cdot(2p+1)]$	$+0.0_34242$	*$+0.0_3395$*
g_7	—	*-0.0_456*	*-0.0_450*
g_8	—	*$+0.0_565$*	*$+0.0_555$*
s_0	0.3443	0.5828	0.7707
η_0	0.9244	1.0542	1.1950
h_0	1.0000	0.5000	0.3333
s_0[20]	0.38	0.638	0.818
η_0[20]	0.943	1.26	1.50
h_0[20]	1.000	0.500	0.333

[18] In the numerical coefficients the subscripts have the meaning indicated by $0.0_47 = 0.00007$.

[19] The coefficients in italics were obtained by extrapolation from the previous ones.

[20] These solutions were obtained by an approximate method. (See Section VI.)

TABLE IIc. *Plasma solutions for Case 3 in text.*

Cases (P, C, S) SJ:

Plasma equation: $s^2=2(1+\beta)\ (1-\epsilon^{-\eta})$

$q=0.895\ l_p(kT_g m_p)^{-1/2}$ $\alpha=s/r=(J/n_0 qkT_e)^{1/2}$

Constant	Case: PSJ	CSJ	SSJ
β	0	1	2
s_0	1.414	2.000	2.449
η_0	∞	∞	∞
h_0	1.0000	0.5000	0.3333

Cases (P, C, S) MJ:

Plasma equation: $s^3=(3/2)\ (1+\beta^2)\ (1-\epsilon^{-2\eta})$

$q'=1.2\ (l_p/m_p)^{1/2}$ $\alpha=s/r=(J^2/n_0^2 q'^2 kT_e)^{1/2}$

Constant	Case: PMJ	CMJ	SMJ
β	0	1	2
s_0	1.145	1.816	2.378
η_0	∞	∞	∞
h_0	1.000	0.500	0.333

TABLE IId. *Plasma solutions for Case 4 in text.*

Cases (P, C, S) $S\lambda$:

$q=0.895\ l_p(kT_g m_p)^{-1/2}$ $\alpha=s/r=(\lambda/qkT_e)^{1/2}$

	Case: $PS\lambda$	$CS\lambda$	$SS\lambda$
Plasma equation	$\epsilon^{-\eta}=\cos s$	$\epsilon^{-\eta}=J_0(s)$	$\epsilon^{-\eta}=(\sin s)/s$
s_0	1.571	2.405	3.142
η_0	∞	∞	∞
h_0	0.6366	0.2159	0.1013

Cases (P, C, S) $M\lambda$;

$q'=1.2\ (l_p/m_p)^{1/2}$ $\alpha=s/r=(\lambda^2/q'^2 kT_e)^{1/3}$

Plasma solution for case $CM\lambda$ only:

$$\epsilon^{-\eta}=1-(1/12)s^2-0.0_3 69444 s^2-0.0_4 79089 s^9-0.0_6 32511 s^{12}-0.0_6 16513 s^{15}-\cdots$$

$s_0=2.154$ $\eta_0=\infty$ $h_0=\cdots$

σ_m being defined for the cylindrical case just above (20) and for any case as the coefficient of the $(m+1)^{st}$ term of the series which occurs as integrand in the course of the original solution. In this way I_p is seen to involve a new dimensionless constant which can be chosen to correspond to current density just as s_0 corresponds to distance and η_0 to voltage

$$h_0=s_0^{-\beta-1}\sum_0^\infty(-1)^m\sigma_m\eta_0^{m+(\beta+1)/2}/(2m+\beta+1)$$

so that

$$I_p=h_0 ean_0\lambda \tag{51}$$

which will be called an *ion current equation*. In Cases 2 and 3 ($N_s=J$, constant) it is readily seen that Eq. (49) can be integrated directly giving the ion current equation

$$I_p=h_0 eaJ, \tag{52}$$

where $h_0 = 1/(1+\beta)$. In Case 4 when the mean free paths are short Eq. (49) again applies. The values of h_0 for this and other cases are given in Table II. Another significance may be attributed to h_0 in Cases 1 and 4, namely that

$$(1+\beta)h_0 = \text{average value of } \epsilon^{-\eta} \tag{53}$$

taken throughout the plasma. Thus in the cylindrical case the total number of electrons N_e per unit length of tube is given by

$$N_e = 2h_0 n_0 \pi a^2 \tag{54}$$

whence the total ion current

$$2\pi a I_p = N_e e\lambda = 2\pi a^2 h_0 n_0 e\lambda$$

which checks Eq. (51). The plane and spherical cases work out similarly.

Both when the ionization rate is uniform and when it is proportional to electron density, a simultaneous solution of the plasma balance equation and the equations involving ion current gives in long free path cases

$$I_p = s_0 h_0 e n_0 (2kT_e/m_p)^{1/2} \tag{55A}$$

$$= 8.787 \times 10^{-14} s_0 h_0 n_0 (T_e m_e/m_p)^{1/2} \text{ amp}\cdot\text{cm}^{-2} \tag{55B}$$

and in short free path cases

$$I_p = s_0^2 h_0 e n_0 q k T_e/a \tag{55.5}$$

These types of equation will also be called *ion current equations*.

It will be noted that the ion current equation affords a method by which the electron density at the potential maximum in the plasma may be determined. Solving for n_0 in Case $CL\lambda$ Eq. (55B) becomes

$$n_0 = 4.21 \times 10^{13} (m_p/m_e)^{1/2} I_p T_e^{-1/2}.$$

This method was checked experimentally against the electron and ion densities determined from the positive and negative branches of the volt-ampere characteristic of a fine probe as discussed in Sections VI and VII. For this purpose three runs were made with the tube which Killian used. The results are shown in column 5 of Table III.

TABLE III

Arc Current, (amp.)	Bulb temp.	Electron temp. (°K)	$I_p \times 10^4$ (amps/cm²)	$n_0 \times 10^{-10}$		
				by Eq. (55)	by $i^2 - V$ plot	
					for ions	for electrons
5.0	15.5°C	20,600	4.52	8.04	16.3	8.07
5.0	0°C	27,800	4.57	7.0	16.4	6.2
1.0	15.5°C	23,300	0.84	1.40	3.12	1.54

The value of I_p for the 1.0 amp. case has been corrected for the wall sheath thickness in accordance with Section VI so that I_p refers to the sheath edge as it should rather than the tube wall. The correction amounts to 4.5 percent in this case.

The potential which a non-conducting tube wall bounding the discharge assumes can now be calculated. Such a wall becomes sufficiently negative so that all the electrons are turned back in the sheath except the small number required to neutralize the positive ion current. By kinetic theory the electron current density reaching the surface of the wall is

$$I_n = en_0\epsilon^{-\eta_w}(kT_e/2\pi m_e)^{1/2}$$

where $\eta_w = -11{,}600\ V_w/T_e$, V_w being the wall voltage with respect to the plasma potential maximum. But if the reflection coefficient of the wall is ρ_e the current actually collected[21] will be only

$$I_n' = (1-\rho_e)I_n.$$

Equating I'_n to I_p as given by Eq. (55) it is found that

$$\eta_w = \ln\left[(m_p/m_e)^{1/2}/2\pi^{1/2}h_0s_0\right] + \ln(1-\rho_e) \tag{56}$$

which in Case $CL\lambda$ and Hg gives

$$\eta_w = 6.45 + \ln(1-\rho_e). \tag{57}$$

Both the voltage on the axis of the positive column and T_e can be determined from the semi-log plot of the volt-ampere characteristic of a fine axial wire. Also, the voltage at which a collector on the tube wall opposite the wire takes zero current is readily measured. The difference of the two voltages multiplied by $11{,}600/T_e$ gives η_w. Referring to the original data for collectors F[22] and H upon which Tables III and XIV of L and M–S Parts II and IV are based it is found that $\eta_w = 5.9 \pm 0.2$ for Runs 34b to 37a. In a special test with a positive column of twice the diameter used there it was found that η_w was 6.13 for one pair and 5.73 for another pair of electrodes, again giving an average of 5.9. A reasonable value for ρ_e is thought to be 0.15 leading to the theoretical value 6.3 for η_w. The agreement with experiment is considered to be good particularly in view of the effect of the collector support.[22]

This is, perhaps, as good a place as any to point out that in many cases the plasma theory as so far developed applies to the plasma in the neighborhood of an anode almost as well as to electrodes drawing less electron current. The only necessary condition is that the electron current density reaching the anode shall be small compared to the random current density at the sheath edge. It has already been pointed out[23] that without violating

[21] It might reasonably be expected that the constant drain of fast electrons by the walls would cause a deficiency of high velocity electrons. No such effect has been found at small distances from a wall or from an electrode considerably less negative than the wall. The explanation of this phenomenon is not known. Were it not for this mechanism which rapidly reestablishes a M.D., slow electrons would accumulate indefinitely at a potential maximum and build up the ionization density to a high value, escaping finally either by recombination or by setting up oscillations. Perhaps the unknown mechanism involves just such oscillations.

