ON THE TRANSITION PROBABILITY BETWEEN TWO STATES WITH POSITIVE ENERGY IN A CENTRAL FIELD

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Abstract

By the aid of the hypergeometric function with two variables, the transition probability in the sense of the quantum mechanics between two states with positive energy in a central field due to nuclear charge Ne has been calculated. When the transition probability calculated is applied to the intensity of continuous x-rays, the spectral energy distribution, which is nearly independent of the frequency of the radiation emitted, has been obtained. As the result of the present computation, the dependence of the intensity for the isochromat I_ν on the applied voltage V is somewhat flatter than the result expected from the law $I_\nu \sim 1/V$. The isochromat experiments by Kuhlenkampff for the continuous x-rays emitted from a thin target show a good agreement with the values computed here. The polarization and the angular intensity distribution of continuous x-rays have been discussed.

When we want to attack the problem of the intensity of continuous x-rays by the wave mechanics, we have to calculate the transition probablity between two so-called hyperbolic orbits defined by (E_1, k_1, m_1) and (E_2, k_2, m_2), where E_1 and E_2 are positive. The emission spectrum of continuous x-rays has been investigated by a number of physicists theoretically[1] and experimentally.[2] The special points to be solved theoretically are the dependency of the spectral energy distribution of the polarization and of the angular intensity distribution for the continuous x-ray spectrum, on the velocity of cathode rays, and on the material used as the target.

The object of the present paper is to compute the transition probability between two states with positive energy in a central field due to a nucleus of charge Ne, in order to get the relation between the intensity of continuous x-rays and the velocity of the cathode ray electrons and the material of the target, from which the spectral energy distribution can be obtained.

With regard to the angular intensity distribution of continuous x-rays, even by the classical quantum theory it has been rather hard to solve without any ambiguity. We should attack this problem by taking Dirac's idea for an electron which encounters a nucleus with the resultant emission of radiation, but it seems to be difficult to get a result without some assumption. The author would like to leave this problem to be solved in the future.

2. Emission of Continuous X-ray Radiation from an Infinitely Thin Target

In the case of the continuous x-ray spectrum emitted from an infinitely thin target bombarded by cathode rays, we may take the cathode ray electron

[1] A. Sommerfeld, Phys. Zeits. **10**, 969 (1909); H. A. Kramers, Phil. Mag. **46**, 836 (1923); G. Wentzel, Zeits. f. Physik **27**, 257 (1924).

[2] Handbuch der Physik, Bd. xxiii, Chapter 4; H. Kuhlenkampff, Ann. d. Physik **69**, 548 (1922); **87**, 597 (1928); W. Duane, Proc. Nat. Acad. Sci. **13**, 662 (1927); **14**, 450 (1928).

and the nucelus of an atom in the target as one system. If we may speak of a model-like atom, the cathode-ray electron might be assumed to be on an hyperbolic orbit whose focus is the nucleus of the atom. The continuous spectrum is emitted by the electron's transition between the two hyperbolic orbits. The initial state is given by the applied voltage V for the cathode rays—the energy $W_1 = mc^2[1/(1-\beta_1^2)^{1/2}-1] = eV$ where $\beta_1 = v_1/c$—and also direction relative to the atom. When we use the suffix 1 and 2 for the initial and final state respectively, the principle of energy conservation gives

$$W_1 = h\nu + W_2 + \tfrac{1}{2}Mv^2. \tag{1}$$

Since the final momentum of the atom is of the same order of magnitude as the momentum of the electron and since its mass M is of the order of 10^4 times as great, its kinetic energy $\frac{1}{2}Mv^2$ will be negligible compared with that of the electron.

From Einstein's idea, we can generally express the intensity I of the continuous radiation per electron emitted from unit cross section of an infinitely thin target per unit time, by the aid of the transition probability $A_{E_2}^{E_1}$ between two hyperbolic orbits $(E_1 = W_1/Rh, k_1, m_1)$ and (E_2, k_2, m_2):

$$I = N_0 A_{E_2}^{E_1} h\nu, \tag{2}$$

where N_0 is the number of atoms per unit cross section of the target in the initial state at a time $t=0$. So far as we are concerned with the total intensity without considering a special direction of the observation of the radiation, the intensity is given by (2) and the transition probability $A_{E_1}^{E_2}$ between two hyperbolic orbits in an atom of nuclear charge Ne can be calculated from the matrix element[3]:

[3] Compare the expression for the hydrogen atom, Y. Sugiura, J. d. Physique, **8,** 113 (1927). The applicability of the matrix element (3) is restricted to the case where c/ν is large compared to the region of space over which the wave functions ψ differ appreciably from zero. When we want to solve the intensity problem of continuous x-rays, since c/ν is quite small and ψ's do not converge to zero very rapidly, we have to take into account the phase variation inside of the atom. (I should like to express sincere thanks to the Editors of the Physical Review for their kind comments on this point.) Consequently we should have to carry out the integral

$$Q_{1,2} = \int q e^{-2\pi i\nu_{1,2} ur/c}\, \psi_1 \psi_2{}^* d\tau$$

instead of (3), in order to obtain the retarded coordinates matrix element required in the present problem of x-rays, corresponding to the radiation emitted by the transition between two states defined by the normalized wave functions ψ_1 and ψ_2 in the direction of the unit vector u, r being the radius vector. We have here to note that we are not treating the perturbation due to radiation in the present calculations, as has been done by A. Rubinowicz (Phys. Zeits. **29,** 817 (1928); Zeits. f. Physik **53,** 267 (1929)).

Taking the direction of the observation of radiation as the z-axis, and expanding the exponential factor $\exp[-2\pi i\nu_{1,2}Z/c]$ into series, we can easily find that the selection principle for k, $\Delta k = \pm 1$, does not hold for the second term in the expansion, the corresponding selection being $\Delta k = 0$ or ± 2. The third and higher terms may be assumed to give very small contribution to the sum, $Z = Z_1 + Z_2 + Z_3 + \cdots$, the first term Z_1 of which and X-, Y-matrix components are the same as those given by (3). Corresponding to the transitions $\Delta k = \pm 1$, which

$$\left.\begin{matrix}X\\Y\\Z\end{matrix}\right\}_{E_1,E_2}=\left(\frac{(k_1-\frac{1}{2})(k_2-\frac{1}{2})\Gamma(k_1-m_1)\Gamma(k_2-m_2)}{4\pi^2\Gamma(k_1+m_1)\Gamma(k_2+m_2)}\right)^{1/2}$$

$$\int_0^\infty\int_0^\pi\int_0^{2\pi}\begin{matrix}x\\y\\z\end{matrix}\,U(E_1,r)U(E_2,r)r^2dr$$

$$P_{k_1-1}^{m_1}(\cos\theta)P_{k_2-1}^{m_2}(\cos\theta)\sin\theta d\theta e^{i(m_1-m_2)\phi}d\phi, \tag{3}$$

where

$$U(E,\rho)=\frac{i\rho^{k-1}}{2a_0^{3/2}}f(E)\int_{-iE^{1/2}}^{iE^{1/2}}e^{z\rho}(z-iE^{1/2})^{k-1-iN/E^{1/2}}(z+iE^{1/2})^{k-1+iN/E^{1/2}}dz,$$

$$f^2(E)=\frac{e^{2\pi N/E^{1/2}}(e^{2\pi N/E^{1/2}}-1)}{2\pi^2N(2E^{1/2})^{2k-2}(\Gamma(k))^2\prod_{\mu=1}^{k-1}\left(1+\frac{N^2}{\mu^2E}\right)},$$

$$\rho=r/a_0,\quad a_0=h^2/4\pi^2me^2,\quad\text{and}\quad k_1,k_2=1,2,3,\cdots.$$

Taking the sum of the matrix amplitudes square, we can express the intensity $I_{k_1k_2}$ for each k_1 and k_2:

$$I_{k_1k_2}=N_0\frac{8\pi^2e^2hR^3}{3mc^3a_0^2}\left(\frac{\nu}{R}\right)^4\frac{1}{g_{k_1}}d\sum_{m_1m_2}(X^2+Y^2+Z^2)_{E_1E_2}, \tag{4}$$

where $g_{k_1}=2k_1-1$ is the statistical weight of the initial state, and R is the Rydberg constant. On the other hand

$$\frac{d}{dE_2}\sum_{m_1m_2}(X^2+Y^2+Z^2)_{E_1E_2}=a_0^2k\left[\int_0^\infty a_0^3\rho^3U(E_1,\rho)U(E_2,\rho)d\rho\right]^2 \tag{5}$$

$$=a_0^2k\,J^2_{k_1k_2},$$

k being the smaller value of k_1 and k_2. We obtain therefore the intensity of the radiation which lies between ν and $\nu+d\nu$

$$Id\left(\frac{\nu}{R}\right)=N_0\frac{8\pi^2e^2h}{3mc^3}R^3\sum_{k_1k_2}\frac{k}{2k_1-1}\left(\frac{\nu}{R}\right)^4J^2_{k_1k_2}dE_2, \tag{6}$$

where according to the energy principle (1), $E_2=E_1-\nu/R$.