[22] The ideal collector would be axial, but the fact that F is not axial is unimportant compared to the errors introduced by the lead-in structure in lowering the plasma potential at the supported end of the collector.

[23] L and M-S., Part IV, pp. 766, 767.

this condition anodes of reasonable area are capable of collecting the full arc current.

We are now in a position to discuss the magnitude of the ion currents received by two equal plane collectors arranged back to back in the positive column of a discharge so that one electrode J faces the cathode, the other K, the anode. Fig. 5 gives the volt-ampere characteristics [24] in Hg vapor saturated at 16°C of two such electrodes. Each was allowed to float while the characteristic of the other was being taken. Both were square plates 0.95 cm on a side spaced 0.08 cm apart in a tube of 3.2 cm diameter carrying

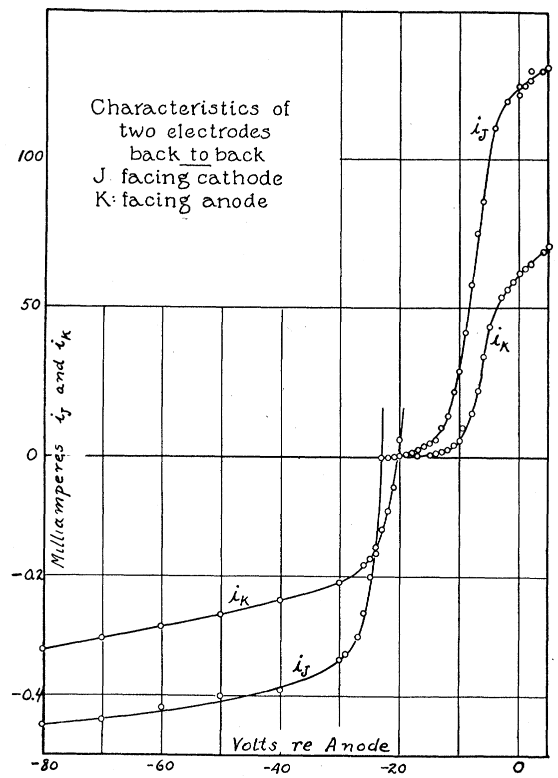

Fig. 5. Note change in current scale at zero.

an arc current of 0.60 amps. The arc gradient was 0.24 v·cm^{-1}. It is to be noted that not only did J receive the larger electron current at positive voltages but also the larger ion current at negative voltages. K, which was completely exposed to any longitudinal drift of the ions toward the cathode captured fewer ions than J which, on the random-ion-current theory, could only receive the random component of the ion current. The present theory, however, explains this quite readily. We observe that the larger electron current to J arises from the very appreciable ratio of drift to random elec-

[24] From unpublished data taken by I. Langmuir and H. M. Mott-Smith, Jr.

tron current density. This not only increases the density of electrons in the plasma near J above normal but decreases the density near K below normal. Positive space charge arising from an excess of ions if present would set up potential differences tending to wipe this out, but such an excess has little tendency to occur because the generation of the positive ions is it self proportional to the electron density. Thus the density of ionization as a whole is less near K than near J, and the smaller ion current to K than to J at negative electrode voltages is to be expected. The relative difference between the two ion currents is smaller than between the two electron currents and this probably arises from the effect of the longitudinal field in the column which tends to increase the volume from which ions can reach K and to decrease the volume contributing ions to J. (This longitudinal field of 0.24 $v\cdot cm^{-1}$ is small compared to the average radial plasma field. The electron temperature was approximately 30,000°K. In case $CL\lambda$ $\eta_0=1.155$ whence the potential difference between tube axis and plasma edge is 1.155(30,000/11,600) = 3v, a difference equal to that found in 12.5 cm along the axis. It is interesting to note that the displacement of the curves shows that the presence of the dual electrode causes a potential difference of some 3v in less than a millimeter distance.)

The ratio of random electron current to drift current. The quotient of the density of the ion current to the wall in a positive column and the arc current itself leads directly to the ratio of random electron current density I_e, to drift current density, I_x. From kinetic theory

$$I_e = en_0(kT_e/2\pi m_e)^{1/2}$$

at the tube axis. Combining this with Eq. (55A) we have

$$I_e/I_p = (m_p/m_e)^{1/2}/2s_0h_0\pi^{1/2}.$$

The average drift current over the tube cross-section is $i_B/\pi a^2$ and taking account of the actual distribution by using Eq. (53) we have

$$I_x = i_B/2h_0\pi a^2$$

at the tube axis. It follows then, that

$$I_e/I_x = (\pi^{1/2}/s_0)(m_p/m_e)^{1/2}a^2I_p i_B/.. \qquad (57.5)$$

The approximate equality of I_p for electrodes of various sizes and in various positions. One of the experimental facts which favored the random ion current belief was the observation that at low gas pressures electrodes variously disposed in a tube excited from a hot cathode received ion current densities (corrected to sheath area) which differed usually less than 2 to 1 in ratio even though one electrode might be in the center, the other on the wall of the bulb. Experiment thus shows that the shape of the plasma boundary at a certain place has no great effect on the ion current density to that place. The theory indicates the same result, for Eq. (55A) shows that for plasmas of different sizes and varying in shape from the plane to the spherical but all having the same maximum ionization intensity, I_p is proportional to

s_0h_0. From Table IIb it is seen that the variation involved is only from 0.34 to 0.26.

The average ion velocity at s_0. Later it will be convenient to know the average velocity of the ions as they pass from the plasma into the sheath. The η equivalent to this velocity will be denoted by $\bar{\eta}_0$ and can be calculated readily. We have

$$I_p = ne\bar{v}_p, \quad n = n_0\epsilon^{-\eta_0}, \quad \bar{v}_p = (2kT_e\bar{\eta}_0/m_p)^{1/2}$$

where $\bar{v}_p$ is the mean ion velocity. Combining these, we find

$$I_p = en_0\epsilon^{-\eta_0}(2kT_e\bar{\eta}_0/m_p)^{1/2}$$

and eliminating I_p with Eq. (55A) it is found that

$$\bar{\eta}_0 = h_0^2s_0^2\epsilon^{2\eta_0} \tag{58}$$

in long free path cases. In Case $CL\lambda\bar{\eta}_0 = 0.7359$.

The magnitude of the neglected term in the plasma equation. Eq. (55) makes it possible to calculate the coefficient of the neglected term in Eq. (11) or (13) which will be denoted by A, for any given case. Using Eqs. (45) and (55) to eliminate α and n_0 respectively, we have

$$A = h_0s_0^3(2kT_e)^{3/2}/8\pi ea^2m_p^{1/2}I_p \tag{59A}$$

for long free path cases. Introducing the numerical values of known constants and expressing I_p in amperes·cm^{-2} this becomes

$$A = 4.210\times 10^{-12}h_0s_0^3(m_e/m_p)^{1/2}T_e^{3/2}/a^2I_p \tag{59B}$$

and for Case $CL\lambda$ in mercury

$$A = 1.119\times 10^{-15}T_e^{3/2}/a^2I_p. \tag{60}$$

Thus in Run 37a[25] where the arc current was 8.0 amps, $T_e = 19{,}500°$ $a = 1.6$ cm, and $I_p = 2.29\times 10^{-3}$, and we find $A = 5.19\times 10^{-7}$. In Run 34b at the other extreme of this group of runs the arc current was 0.5 amp. In this case $T_e = 27{,}500°$ and $I_p = 0.17\times 10^{-3}$, whence $A = 1.17\times 10^{-5}$. From Table IIa it is readily found that $\nabla_s^2\eta$ at the potential maximum has its largest value 4.94 in the plane case. We thus confirm the smallness of $A\nabla_s^2\eta$ at the origin relative to the other terms of Eq. (13). How far out this term may be neglected will be discussed in the next section.

IV The Sheath Edge and Sheath

The limit of validity of the plasma equation. We have already noted that the approximation which gives the plasma equation fails at some value of s less than s_0 because of the fact that $\Delta_s^2\eta$ becomes infinite at s_0. Physically of course, this is just the type of development that is necessary to give a sheath. Accordingly, the problem of carrying the solution up to and past s_0 is the problem of the sheath edge and sheath.

[25] L. and M-S. Table III, Part II.

Only the low pressure case will be analyzed in this section and that comparatively roughly because the plasma-sheath transition is inherently more complicated than either plasma or sheath alone. The high pressure case might be covered by the assumption that the ion velocity was proportional to the electric field, but there is a wide range of pressures for which the ion may drift in the plasma yet fall freely through the greater part of a thin sheath.

The first question which arises concerns the point at which the plasma solution should be abandoned. We may, without definitely committing ourselves for the present, say that we shall have to do this when the Poisson term, neglected in the plasma solution, becomes equal to a certain fractional part, ϕ, of either of the other two terms, that is when

$$A d^2\eta/ds^2 = \phi\epsilon^{-\eta}. \tag{61}$$

Here $\nabla_s{}^2\eta$ has been replaced by $d^2\eta/ds^2$ since the only sheaths which will be considered are those which are thin compared to the radius of the tube.

The point on the plasma solution at which this relation is satisfied is designated on Fig. 6 by η_ϕ, s_ϕ. The coordinates s_ϕ and η_ϕ are most easily

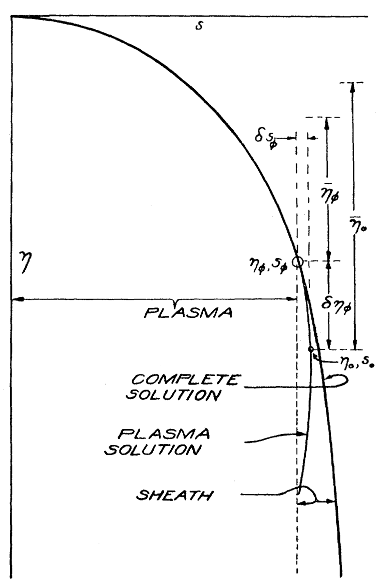

Fig. 6. The relation of the plasma and complete solutions of the plasma-sheath integral equation.

expressed by their differences δs_ϕ and $\delta\eta_\phi$ from s_0 and η_0 respectively. In order to evaluate δs_ϕ and $\delta\eta_\phi$, $d^2\eta/ds^2$ must first be expressed with η as the independent variable. We have

$$d\eta/ds = (ds/d\eta)^{-1}, \quad d^2\eta/ds^2 = -(d^2s/d\eta^2)(ds/d\eta)^{-3} \tag{62}$$

A Taylor Expansion gives

$$\delta s = \cdot (\tfrac{1}{2}) s_0'' \delta\eta^2 [1 + \delta\eta(\cdot\cdot\cdot)] \tag{63}$$

where $\delta s = s - s_0$, $\delta\eta = \eta - \eta_0$, s_0'' denotes the value of $d^2s/d\eta^2$ at s_0, and the term involving s_0' is absent since $s_0' = 0$ by definition of s_0. Neglecting all but the lowest order terms in $\delta\eta$ we also have

$$ds/d\eta = s' = s_0''\delta\eta$$

$$d^2s/d\eta^2 = s'' = s_0''.$$

Substitution in Eq. (62) gives

$$d\eta/ds = s_0''^{-1}\delta\eta^{-1} \tag{64}$$

$$d^2\eta/ds^2 = -s_0''^{-2}\delta\eta^{-3} \tag{65}$$

In addition,

$$\epsilon^{-\eta} = \epsilon^{-\eta_0}$$

to the same degree of approximation.

Then $\delta\eta_\phi$, the value of $-\delta\eta$ for which Eq. (61) is satisfied, (the minus sign being inserted simply to make $\delta\eta_\phi$ intrinsically positive), is found through substituting the last two equations into Eq. (61). It is found to be given by

$$\delta\eta_0{}^3 = A\epsilon^{\eta_0}/s_0''^2\phi \tag{66}$$

The corresponding value, δs_ϕ, of $-\delta s$ can be found from Eq. (63). In case $CL\lambda$

$$s_0'' = -0.635$$

so that using A from Eq. (60).

$$\delta\eta_\phi = 2.064 \times 10^{-5} T_e{}^{1/2} a^{-2/3} I_p{}^{-1/3} \phi^{-1/3}$$

and

$$\delta s_\phi = 1.35 \times 10^{-10} T_e a^{-4/3} I_p{}^{-2/3} \phi^{-2/3}$$

for mercury.

As example we may take the two runs already used. Noting that $\epsilon^{\eta_0}/s_0''^2 = 7.88$ in Case $CL\lambda$, the values of A already found substituted in Eq. (66) give immediately

$$\delta\eta_\phi = 0.0160/\phi^{1/3} \quad \text{and} \quad \delta\eta_\phi = 0.045/\phi^{1/3}$$

for the 8 amp. and 0.5 amp. arcs respectively. Thus for $\phi = 0.05$ we have

$$\left.\begin{array}{l}\delta\eta_\phi = 0.045 \\ \delta s_\phi = 6.4 \times 10^{-4}\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l}\delta\eta_\phi = 0.122 \\ \delta s_\phi = 47 \times 10^{-4}.\end{array}\right.$$

The extension of the general plasma-sheath equation, Eq. (4), past η_ϕ, s_ϕ is very much more complicated than the solution of the plasma itself. In addition, the only theoretical result which finds application at present is the sheath thickness, and that occurs only as the correction to the discharge tube radius which gives the plasma radius. Accordingly, the present treatment of the problem will be only approximate. The solution involves three simplifications of the general plasma-sheath equation. The first consists in the replacement of $\Delta^2 V$ by d^2V/dr^2 as already noted in connection with Eq. (61). The second concerns the ion charge term of the general equation and results in the elimination of the integral. The third consists in dropping the electron charge term when it becomes negligible.

Regarding the second simplification it is to be noted that in λ cases the low and rapidly decreasing electron density in the wall sheath suppresses the generation of ions there, and in any case, the thinness of the sheath will render its ion contribution negligible compared to that from the plasma. Hence the ion current density through the sheath can be assumed to be constant and to have the value calculated for s_ϕ. This value is equal to I_p, the value at s_0, to within a quantity of the order of δs_ϕ or $\delta\eta_\phi^2$. To calculate the ion space charge the velocities of the ions must also be known. The plasma equation enables us to find the single velocity which is equivalent as regards space charge to the actual distribution of ion velocities at s_ϕ. The same steps that yield Eq. (58) lead to

$$\bar{\eta}_\phi = s_0^2 h_0^2 \epsilon^{2\eta_\phi} \tag{67}$$

where η_ϕ is the η corresponding to the equivalent single velocity. A serious difficulty enters here, for it immediately appears that $\bar{\eta}_\phi$ is characteristic of η_ϕ, s_ϕ only. For example, $\bar{\eta}_0$, corresponding to the single equivalent velocity of the stream at η_0, s_0, is not $\bar{\eta}_\phi+\delta\eta_\phi$, but by combining Eqs. (57) and (67) it is found that

$$\bar{\eta}_0 = \bar{\eta}_\phi + 2s_0^2 h_0^2 \epsilon^{2\eta_0}\delta\eta_\phi = \bar{\eta}_\phi + 1.47\delta\eta_\phi \tag{68}$$

in Case $CL\lambda$. This is indicated qualitatively in Fig. 6. If $\bar{\eta}_\phi$ and $\bar{\eta}_0$ represented mean potentials of origin of the ions, the expected relation would have been correct. Actually the average here encountered is the reciprocal square of the mean reciprocal square root of $\eta_\phi-\eta$ and $\eta_0-\eta$ respectively. In the present solution, however, we shall treat η_ϕ like a simple mean value rather than the complicated average which it really represents. The positive ion space charge in the sheath then becomes $(m_p/2kT_\epsilon\eta_s)^{1/2}I_p$ where $\eta_s=\eta-\eta_\phi+\bar{\eta}_\phi$. By Eq. (55A) this becomes $en_0s_0h_0\eta_s^{-1/2}$ and using this in place of the integral term in Eq. (4) we have

$$d^2V/dr^2 - 4\pi e n_0 \epsilon^{-\eta} + 4\pi e n_0 s_0 h_0 \eta_s^{-1/2} = 0$$

which can be reduced to

$$A d^2\eta/ds^2 + \epsilon^{-\eta} - s_0 h_0 \eta_s^{-1/2} = 0, \tag{69}$$

A being given by Eq. (59A).

In order to bring this equation into a form which has already been treated by Langmuir[26] it is convenient to substitute for $s(=\alpha r)$ a new independent variable ξ, which is defined by the following series of equations

[26] I. Langmuir, Phys. Rev. **33**, 976–980 (1929) Eq. (68).

$$d\xi = dr/x_p \tag{70}$$

$$I_p = ((2)^{1/2}/9\pi)(e/m_p)^{1/2} W_e^{3/2}/x_p^2 \tag{71}$$

$$W_e = kT_e/e = T_e/11{,}600 \tag{72}$$

$d\xi$ rather than ξ being defined since the origin of ξ's is better left indefinite for the present. The physical significance of W_e and x_p are evident from their relation to electron temperature and the simple space charge equation respectively. As the new variable appears in the first term only of Eq.(69), the substitution is most easily carried out by means of the relations

$$ds^2 = \alpha^2 dr^2 = \alpha^2 x_p^2 d\xi^2$$

followed by substitutions for x_p^2 and W_e from Eqs. (71) and (72). It is found that

$$A/ds^2 = 9s_0h_0/4d\xi^2 \tag{73}$$

so that

$$d^2\eta/d\xi^2 = (4/9)(\eta_s^{-1/2} - \epsilon^{-\eta}/s_0h_0)$$

Integrating once, it is found that

$$(d\eta/d\xi)^2 - (d\eta/d\xi)_\phi^2 = (16/9)\eta_s^{1/2} - \bar{\eta}_\phi^{1/2} - (1 - \epsilon^{-\eta_s + \bar{\eta}_\phi})\epsilon^{-\eta_\phi}/2s_0h_0].$$