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give the principal contribution to the Z-component, the matrix elements are equal to those calculated by (3), because the second term in the above expansion vanishes for these transitions, higher terms being neglected. On the other hand corresponding to the transitions $\Delta k=0, \pm 2$, the X-, Y- and Z_1-components vanish, and, in a first approximation, only the Z_2-component remains in the matrix element, which is of smaller order of magnitude than the matrix element corresponding to the transitions $\Delta k=\pm 1$. In a first approximation required here, the matrix element (3) may, therefore, give a sufficient result to the present problem.

Strictly speaking, as will be discussed later in the section 5, regarding the polarization and the angular intensity distribution of the continuous x-rays, we ought to apply the Dirac relativistic expressions for the wave functions to the present calculations, where the idea of retarded matrix element will play a great rôle.

When we specify the direction of the cathode-ray electron and the direction of the observation of the radiation, we have to calculate the square of the component of the electric moment in the direction of the observation. This problem will be discussed later.

3. Square of Matrix Amplitudes for Continuous X-Ray Spectrum

In order to calculate the transition probability between two hyperbolic orbits (E_1, k_1, m_1) and (E_2, k_2, m_2), we have to integrate (5)

$$J=\int_0^\infty a_0^3\rho^3 U(E_1,\rho)U(E_2,\rho)d\rho.$$

Putting $z=iE^{1/2}u$ in the integral for $U(E,\rho)$, we can express $U(E\rho)$ by the confluent hypergeometric function $M_{K,\mu}$ as follows:

$$U(E,\rho)=\frac{i(-1)^{k-1-iN/E^{1/2}}}{2\rho a_0^{3/2}}f(E)(2iE^{1/2})^{k-1}$$
$$\frac{\Gamma\left(k+\frac{iN}{E^{1/2}}\right)\Gamma\left(k-\frac{iN}{E^{1/2}}\right)}{\Gamma(2k)}M_{-iN/E^{\frac{1}{2}},k-1/2}(2iE^{1/2}\rho). \tag{7}$$

Since

$$\int_0^\infty e^{-i(E_1^{1/2}+E_2^{1/2})\rho}\rho^n d\rho=\frac{\Gamma(n+1)}{\{i(E_1^{1/2}+E_2^{1/2})\}^{n+1}},$$

$$J=\frac{-(-i)^{k_1+k_2}e^{-\pi N(1/E_1^{1/2}+1/E_2^{1/2})}}{4(E_1^{1/2}+E_2^{1/2})^2}f(E_1)f(E_2)(2E_1^{1/2})^{k_1-1}(2E_2^{1/2})^{k_2-1}$$
$$\left(\frac{2E_1^{1/2}}{E_1^{1/2}+E_2^{1/2}}\right)^{k_1}\left(\frac{2E_2^{1/2}}{E_1^{1/2}+E_2^{1/2}}\right)^{k_2}\cdot\frac{\Gamma(k_1+k_2+2)}{\Gamma(2k_1)\Gamma(2k_2)}\Gamma\left(k_1+\frac{iN}{E_1^{1/2}}\right)$$
$$\Gamma\left(k_1-\frac{iN}{E_1^{1/2}}\right)\Gamma\left(k_2+\frac{iN}{E_2^{1/2}}\right)\Gamma\left(k_2-\frac{iN}{E_2^{1/2}}\right)$$
$$\cdot\sum_{n_1=0}^\infty\sum_{n_2=0}^\infty\frac{(k_1+k_2+2,n_1+n_2)\left(k_1+\frac{iN}{E_1^{1/2}},\ n_1\right)\left(k_2+\frac{iN}{E_2^{1/2}},\ n_2\right)}{(2k_1,n_1)(2k_2,n_2)(1,n_1)(1,n_2)}$$
$$\left(\frac{2E_1^{1/2}}{E_1^{1/2}+E_2^{1/2}}\right)^{n_1}\left(\frac{2E_2^{1/2}}{E_1^{1/2}+E_2^{1/2}}\right)^{n_2}, \tag{8}$$

where the symbol

$$(\lambda,\ n)=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$$

The above double summation is the hypergeometric function of the second type $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$ with two variables,[4] whose parameters are given by

$$\alpha = k_1 + k_2 + 2$$
$$\beta = k_1 + iN/E_1^{1/2}, \qquad \beta' = k_2 + iN/E_2^{1/2}$$
$$\gamma = 2k_1, \qquad \gamma' = 2k_2$$
$$x = \frac{2E_1^{1/2}}{E_1^{1/2} + E_2^{1/2}}, \qquad y = \frac{2E_2^{1/2}}{E_1^{1/2} + E_2^{1/2}}.$$

Since

$$\Gamma\left(k + \frac{iN}{E^{1/2}}\right)\Gamma\left(k - \frac{iN}{E^{1/2}}\right) = \frac{2\pi N e^{\pi N/E^{1/2}}(\Gamma(k))^2}{E^{1/2}(e^{2\pi N/E^{1/2}} - 1)} \prod_{\mu=1}^{k-1}\left(1 + \frac{N^2}{\mu^2 E}\right), \quad k \geq 1,$$

we get

$$J = -(-i)^{k_1+k_2} \frac{N\Gamma(k_1+k_2+2)\Gamma(k_1)\Gamma(k_2)x^{k_1}y^{k_2}}{2\Gamma(2k_1)\Gamma(2k_2)(E_1E_2)^{1/2}(E_1^{1/2}+E_2^{1/2})^2}$$
$$\left(\frac{\prod_{\mu_1=1}^{k_1-1}\left(1 + \frac{N^2}{\mu_1^2 E_1}\right)\prod_{\mu_2=1}^{k_2-1}\left(1 + \frac{N^2}{\mu_2^2 E_2}\right)}{(1 - e^{-2\pi N/E_1^{1/2}})(1 - e^{-2\pi N/E_2^{1/2}})}\right)^{1/2} F_2\left(k_1 + k_2 + 2,\ k_1 + \frac{iN}{E_1^{1/2}},\right.$$
$$\left. k_2 + \frac{iN}{E_2^{1/2}},\ 2k_1, 2k_2, x, y\right) \tag{9}$$

where

$$x = \frac{2E_1^{1/2}}{(E_1^{1/2}+E_2^{1/2})}, \quad y = \frac{2E_2^{1/2}}{(E_1^{1/2}+E_2^{1/2})}$$

According to the selection principle for k, we have two cases:

i) $\qquad k_1 = k, \qquad k_2 = k+1,$

ii) $\qquad k_1 = k+1, \qquad k_2 = k.$

Regarding β and β', F_2 is quite symmetrical, so that by interchanging E_1 and E_2 in J, we can obtain the value of J for the second case from the first. Putting $k_1 = k$, $k_2 = k+1$, we get[5]

$$F_2\left(2k+3, k + \frac{iN}{E_1^{1/2}},\ k+1 + \frac{iN}{E_2^{1/2}},\ 2k, 2k+2, x, y\right)$$
$$= \frac{2k-1}{(2k+2)(k-1+iN/E_1^{1/2})} \frac{\partial}{\partial x} F_2\left(2k+2, k-1 + \frac{iN}{E_1^{1/2}},\ k+1 + \frac{iN}{E_2^{1/2}},\right.$$
$$\left. 2k-1, 2k+2, x, y\right)$$

[4] P. Appell et J. Kampé de Fériet, Fonctions Hypergéométriques et Hypersphériques, p. 13, Paris, 1926.