Evaluating $(d\eta/d\xi)_\phi$ by means of Eqs. (73), (61), (64), and (65) we find

$$(d\eta/d\xi)_\phi^2 = (4\epsilon^{-\eta_\phi}/9s_0h_0)\phi\delta\eta_\phi.$$

In order to proceed, a further approximation now becomes necessary. The quantities η_ϕ and $\bar{\eta}_\phi$ must be replaced by η_0 and $\bar{\eta}_0$ throughout both the above equations. Thus η_s which was equivalent to $\eta - \eta_\phi + \bar{\eta}_\phi$ is redefined as $\eta - \eta_0 + \eta_0$. Naturally, $d\eta_s$ can be written for $d\eta$ in the derivatives. Making use of Eq. (58) to eliminate s_0h_0, the differential equation then becomes

$$(d\eta_s/d\xi)^2 - (4/9\bar{\eta}_0^{1/2})\phi\delta\eta_\phi = (16/9)\left[\eta_s^{1/2} - \bar{\eta}_0^{1/2} - (1 - \epsilon^{-\eta_s + \bar{\eta}_0})/2\bar{\eta}_0^{1/2}\right] \tag{74}$$

The right member now vanishes when $\eta = \eta_0$ (i.e. $\eta_s = \bar{\eta}_0$). The left member then tells us that the electric field at η_0, s_0 has the value which we had supposed it to have at η_ϕ, s_ϕ. Thus the effect of the last approximation has been to transfer the beginning of the sheath solution to η_0, s_0 as regards ion and electron space charge concentrations and ion velocities but to retain the correct value of initial electric field.

The next integration has to be performed in three steps, namely; *A*. By expansion in a Taylor Series at $\eta_s = \bar{\eta}_0$, giving

$$\xi = (3/2)\bar{\eta}_0^{3/4}(\bar{\eta}_0 - \tfrac{1}{2})^{-1/2}\left\{\ln\frac{\eta_s - \bar{\eta}_0}{\bar{\eta}_0} - (1/2)\ \ln\ \frac{\phi\delta\eta_\phi}{4\bar{\eta}_0(\bar{\eta}_0 - \frac{1}{2})}\right\}$$

for the range $\bar{\eta}_0 < \eta_s \leqq 2\bar{\eta}_0$. Putting $\bar{\eta}_0 = 0.736$ for Case *CL* λ the value of ξ for $\eta_s = 2\bar{\eta}_0$ becomes

$$\xi\Big]_{1.472} = -0.449 - 1.23\ln(\phi\delta\eta_\phi).$$

B. By quadrature of Eq. (74) neglecting $(4/9\bar{\eta}_0^{1/2})\phi\delta\eta_\phi$, giving for Case $CL\lambda$

η_s	1.472	2.944	4.416
$\xi-\xi]_{1.472}$	0	2.82	4.40

in the range $2\bar{\eta}_0 \leqq \eta_s \leqq 6\bar{\eta}_0$.

C. By integration of Eq. (74) neglecting the exponential (electron space charge) giving, finally

$$\xi=(\eta_s^{1/2}+2.88)(\eta_s^{1/2}-1.44)^{1/2}-0.10-1.23\ln(\phi\delta\eta_\phi) \tag{75}$$

for Case $CL\lambda$ when $\eta_s \geq 6\bar{\eta}_0$, ξ being measured from s_0.

In Run 37a already cited, it was found that $\delta\eta_\phi=0.045$ when $\phi=0.05$. We have also found that the potential of the tube wall is given by $\eta_w=6.30=\eta_s+\eta_0-\bar{\eta}_0$. whence $\eta_s=5.88$ and

$$\xi_w=5.16-1.23\ln(\phi\delta\eta_\phi) \tag{76}$$

where ξ_w measures the sheath thickness on a non-conducting wall. Thus $\xi_w=12.7$ whereas the same ion current space charge limited in the absence of electrons would give

$$\xi_w=(5.88)^{3/4}=3.78.$$

This sheath, then, is 3.3 times what might be called the normal thickness. In Run 346 (0.5 amp arc) $\delta\eta_\phi=0.12$ for $\phi=0.05$ and $\xi_w=11.4$.

The exact sheath solution would not involve ϕ which is a measure of the error tolerated in the plasma solution before it is abandoned, and consequently the presence of ϕ can be used to estimate the degree of approximation involved in Eq. (75). Since $\delta\eta_\phi$ varies with $\phi^{-1/3}$, the term which is variable in ϕ becomes $-0.81\ln\phi$. Thus a two-fold change in ϕ causes a change in ξ of only ± 0.56 which is less than errors introduced in the integration of Eq. (74).

The relation between ξ_w and the tube radius is most readily obtained from Eq. (73). Putting ρ for $\Delta s/s_0$, the fraction of the tube radius occupied by the sheath, this gives

$$\rho=(2/3s_0)(A/s_0h_0)^{1/2}\xi_w. \tag{77}$$

In Case $CL\lambda$ this becomes

$$\rho=1.662A^{1/2}\xi. \tag{78}$$

Whence, in Run 37a, $\rho=0.015$ and in Run 34b, $\rho=0.065$

V. Effect on the Plasma of an Ion Temperature

Up to this point the theoretical treatment of the plasma has been based on the assumption that newly formed ions start from rest, whereas it is most reasonable to suppose that they actually possess the velocity distribution characteristic of the gas atoms from which they have just been formed. W. Schottky's treatment[27] of the short mean free path case includes consideration of such an ion temperature. Here the long free path case is handled. In view of the result that the difference in voltage between the potential maximum and the sheath edge is of the order of $T_e/11{,}600$ in the cases so far discussed, whereas any ion temperature effects would probably be confined to a voltage difference of $T_p/11{,}600$, $(T_p=T_g)$ this assumption appears to be justified in the main. Only in the neighborhood of the potential maximum could the ion temperature conceivably influence the result appreciably, and it is possible to evaluate there the first and second order corrections from this cause.

Case *PLJ* will serve as the example. As before η denotes $-eV/kT_e$, but in addition it is convenient to introduce μ to denote $-eV/kT_p$. The plasma-sheath equation now takes the form

$$A'\nabla_q^2\mu+\epsilon^{-\eta}-\pi^{1/2}\left\{\int_0^q \exp(\mu-\mu_z)\left[1-P((\mu-\mu_z)^{1/2})\right]dq_z + \int_q^\infty \exp(\mu-\mu_z)dq_z\right\}=0 \qquad (79)$$

where $P(x)=2\pi^{-1/2}\int_0^x\epsilon^{-t^2}dt$, and $q=s(T_e/T_p)^{1/2}$ takes the place of s in the previous analyses. We shall not give the derivation of this equation. Let it suffice to point out that the first integral gives the density of the accelerated ions originating at values of η_z(or μ_z) less than η(or μ) and the second integral gives the density of the retarded ions originating at values of η_z(or μ_z) greater than η(or μ) and that these expressions include the flow of ions across the potential maximum. The two integrals can be rearranged to give

$$\pi^{1/2}\int_0^\infty \exp(\mu-\mu_z)dq_z-\pi^{1/2}\int_0^q \exp(\mu-\mu_z)P((\mu-\mu_z)^{1/2})dq_z.$$

Assuming the solution

$$\left.\begin{aligned}\mu&=Bq^2+Cq^4+\cdots\\ \mu_z&=Bq_z^2+Cq_z^4+\cdots\end{aligned}\right\}, \qquad (80)$$

the first integral immediately above, which will be denoted by H_1, becomes

$$H_1=\pi^{1/2}\epsilon^\mu\int_0^\infty \exp(-Bq_z^2-Cq_z^4-\cdots)dq_z.$$

Now, to the present degree of approximation

$$H_1=H_1\Big|_{C=0}+C(dH_1/dC)_{C=0}$$

[27] Schottky, Phys. Zeits. **25**, 342 (1924).

by a Taylor Expansion, so that

$$H_1=\pi^{1/2}\epsilon^{\mu}\left[\int_0^{\infty}\epsilon^{-Bq_z^2}dq_z+C\int_0^{\infty}-q_z^4\epsilon^{-Bq_z^2}dq_z\right]$$

and

$$H_1=(\pi/2B^{1/2})\epsilon^{\mu}[1-3C/4B^2]=(\pi/2B^{1/2})(1+\mu)[1-3C/4B^2].$$