[5] Formulae of the hypergeometric functions with two variables can be found in the book by P. Appell, l. c.

$$=\frac{2k-1}{(2k+2)(k-1+iN/E_1^{1/2})}\frac{\partial}{\partial x}F_1\left(k-1+\frac{iN}{E_1^{1/2}},\ k+1-\frac{iN}{E_2^{1/2}},\ k+1+\frac{iN}{E_2^{1/2}},\ 2k-1,x,\frac{x}{1-y}\right)$$

$$=\frac{(1-y)^{-k-1-iN/E}}{2k+2}$$

$$\left\{\begin{aligned}&\left[\left(k+1-\frac{iN}{E_2^{1/2}}\right)F_1\left(k+\frac{iN}{E_1^{1/2}},\ k+2-\frac{iN}{E_2^{1/2}},k+1+\frac{iN}{E_2^{1/2}},2k,x,\frac{x}{1-y}\right)\right.\\&+\frac{(k+1+iN/E_2^{1/2})}{1-y}F_1\left(k+\frac{iN}{E_1^{1/2}},\ k+1-\frac{iN}{E_2^{1/2}},\ k+2+\frac{iN}{E_2^{1/2}},\right.\\&\left.\left.\qquad 2k,x,\frac{x}{1-y}\right)\right]\end{aligned}\right\} \quad (10)$$

By the aid of the following relations:

$$\frac{\beta}{\gamma}xF_1(\alpha+1,\beta+1,\beta',\gamma+1,x,y)+\frac{\beta'}{\gamma}yF_1(\alpha+1,\beta,\beta'+1,\gamma+1,x,y)$$
$$=F_1(\alpha+1,\beta,\beta',\gamma,x,y)-F_1(\alpha,\beta,\beta',\gamma,x,y),$$

$$(y-1)^3F_1(\alpha,\beta,\beta',\gamma,x,y)=\frac{(\gamma-\alpha)(\gamma+1-\alpha)(\gamma+2-\alpha)}{\gamma(\gamma+1)(\gamma+2)}y^3F_1(\alpha,\beta,\beta',\gamma+3,x,y)$$
$$-3\frac{(\gamma-\alpha)(\gamma+1-\alpha)}{\gamma(\gamma+1)}y^2F_1(\alpha,\beta,\beta'-1,\gamma+2,x,y)+3\frac{\gamma-\alpha}{\gamma}yF_1(\alpha,\beta,\beta'-2,\gamma+1,x,y)$$
$$-F_1(\alpha,\beta,\beta'-3,\gamma,x,y),$$

$$F_1(\alpha,\beta,\beta',\beta+\beta',x,y)=(1-y)^{-\alpha}F\left(\alpha,\beta,\beta+\beta',\frac{y-x}{y-1}\right),$$

we can reduce the double summation in F_2 into the sum of the single summations F's, remembering $x+y=2$ in the present case:

$$F_2\left(2k+3,k+\frac{iN}{E_1^{1/2}},k+1+\frac{iN}{E_2^{1/2}},2k,2k+2,x,y\right)$$
$$=\frac{(2k-1)(1-y)^{-k-1-iN/E_2^{1/2}}}{(2k+2)x}\left\{F_1\left(\alpha,\beta,\beta',\gamma-1,x,\frac{x}{1-y}\right)\right.$$
$$\left.-F_1\left(\alpha-1,\beta,\beta',\gamma-1,x,\frac{x}{1-y}\right)\right\}$$

$$=\frac{(-1)^k(2k-1)}{(2k+2)}\left(\frac{x}{1-y}\right)^2e^{-\pi N/E_1^{1/2}}(1-y)^{-iN(1/E_2^{1/2}-1/E_1^{1/2})}$$

$$\left.\begin{cases} \dfrac{(\gamma-\alpha)(\gamma+1-\alpha)}{(\gamma-1)\gamma(\gamma+1)}\{(\gamma-1-\alpha)(1-y)F(\alpha,\beta,\gamma+2,xy) \\ \qquad\qquad +(\gamma+2-\alpha)F(\alpha-1,\beta,\gamma+2,xy)\} \\ -3\left(\dfrac{1-y}{x}\right)\dfrac{\gamma-\alpha}{(\gamma-1)\gamma}\{(\gamma-1-\alpha)(1-y)F(\alpha,\beta,\gamma+1,xy) \\ \qquad\qquad +(\gamma+1-\alpha)F(\alpha-1,\beta,\gamma+1,xy)\} \\ +3\left(\dfrac{1-y}{x}\right)^2\dfrac{1}{\gamma-1}\{(\gamma-1-\alpha)(1-y)F(\alpha,\beta,\gamma,xy) \\ \qquad\qquad +(\gamma-\alpha)F(\alpha-1,\beta,\gamma,xy)\} \\ -\left(\dfrac{1-y}{x}\right)^3\{(1-y)F(\alpha,\beta,\gamma-1,xy)+F(\alpha-1,\beta,\gamma-1,xy)\} \end{cases}\right\} \quad (11)$$

where

$$\alpha=k+iN/E_1^{1/2},\quad \beta=k+1-iN/E_2^{1/2},\quad \beta'=k+1+iN/E_2^{1/2},\quad \gamma=2k.$$