The second integral expands to

$$H_2=2\int_0^{q}[(\mu-\mu_z)^{1/2}+(2/3)(\mu-\mu_z)^{3/2}+\cdots]dq_z.$$

Using the assumed solution we have

$$H_2=2\int_0^{q}[(Bq^2+Cq^4-Bq_z^2-Bq_z^4)^{1/2}+(2/3)(Bq^2+Cq^4-Bq_z^2-Bq_z^4)^{3/2}]dq_z$$

and applying the same method as before

$$H_2=(\pi B^{1/2}q^2/2)(1+Bq^2/2+5q^2C/8B).$$

Eq. (80) can be written

$$q^2=(\mu/B)(1-\mu C/B^2+\cdots). \tag{81}$$

Using this to express H_2 in terms of μ we have

$$H_2=(\pi\mu/2B^{1/2})(1+\mu/2-3C\mu/8B^2).$$

If T_p/T_e be noted by τ then $\eta=\tau\mu$ and Eq. (79) can now be written

$$A'\nabla_q^2\mu+\epsilon^{-\tau\mu}-H_1+H_2=0.$$

Neglecting $A'\nabla_q^2\mu$, expanding $\epsilon^{-\tau\mu}$, substituting for H_1 and H_2, and equating sums of coefficients of like powers of μ to zero, it is found that

$$1-(\pi/2B^{1/2})(1-3C/4B^2)=0$$

$$-\tau-(\pi/2B^{1/2})(1-3C/4B^2)+\pi/2B^{1/2}=0$$

whence

$$B=\pi^2/4(1+\tau)^2,\quad C=\pi^4\tau/12(1+\tau)^5.$$

Noting that $q^2=s^2/\tau$ and $\mu=\eta/\tau$ we thus find for Eq. (81)

$$s=[2(1+\tau)\eta^{1/2}/\pi][1-2\eta/3(1+\tau)+\cdots]$$

for the plasma solution in the neighborhood of $s=0$, at which place, as has been pointed out, the ion temperature will have the maximum effect. Comparison with the solution originally obtained (Table II*b*) shows that the distortion introduced by the finite ion temperature is only of the order of T_p/T_e, a very small quantity in most cases.

VI. Potential Distribution in the Plasma
Part II—Internal Plasma

The cases hitherto treated have all dealt with outward motion (in the extreme case parallel motion) of the ions toward an external collector. If, however, a negatively charged electrode is placed in the midst of the plasma this electrode becomes surrounded by a potential maximum and an entirely new condition arises in which the ions generated inside that maximum flow inward while those generated outside flow outward. Among the three plasma shapes already investigated it is evident that in the plane case the introduction of an additional infinite plane electrode parallel to the walls leaves the problem formally unchanged. On the other hand, introducing an axial cylindrical electrode in a cylindrical tube, or a central spherical electrode in a spherical bulb does change the problem. It is readily seen that the lower limit of the integral expressing the ion concentration in the plasma equation, Eq. (14) for instance, must lie at the potential maximum. Thus, to deal with the new cases, the zero limit of the integral has to be replaced by the finite value of s, say S, which corresponds to the radial distance of the potential maximum. Hence, the equation becomes,

$$\epsilon^{-\eta} - s^{-\eta} \int_S^s s_z^\beta \epsilon^{-\eta_z} (\eta - \eta_z)^{-1/2} ds_z = 0. \tag{82}$$

As before, an expansion in series can be attempted,

$$s - S = (2/\pi)\eta^{1/2}(1 + a_1\eta + \cdots)$$

outside the potential maximum and

$$s - S = -(2/\pi)\eta^{1/2}(1 + b_1\eta + \cdots)$$

inside the potential maximum. The coefficients $a_1, a_2, \ldots, b_1, b_2, \ldots$, are functions of S. Unfortunately, for the smaller values of S these series do not converge for all values of η less than η_0 with the result that rigorous solutions would be most complicated even if at all possible. It is questionable whether any involved mathematical investigation which is intended to cover electrodes of all sizes is justifiable.[28]

But the case of a very small cylindrical collector is of particular importance, because fine wires are often used experimentally as probes. In this instance the external plasma, namely that part of the plasma outside the potential maximum, very soon becomes indistinguishable from the plasma about an axial potential maximum and equations already derived can be applied. Inside the potential maximum an approximate mathematical method can be used. This method was employed before the rigorous solution of the long free path plasma equation had been found and applies strictly

[28] In the very short mean free path cases rigorous solutions are readily obtained. These solutions involve the zero order Bessel Functions of both the first and second kind in the cylindrical case and $(\cos s)/s$ as well as $(\sin s)/s$ in the spherical case.

only when ionization is uniform. It gives the comparative results shown in Table II*b*.

The approximation is made by supposing that the ion density at A Fig. (7) is caused by ions starting from the potentials lying on the broken

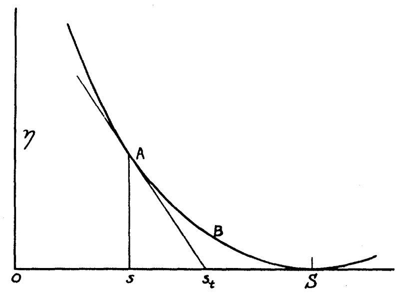

Fig. 7. Illustrating the approximation made in solving the internal plasma case.

line As_tS rather than from the potentials lying on the actual potential curve ABS. As_t is tangent to ABS at A. On this basis

$$\eta_z=\eta+(s_z-s)\eta' \quad \text{when} \quad s<s_z<s_t$$

$$\eta_z=0 \quad \text{when} \quad s_t<s_z<S$$

$$s_t-s=-\eta/\eta'.$$

These are the quantities now to be introduced in the integral of Eq. (82) after making that equation apply to uniform ionization ($\epsilon^{-\eta_z}=1$) and to the cylindrical case ($\beta=1$). The integral breaks up into two parts

$$(-\eta')^{-1/2}\int_s^{s_t} s_z(s_z-s)^{-1/2}ds_z+\eta^{-1/2}\int_{s_t}^{S} s_z ds_z$$

which combine to

$$\eta^{-1/2}[(S^2-s^2)/2-s\eta/\eta'+(\eta/\eta')^2/6].$$

Dropping $(\eta/\eta')^2/6$ because it is always negligible we have the approximate differential equation for an internal cylindrical plasma

$$\epsilon^{-\eta}-\eta^{-1/2}[(S^2-s^2)/2s-\eta/\eta']=0.$$

The substitutions $\sigma=s/S$ and $\zeta=\eta^{1/2}/S$ convert this equation to

$$2\sigma\zeta\epsilon^{-S^2\zeta^2}+\sigma^2+\sigma\zeta d\sigma/d\zeta-1=0. \tag{83}$$

In the limit, when S is zero, $\epsilon^{-S^2\zeta^2}=1$ and

$$2\sigma\zeta+\sigma^2+\sigma\zeta d\sigma/d\zeta-1=0. \tag{84}$$

The solution of this equation which satisfies the boundary conditions at $s=S$, (i.e., at $\sigma=1$) namely, that $\zeta=0$ and $d\zeta/d\sigma=0$ there, is

$$2\zeta(\sigma+\zeta)+\ln(1-2\sigma\zeta)=0 \qquad (85)$$

or in terms of s and η,

$$(2/S^2)(\eta+s\eta^{1/2})+\ln(1-2s\eta^{1/2}/S^2)=0.$$

Fig. 8 is a plot of this limiting form of the plasma equation. With increasing η Eq. (85) rapidly approaches the limiting form

$$2\sigma\zeta=1-\epsilon^{-2\zeta^2-1}=1.$$

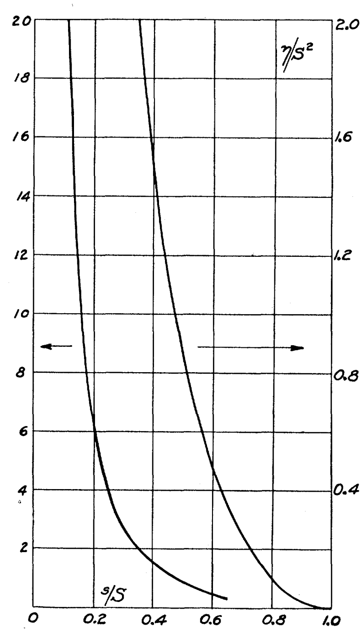

Fig. 8. Limiting form of potential distribution in plasma.

Thus when $\eta^{1/2}/S>1$ the internal plasma solution may be written

$$\eta=S^4/4s^2. \qquad (86)$$

If as before, Eq. (61) be used as the criterion for the failure of the plasma solution as the sheath is approached, the limit of validity is again given by that equation. In our present approximations, however, $\epsilon^{-\eta}$ may be taken as unity and it is found that

$$S^4/s_\phi{}^4=\phi/A, \qquad s_\phi=S(A/\phi)^{1/4}, \qquad \eta_\phi=S^2\phi^{1/2}/4A^{1/2}.$$

Thus, choosing $\phi=0.05$ it is found in the two cases previously cited that

$s_\phi=0.0571S$, $\eta_\phi=78.8S^2$ in the 8.0 amp. arc of Run 37a

$s_\phi=0.123S$, $\eta_\phi=16.3S^2$ in the 0.5 amp arc of Run 34b.