Putting (11) in (9) and multiplying the complex conjugate, we get the square of the integral $J_{k_2=k_1+1}$:

$$J^2_{k_2=k_1+1}=4N^2\left(\frac{\Gamma(k)\Gamma(k+1)}{\Gamma(2k-1)}\right)^2 \frac{E_1x^{2k}y^{2k+2}e^{-2\pi\alpha_1}\prod_{\mu=1}^{k-1}\left(1+\frac{a_1^2}{\mu^2}\right)\prod_{\mu=1}^{k}\left(1+\frac{a_2^2}{\mu^2}\right)}{E_2(\nu/R)^4}P\cdot P^* \quad (12)$$

where $a_1=N/E_1^{1/2}$, $a_2=N/E_2^{1/2}$,

$$\begin{aligned} P=&\frac{(k-ia_1)(k+1-ia_1)}{(2k-1)2k(2k+1)}\begin{Bmatrix} (k-1-ia_1)(1-y)F(k+ia_1,k+1-ia_2,2k+2,xy) \\ +(k+2-ia_1)F(k-1+ia_1,k+1-ia_2,2k+2,xy) \end{Bmatrix} \\ &-3\left(\frac{1-y}{x}\right)\frac{(k-ia_1)}{(2k-1)2k}\begin{Bmatrix} (k-1-ia_1)(1-y)F(k+ia_1,k+1-ia_2,2k+1,xy) \\ +(k+1-ia_1)F(k-1+ia_1,k+1-ia_2,2k+1,xy) \end{Bmatrix} \\ &+3\left(\frac{1-y}{x}\right)^2\frac{1}{2k-1}\begin{Bmatrix} (k-1-ia_1)(1-y)F(k+ia_1,k+1-ia_2,2k,xy) \\ +(k-ia_1)F(k-1+ia_1,k+1-ia_2,2k,xy) \end{Bmatrix} \\ &-\left(\frac{1-y}{x}\right)^3\begin{Bmatrix} (1-y)F(k+ia_1,k+1-ia_2,2k-1,xy) \\ +F(k-1+ia_1,k+1-ia_2,2k-1,xy) \end{Bmatrix}, \end{aligned}$$

and P^* is the complex conjugate of P. $J^2_{k_1=k_2+1}$ can be obtained by interchanging E_1 and E_2 in the expression (12) for $J^2_{k_2=k_1+1}$. k in both expressions for $J^2_{k_2=k_1+1}$ and $J^2_{k_1=k_2+1}$ means the smaller value of k_1 and k_2. Owing to the convergency of the hypergeometric function in (12), it is not practical to use the expression P (12). When we transform the hypergeometric function with the variable xy to that with $1-xy$, by using the following formula:

$$\begin{aligned} F(\alpha,\beta,\gamma,x)=&\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta+1-\gamma,1-x) \\ &+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma+1-\alpha-\beta,1-x), \end{aligned} \quad (13)$$

it is practically sufficient for our purpose to take the first four or five terms of the hypergeometric series. We can moreover simplify the expression P in the following way: since we have the relations[6]

$$F(\alpha,\beta,\gamma-1,x)=\frac{1}{\gamma(\gamma-1)(1-x)}\left[\begin{array}{l}\gamma\{\gamma-1-(2\gamma-\alpha-\beta-1)x\}F(\alpha,\beta,\gamma,x)\\ \qquad+(\gamma-\alpha)(\gamma-\beta)xF(\alpha,\beta,\gamma+1,x)\end{array}\right],$$

$$(\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1,x)=\gamma(1-x)F(\alpha,\beta,\gamma,x)-\{\alpha-1-(\gamma-\beta)x\}F(\alpha,\beta,\gamma+1,x),$$

$$\gamma F(\alpha-1,\beta,\gamma,x)=\gamma(1-x)F(\alpha,\beta,\gamma,x)+(\gamma-\beta)xF(\alpha,\beta,\gamma+1,x),$$

we get

$$P=\frac{F(\alpha,\beta,\gamma+2,xy)}{(\gamma-1)\gamma(\gamma+1)}(\gamma+1-\alpha)\cdot\{(\gamma-\alpha)(\gamma-1-\alpha)x-(\gamma-\alpha)(\gamma-2)$$
$$+(\gamma-1-\alpha)(\gamma+1-\beta)2y-(\gamma+1-\beta)(\gamma-2)\frac{y}{x}+(\gamma+1-\beta)(\gamma-\beta)\frac{y^2}{x}\}$$
$$-\frac{F(\alpha,\beta,\gamma+1,xy)}{(\gamma-1)\gamma}$$
$$\left\{\begin{array}{l}(\gamma-\alpha)(\gamma-1-\alpha)x-(\gamma-\alpha)(\gamma-2)+(\gamma-1-\alpha)(\gamma+1-\beta)2y\\ \quad-(\gamma+1-\beta)(\gamma-2)\frac{y}{x}+(\gamma+1-\beta)(\gamma-\beta)\frac{y^2}{x}-(\gamma+2)(\gamma-\alpha)\frac{1}{x}\\ \quad-\gamma(\gamma+2-\beta)\frac{y}{x^2}+\gamma(\gamma+2)\frac{1}{x^2}\end{array}\right\} \tag{14}$$

where

$$\alpha=k+ia_1,\quad \beta=k+1-ia_2,\quad \gamma=2k,$$
$$x=2E_1^{1/2}/(E_1^{1/2}+E_2^{1/2}),\quad y=2E_2^{1/2}/(E_1^{1/2}+E_2^{1/2}).$$

After some elementary calculations we can find easily

$$P=\frac{-2a_1i}{(\gamma-1)\gamma}\left[\frac{\gamma+1-\alpha}{\gamma+1}F(\alpha,\beta,\gamma+2,xy)-\left(1-\frac{1}{x}\right)F(\alpha,\beta,\gamma+1,xy)\right]. \tag{15}$$

Putting (13) in (15), and taking into account the following formulae:

$$F(\alpha,\beta,\gamma,x)-\frac{\gamma-\alpha}{\gamma}F(\alpha,\beta,\gamma+1,x)=\frac{\alpha}{\gamma}F(\alpha+1,\beta,\gamma+1,x),$$
$$F(\alpha,\beta,\gamma,x)+\frac{\alpha x}{\gamma}F(\alpha+1,\beta+1,\gamma+1,x)=F(\alpha,\beta+1,\gamma,x),$$
$$(\gamma-\alpha-1)F(\alpha,\beta,\gamma,x)=(\beta-\alpha-1)(1-x)F(\alpha+1,\beta,\gamma,x)+(\gamma-\beta)F(\alpha+1,\beta-1,\gamma,x),$$

[6] C. F. Gauss, Werke, Bd. 3, p. 125–162.

we get

$$P=\frac{2a_1\Gamma(2k-1)}{a_2-a_1}\left[\frac{\Gamma(1+i(a_2-a_1))}{\Gamma(k-ia_1)\Gamma(k+1+ia_2)}\right.$$

$$\left\{\begin{array}{l}F(k+1+ia_1,k+1-ia_2,1-i(a_2-a_1),1-xy)\\ \quad -\dfrac{1}{x}F(k+1+ia_1,k-ia_2,1-i(a_2-a_1),1-xy)\end{array}\right\}$$

$$-\frac{\Gamma(1-i(a_2-a_1))(1-xy)^{i(a_2-a_1)}}{\Gamma(k+ia_1)\Gamma(k+1-ia_2)}$$

$$\left.\left\{\begin{array}{l}F(k+1-ia_1,k+1+ia_2,1+i(a_2-a_1),1-xy)\\ \quad -\dfrac{1}{x}F(k+1-ia_1,k+ia_2,1+i(a_2-a_1),1-xy)\end{array}\right\}\right],$$

in which the first term and the coefficient of $(1-xy)^{i(a_2-a_1)}$ of the second are complex conjugate to each other.