In order that errors arising from putting $\epsilon^{-\eta}=1$ should not be serious, it may be insisted that for the present approximation $\eta_\phi<0.25$. It follows that in Run 37a, $S<0.056$ whence $s_\phi<3.2\times10^{-3}$, or in terms of fractional tube radius $s_\phi(\text{int})/s_\phi(\text{ext})<4.1\times10^{-3}$. In Run 34b these quantities come out $S<0.124$, $s_\phi<.0153$, $s_\phi(\text{int})/s_\phi(\text{ext})<0.0197$. Thus, for the present formulae to hold within the approximations stated, the radius of the probe used in Run 37a would have to be somewhat less than 1/250th of the tube radius and in Run 34b somewhat less than 1/50th. Further quantitative relations depend on the possession of an adequate sheath theory for this case.

A striking feature of the internal plasma is the very uniform potential throughout all but a small portion near the internal electrode. Thus Eq. (86) shows that when $s=S^2/2$ then $\eta=S^2$. Accordingly, at least three-quarters of the ions generated in the internal plasma are formed at less than $\eta=S^2$ below the maximum. Even though S be as large as 0.1, the majority of these ions will possess enough thermal energy to cross the potential maximum and escape from the probe vicinity, for the ratio $\tau=T_p/T_e$ is rarely less than 0.01. This means that the trapping of ions by the potential maximum is relatively unimportant in the case of a fine wire probe. Of the ions reaching the probe only a small fraction may originate within the internal plasma; the vast majority have crossed the maximum from the outside. Thus the negatively charged probe can be treated on the present theory just as it was on the old.[29,30]

The important case appears to be that in which the collected current is limited by orbital motion. The equation for the resulting volt-ampere characteristic may be written[30]

$$i_p{}^2=(4A^2I_p{}^2/\pi)(-eV_c/kT_p+1) \tag{87}$$

where i_p is the positive ion current to the probe, A is the probe area, I_p is the positive ion current density in the plasma about the probe, and V_c is the negative probe voltage measured with respect to the potential maximum in the plasma. Thus if the square of the observed current i_p be plotted against the collector voltage V_c a straight line will be obtained whose negative slope Σ is given by

$$I_p/T_p{}^{1/2}=(\pi k/4e)^{1/2}\Sigma^{1/2}/A. \tag{88}$$

In the previous applications of this theory it has been assumed that I_p at the tube axis was substantially equal to its value at the wall, and on this basis values of T_p were calculated which were comparable with T_e. It will be observed, however, that in view of the general kinetic theory relation

$$(I/e)T^{1/2}=n(k/2\pi m)^{1/2}. \tag{89}$$

[29] L and M-S, Part I. The next few equations are taken directly from this article.
[30] H. M. Mott-Smith and I. Langmuir, Phys. Rev. **28**, 727 (1926).

Eq. (88) can say nothing regarding I_p and T_p individually. It can yield only the ion density in the plasma near the probe

$$n_p = (\pi m_p/2e)^{1/2}\Sigma^{1/2}/Ae = 3.32 \times 10^{11}(m_p/m_e)^{1/2}\Sigma^{1/2}/A \text{ cm}^{-3}$$

if Σ is expressed in amp. volt$^{-1/2}$. It might appear possible to use the addition of unity to $-eV_c/kT_p$ in the second factor in the right member of Eq. (87) for determining T_p and I_p, but the extrapolation required and the uncertainty of the zero point of V_c among other factors make such an attempt futile.

Although Langmuir and Mott-Smith have apparently used this method for measuring ion density with success, two difficulties appeared in the tests already tabulated in Table III. The first is evident in the table, namely that the apparent density (Column 6) obtained for the ions in this way has approximately twice the value of the density found either by Eq. (55) or

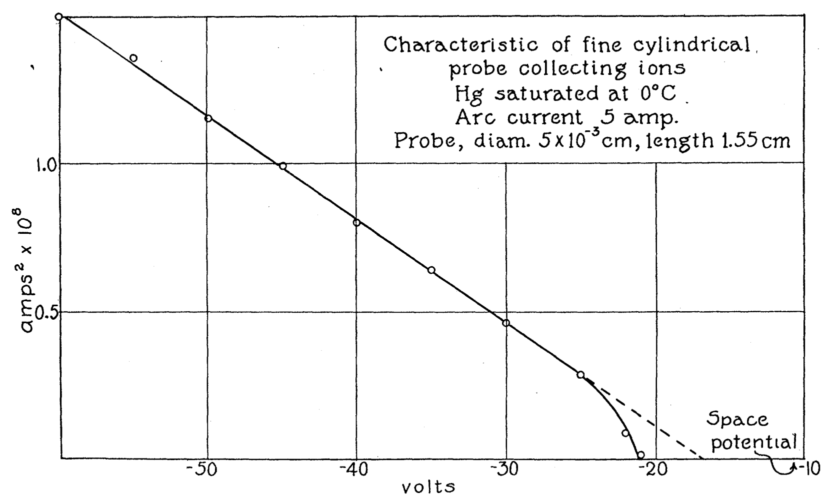

Fig. 9.

from the positive branch of the volt-ampere characteristic of the probe, as treated in Section VII. The second difficulty is illustrated by Fig. 9, where it is seen that the straight line coinciding with the upper portion of the curve cuts the axis not at the space potential as theoretically required, but some 6v. negative to it. Only the first run was normal in this respect. Neither of these difficulties casts serious doubt on the proposed theory; because the questions raised pertain rather to the theory of the collector. Quite apart from any plasma theory the ionization densities as determined from the two ends of the volt-ampere characteristic should agree, and yet there is the two-fold difference between Columns 6 and 7 of Table III.

If probes of greater and greater diameter be employed there are two important effects. The diameter of the potential maximum increases and as a first result the number of ions generated in the internal plasma increases more rapidly than the number crossing inward over the potential maximum. This occurs because the former number is roughly proportional to the cross sectional area of the internal plasma, while the latter is proportional to the perimeter of the cross-section. As a second result the average potential within the internal plasma decreases with respect to the maximum so that a smaller proportion of the ions generated inside the maximum escape. The progression of these two factors thus accomplishes the transition from the condition where the small electrode only intercepts a part of the low temperature random ion current which is flowing around it to the condition where the large electrode determines a volume in the plasma from which it drains all the ions generated there.

VII. Potential Distribution in the Plasma Part III—The Vicinity of a Positively Charged Electrode

It has been pointed out that if an electrode whose exposed area is just sufficient to receive a random electron current equal to the arc current, be used as anode, there is zero anode drop.[31] If the electrode area is greater than this critical area the anode drop is negative, if less than this positive. A collector having an area equal to or greater than this critical area in a discharge tube carrying a fixed arc current can be maintained at any desired potential negative to the plasma without having any material effect on the plasma potential, as this is fixed by the anode. If, now, the collector potential is raised past the anode potential, the collector becomes anode and the plasma potential rises with it, the original anode thus taking the role of negative collector. Only when the electrode area is but a small fraction of the critical area can it be maintained at a voltage considerably positive with respect to the plasma.

The simplest case to consider is that of a small wire in a plasma of large dimensions relative to it. This case has not, as yet, been treated quantitatively since qualitative considerations seem to suffice for the present. The potential distribution is of the type shown in Fig. 10. The space charge in the sheath is made up of orbital electrons and out-going ions which have been generated in the sheath and whose space charge contribution may be very small except near the sheath edge. The electric field decreases to zero (or at least a very small value) at the sheath-edge or plasma potential maximum, and from that point out the potential distribution is approximately normal. There are two disturbing factors, the drain of electrons from the plasma and the ions flowing outward from the sheath. Over wide ranges of potential on a fine wire the former results in only a very small electron deficiency and even when this becomes considerable it is compensated by the proportional decrease in ion generation. The second factor can be compensated by a slight increase in curvature of the plasma potential curve which causes a decrease in the space charge contribution from ions generated immediately outside the sheath edge. Thus a positively charged

[31] L and M-S, Part IV, p. 766.

electrode of this type does not affect more than a small region in the plasma and the theory of electron collection which calls for a straight line $i_e^2 - V$ plot as previously outlined applies.[32] Ionization densities calculated in this way have already been given in comparison with the results of other methods in Table III.

It is probable that the thickness of the electron sheaths in such cases is considerably greater than the value obtained from the ordinary space charge equation because of neutralization of electron space charge by ions generated within the sheath. They are generated there at a greater rate per electron than at the plasma potential maximum because of the higher electron velocities.[33] But this greater rate of generation does not lead to

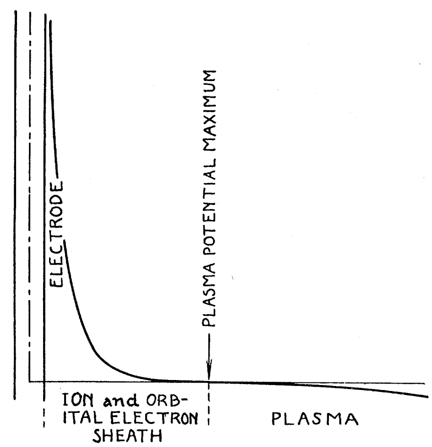

Fig. 10. Potential near a positively charged cylinder.

a net positive charge as the large electric field rapidly gives the ions considerable velocities. At the plasma potential maximum, however, the space charge neutralization is practically complete. Accordingly, near the sheath edge (plasma potential maximum) and for some distance within it there will be appreciable neutralization of electron space charge by the ions. The resulting increase in the sheath thickness may be sufficient to make orbital motion the factor limiting the current to the electrode even with electrodes of comparatively large diameter. Such a collector has actually been found to give an approximately linear $i_e^2 - V$ plot at small positive potentials.[34]

A more complicated case is that of a small positively charged electrode on the tube wall. Collector H,[35] a square plate 1.9 cm on a side, bent to fit the wall of the 3.2 cm diameter discharge is such an electrode. Fig. 4[35] shows that with 2 amps. leaving the cathode, a constant current of 1 amp reached this collector in the voltage range -10 to -5v measured with res-

[32] L and M-S, Part I, p. 455, and Part III, Fig. 10.
[33] See Section VIII (3).
[34] L and M-S, Part III, Fig. 11.
[35] L and M-S, Part II, Figs. 3 and 4.

pect to the anode. The present theory makes a rough calculation of the magnitude of this current possible.[36] As the potential of H was increased up to -10v, the maximum plasma potential opposite it in the discharge tube remained fixed at -10v. (Strictly speaking, of course, there is no potential maximum in the plasma near H because of the arc gradient, but the resulting differences of potential over the width of H are so small compared to the radial plasma potential differences that the arc gradient is neglected in the present discussion.) If, as the potential of H with respect to anode was still further raised (the total arc current being maintained constant at 2 amp.), the plasma had tended to rise with it, the whole arc current would have flowed to it immediately. This is what would occur with a collector somewhat larger than H, as was pointed out at the beginning of this section. But as H varies from -10 to -5v the maximum plasma potential opposite remains practically constant at -10v. The existence of the saturation current is then accounted for in the usual way by the formation of an electron sheath over H so that the whole random electron current crossing the sheath edge is captured by H, whereas no ions can reach it.

A fairly good approximation to the conditions in the adjacent plasma when H is drawing its saturation current would be obtained if H were replaced by an orifice in the tube wall opening into a second discharge tube identical with the first in size as well as excitation. The most important difference is that in this hypothetical case electrons travel both ways through the orifice, but this is not thought to be an essential difference since the actual one-way flow results to a first approximation only in lowering the electron density to one-half, the ion density dropping in proportion due to the proportional ionization. Our hypothetical picture shows that for small openings there is a saddle-like potential distribution in the hole, the pommel lying in one tube, the cantle in the other. As the size of the hole is increased the plasma maxima in the two tubes approach each other along the hole axis, finally merging into a single maximum at the center of the hole. This size undoubtedly corresponds to the size at which H would become anode as soon as it reached plasma potential. Since this did not happen, the plasma maximum lies some distance from the edge of the sheath on H. Thus we may adopt the plasma maximum as the upper limit of the plasma potential in contact with the sheath edge at the center of H.

The radial motion of the ions over the cross-section of the hypothetical hole is so similar to their motion in a cylinder that we are led to use the same mean value of $\epsilon^{-\eta}$ over the sheath edge of H as over the cross-section of a cylinder, namely, $2h_0$ [Eq. (53)]. Then applying the Boltzmann Equation to the electrons at the sheath edge the current density is found to be

$$e^{-\eta}n_0e(kT_e/2\pi m_e)^{1/2}$$

at any point of H and to average

$$I_e=2h_0n_0(kT_e/2\pi m_e)^{1/2}$$

over the whole of H.

[36] See also the treatment of this given by I. Langmuir, Phys. Rev. **33**, (1929).

The use of the Boltzmann Equation in this case where accelerating fields are involved needs some justification. The hole analogy indicates that the plasma equipotentials are perpendicular to H at the sheath edge, so that with respect to tube radii the plasma potential at H is a minimum. In addition the accelerating field from the edge of H toward its center is similar in nature to the accelerating field toward the axis of the discharge tube which does not lead to any apparent discrepancies, and the flow of electrons toward H across the sheath edge is analogous to their longitudinal drift in the discharge tube. The chief effect of the accelerating field at H is to reduce the electron density at the sheath edge everywhere to one-half normal, the effect of which has already been dealt with.

Combining the equation for I_e with Eq.(55) the ratio of I_e to I_p is found to be

$$\begin{aligned} I_e/I_p &= (m_p/m_e)^{1/2}/\pi^{1/2}s_0 \\ &= 0.731(m_p/m_e)^{1/2} \end{aligned} \tag{90}$$

which, in the case of Hg gives $I_e/I_p = 444$, as an upper limit. This is to be compared with the average experimental value 380 of six low pressure runs.[37]

VIII. General Arc Relations

It will be shown in this section that the plasma balance equation completes the number of relations necessary to determine all the variables of the positive column of an arc as a function of one of them. Although all these relations are recognized qualitatively, the complete quantitative formulation of certain ones is lacking. For this reason the discussion undertaken here is to be regarded as suggestive of the possibilities offered by the theory and also as serving, perhaps, to define the problems still awaiting solution.

The variable quantities involved in the positive column of an arc may be divided into two classes, the independent and the dependent. Among the former belong the gas used, the tube radius a, the gas pressure p_g, and the wall temperature, which, in case the atomic mean free path is comparable with a, may be used for the gas temperature T_g. One of the arc variables proper must also be included in this category—experimentally it is usually the total arc current i_B. The dependent variables are, therefore, the axial electric field Z, the electron density in the axis n_0, the electron temperature T_e, the positive ion current density at the wall I_p, and the number of ions generated per electron per second, λ. These variables are five in number and five equations will be required for their complete determination. These equations, involving various more or less accurately known relations and constants will be discussed individually for the low pressure (Case $CL\lambda$) arc.

(1) *The Plasma Balance Equation.* Eq. (46). This is the essentially new equation given by the present theory. When the wall sheath is not thin the tube radius a must be corrected by using Eq. (77). Further, this equation

[37] L and M-S, Part II, Table III, Runs 34b to 37a.

only applies strictly when ions are formed by a one-stage process, for only in that case is the ion generation at a point proportional to electron density. In the hypothetical case that ion generation is entirely a two stage process, the rate would tend toward proportionality to the square of the electron density. Now the transition from uniform ion generation to ion generation proportional to the first power of electron density causes s_0 to go from 0.5828 to 0.7722. Hence the further complete transition to proportionality to the square of the electron density would probably cause a further comparable increase in s_0. It seems probable, however, that not until high arc current densities and high gas pressures as well are reached will this effect on s_0 become considerable, for s_0 depends on the distribution of the ion production through the tube cross-section and not on its magnitude. Thus the ionization of excited atoms may be contributing effectively to the total ionization, thereby causing the "constant" λ to increase with i_B, but as long as these excited atoms are distributed with fair uniformity over the tube cross-section (as in the case at low pressure) s_0 will be but slightly affected.

(2) *The Ion-Current Equation*, Eq. (55A). Deviations from the one-stage ionization process will also affect the accuracy of this equation. But since s_0h_0 changes only from 0.2914 to 0.2703 in passing from uniform to proportional ion generation this equation is much less dependent on ionization mechanism than is the plasma balance equation.

(3) *The Ion Generation Equation*. In accounting for the ion generation Killian[38] has assumed that the ionizing is done by those electrons in a normal M. D. which possess velocities greater than the equivalent of the ionization potential V_i of the gas. As Killian points out, it is sufficient for this purpose to represent the ionization probabilities at the lower voltages only and these are given accurately enough by the relation

$$P=\beta(p_g/T_g)(W-V_i) \tag{91}$$

where P is the number of ions generated per electron per cm of path at gas pressure p_g and temperature T_g (the really significant variable is density) and where W is the equivalent voltage of the ionizing electrons, V_i is the gas ionization potential, and β is an experimentally determined constant. It is the slope of the P vs. W curve reduced to unit pressure and temperature. Using W_e for $T_e/11{,}600$ the calculation of λ on the basis stated gives

$$\lambda=6.70\times10^7\beta(p_g/T_g)W_e^{3/2}(2+V_i/W_e)\epsilon^{-V_i/W_e} \tag{92}$$

Judging from the shape of the experimental probability curves[39] it seems more reasonable to assume that initially P is proportional to the excess velocity of the electron rather than its excess energy. The only effect of this assumption is to change the 2 in the parenthesis of Eq. (92) to 3/2. Thus, putting

$$f_\lambda=6.70\times10^7(W_e/V_i)^{3/2}(3/2+V_i/W_e)\epsilon^{-V_i/W_e} \tag{93}$$

[38] Forthcoming article in THE PHYSICAL REVIEW.

[39] K. T. Compton and C. C. Van Voorhis, Fig. 6, Phys. Rev. **26**, 436 (1925) and T. J. Jones, Fig. 2, Phys. Rev. **29**, 822 (1927).

and

$$\alpha_\lambda = \beta V_i^{3/2}(p_g/T_g) \tag{94}$$

we have

$$\lambda = \alpha_\lambda f_\lambda . \tag{95}$$

A plot of f_λ facilitating calculations of λ is given in Fig. 11. From Compton and Van Voorhis $\beta p_g/T_g$ is 1.4 where $p_g = 1$ mm $= 1330$ baryes and T_g is room temperature, that is about 300°K.[40] Thus $\beta = 0.31$ for mercury vapor.

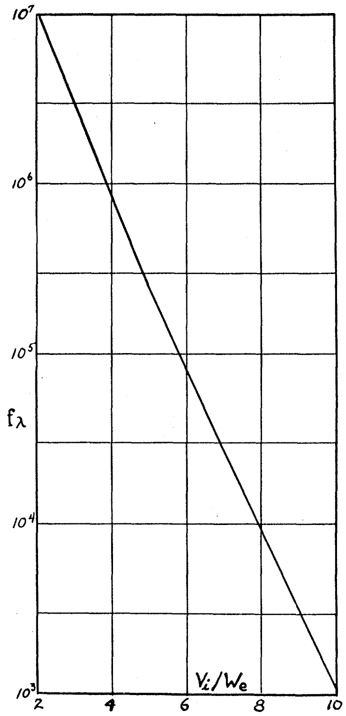

Fig. 11. Variation of relative ionizing power of electrons with their equivalent voltage.

Also, $V_i = 10.4$. For the case illustrated in Fig. 3 we may assume $T_g = 400$°K roughly. Correcting 0.27 baryes, the vapor pressure of Hg which is saturated at 1.4°C, for thermal effusion it is found that $p_g = 0.33$ baryes, whence $\alpha_\lambda = 0.0086$. Since $T_e = 38{,}800$°K, $V_i/W_e = 3.11$ and from Fig.(11) $f_\lambda = 2.5 \times 10^6$ giving $\lambda = 2.1 \times 10^4$. Eq. (48) can also give avalue of λ. Using $a = 3.1$ it is found that $\lambda = 4.5 \times 10^4$ which is rather satisfactory agreement in view of (a), large uncertainties in β, and (b), the large errors in f_λ which arise from small errors in T_e.

Thus the plasma balance equation and the ion generation equation together constitute a pair of simultaneous equations in the variables λ and T_e

[40] By letter K. T. Compton has explained that the pressure of 1 mm, given in Fig. 6 (loc. cit.) corresponds to the initial temperature of the ionization compartment before any heating occurred there.

only, which should fix these two arc variables irrespective of the others. The simultaneous graphical solution of the two equations is given by Fig. 12. The relatively small changes in T_e for large changes in a and α_λ is evident.

The fact, however, that T_e does vary with arc current, decreasing in general with increasing current, indicates that two-stage ionization processes contribute appreciably to the total ionization. How important such processes may be is shown by a comparison of the values of λ given in the last two columns of Table I. These values were calculated in the same way as the values of λ for Killian's results. The fifth column gives the rate of ion generation necessary to maintain the plasma, the last column the rate at which the one-stage process can supply ions. The rapid failure of this source of ionization with increasing arc current is evident.

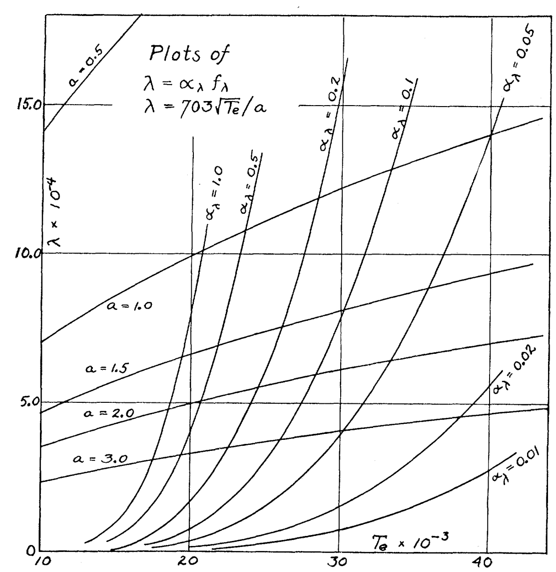

Fig. 12. Simultaneous solution of plasma-balance and one-stage ionization equations.

Unfortunately a new element of uncertainty has been introduced in the attempt to confirm Eq. (95) further by the additional experiments already mentioned in connection with Table III. It is to be noted that the electron temperatures there listed are considerably lower than those found by Killian, who for instance found 38,800° under apparently the same conditions as those which gave us 27,800° as listed in the second row of Table III.

(4) *Mobility Equation.* Killian has used[38] Langevin's Mobility Equation

$$\mu_e = 0.75 e l_e / m_e \bar{v}_e \qquad (96)$$

where μ_e is the electron mobility, l_e the electron mean free path and $\bar{v}_e$ the average thermal velocity, to calculate the electron mean free path from the arc current i_B, arc gradient Z, and N the number of electrons per cm length of column through the relation

$$i_B = Ne\mu_e Z \tag{97}$$

The value so obtained checks well with the accepted value. Combining these two equations and noting at the same time that $\bar{v}_e = 2(2kT_e/\pi m_e)^{1/2}$ and that N_e is given by Eq. (54), it is found that

$$i_B = 0.75\pi^{3/2}h_0 e^2 a^2 n_0 l_e Z/(m_e kT_e)^{1/2} \tag{98}$$

$$= 8.7\times 10^{-10} a^2 n_0 l_e Z/T_e^{1/2} \tag{99}$$

for Hg in practical units.

(5) *Energy Balance Equation.* Although the types of energy loss in the positive column are probably known, their quantitative formulation is not possible as yet. The power input is, of course, $i_B Z$ per cm of tube. Of this, the kinetic energy of the ions striking the wall accounts for approximately $2\pi a I_p(\eta_w - 0.3)kT_e/e$ ergs · sec−1, the 0.3 being estimated as the average potential drop in the plasma. The electrons striking the wall account for $2\pi a I_p 2kT_e/e$ and the heat of recombination for $2\pi a I_p V_i$ watts. In addition, the ions will, on the average, have fallen through various distances parallel to the axis before striking the wall and will therefore dissipate additional energy there. The energy which the electrons lose in their elastic collisions is probably negligible,[38] but this cannot be true of their inelastic collisions which do not lead to ionization, those which do lead to ionization having already been counted at the wall. Certain unknown probabilities are involved in these processes but no new arc variables.

The power immediately accounted for is thus

$$2\pi a I_p[8.0T_e/11{,}600 + V_i] \text{ watts.}$$

Applying this to two typical cases, a 1 amp and an 8 amp arc at low pressure[41] it is found that only 0.31 to 0.28 of the energy is accounted for in this way, necessitating a detailed investigation of the radiation loss and of the other more obscure factors.

Schottky' treatment of the positive column. In his arc theory Schottky[42] combines his plasma solution directly with the plausible assumption that $N_e\lambda$ is proportional to $i_B Z$ for different tube diameters to obtain the important result that arc gradient is inversely proportional to tube diameter if electron and ion mobilities remain constant. Certain experiments of Claude in neon in which the product Za [Schottky's $(\partial V/\partial Z)\cdot R$] was found to be constant appear to confirm the original assumption. The practical value of this treatment cannot be denied, but from a more fundamental point of view such a short cut evades some of the basic relations in an arc, all of which must be woven into any comprehensive theory.

[41] Runs 35a and 37a of L and M-S, Part II, Table III.
[42] W. Schottky, Phys. Zeits. **25**, 635 (1924).

Stability and oscillations. The plasma balance equation represents an equilibrium but it is not obvious that it is a stable equilibrium. Instability of one type in a mercury cathode arc is certainly shown by its negative resistance. But there seem to be other possibilities also. Thus if we suppose that due to statistical fluctuations in the electron velocities the rate of generation of ions in a certain cross section of a positive column is momentarily $\lambda+\delta\lambda$ what will ensue? The plasma balance equation tells us that $\lambda+\delta\lambda$ corresponds to a smaller tube diameter than λ, that is, that the plasma field is stronger and that the ions will, consequently, flow out faster. But this increased field and increased positive ion density result in an increased potential at the tube axis in this region, at least while the excess ions are flowing away. This, in turn, by unduly accelerating electrons causes further excess ionization in this cross-section. At the same time, to the anode side of this region there will be a deficiency of ionization, for there the arc gradient will be less than normal. Will such a lump of excess ionization be dissipated more rapidly than renewed? Will the general drift of ions toward the cathode carry it as a wave in that direction?

If a positive column is maintained by an anode of such size that it has a negative anode drop, there is an absolute potential maximum in the plasma near the anode. At that place one should expect to find more and more electrons trapped because of inelastic collisions made nearby. This would mean a progressive decrease in their effective temperature. Does the same mechanism which accomplishes the rapid recovery of a disturbed M. D. elsewhere in the positive column operate in this region also, preventing the average electron energy from decreasing? Or do the electrons accumulate to a degree where new forces predominate which allow them to dissipate once more?

A study along the lines indicated by these and possibly other similar questions may lead to further insight into the oscillation behavior of arcs.