The well known expansion formulae[7] for the logarithm of the Gamma-function give

$$\log\Gamma(1+x)=\frac{1}{2}\left\{\log\frac{\pi x}{\sin\pi x}-\log\left(\frac{1+x}{1-x}\right)\right\}+\sum_{r=0}^{\infty}\frac{1-s_{2r+1}}{2r+1}x^{2r+1},\qquad |x|<2,$$

$$\log\Gamma(1+x)=\left(x+\frac{1}{2}\right)\log x-x+\frac{1}{2}\log 2\pi+\sum_{r=0}^{\infty}\frac{B_{r+1}}{(2r+1)(2r+2)x^{2r+1}},$$

where s_1 is Euler's constant and $s_n=1/1^n+1/2^n+\dots$, and B_1, B_2, ... are Bernoulli's numbers. Hence, we find from (16)

$$P=\frac{\Gamma(2k-1)}{\Gamma(k)\Gamma(k+1)}\left[\frac{2a_1e^{2\pi a_1}(1-e^{-2\pi a_1})(1-e^{-2\pi a_2})}{\pi a_2(a_2-a_1)(1-e^{-2\pi(a_2-a_1)})}\right]^{1/2} \tag{17}$$

$$\left\{e^{-i(\theta_{a_2-a_1}+\theta_{a_1}-\theta_{a_2})}(A+iB)-e^{i(\theta_{a_2-a_1}+\theta_{a_1}-\theta_{a_2}+2\theta)}(A-iB)\right\},$$

where

$$2\theta=(a_2-a_1)\log(1-xy),$$

$$\theta_\sigma=\arctan\sigma-\sum_{r=0}^{\infty}(-1)^r\frac{1-s_{2r+1}}{2r+1}\sigma^{2r+1}\qquad \sigma<2,$$

$$=\sigma\left(1-\log\sigma+\sum_{r=0}^{\infty}\frac{B_{r+1}}{(2r+1)(2r+2)\sigma^{2r+2}}\right)-\frac{\pi}{4},$$

$$A+iB=\left[\prod_{\mu=1}^{k-1}\left(1-\frac{ia_1}{\mu}\right)\prod_{\mu=1}^{k}\left(1+\frac{ia_2}{\mu}\right)\right]^{-1}$$

$$\left\{\begin{array}{l}F(k+1+ia_1,k+1-ia_2,1-i(a_2-a_1),1-xy)\\ -\dfrac{1}{x}F(k+1+ia_1,\ k-ia_2,\ 1-i(a_2-a_1),\ 1-xy)\end{array}\right\}.$$

[7] N. Nielsen, Handbuch der Theorie der Gammafunktion, p. 38 and 208.

Multiplying P by its complex conjugate, we obtain

$$P\cdot P^*=\left(\frac{\Gamma(2k-1)}{\Gamma(k)\Gamma(k+1)}\right)^2\frac{8a_1e^{2\pi a_1}(1-e^{-2\pi a_1})(1-e^{-2\pi a_2})}{\pi a_2(a_2-a_1)(1-e^{-2\pi(a_2-a_1)})}(A\sin\Theta-B\cos\Theta)^2, \quad (18)$$

where

$$\Theta=\theta_{a_2-a_1}+\theta_{a_1}-\theta_{a_2}+\theta.$$

Putting (18) in (12), we get the square of the integral required for the case $k_2=k_1+1$:

$$J^2_{k_2=k_1+1}=\frac{2^7NE_1E_2}{\pi(\nu/R)^5(E_1^{1/2}+E_2^{1/2})}(xy)^{2k}\frac{\prod_{\mu=1}^{k-1}\left(1+\frac{a_1^2}{\mu^2}\right)\prod_{\mu=1}^{k}\left(1+\frac{a_2^2}{\mu^2}\right)}{1-e^{-2\pi(a_2-a_1)}}(A\sin\Theta-B\cos\Theta)^2. \quad (19)$$

Similarly the square of the integral for the case $k_1=k_2+1$ becomes

$$J^2_{k_1=k_2+1}=\frac{2^7NE_1E_2}{\pi(\nu/R)^5(E_1^{1/2}+E_2^{1/2})}(xy)^{2k}\frac{\prod_{\mu=1}^{k}\left(1+\frac{a_1^2}{\mu^2}\right)\prod_{\mu=1}^{k-1}\left(1+\frac{a_2^2}{\mu^2}\right)}{1-e^{-2\pi(a_2-a_1)}}(A'\sin\Theta-B'\cos\Theta)^2, \quad (20)$$

where

$$A+iB'=\left[\prod_{\mu=1}^{k}\left(1-\frac{ia_1}{\mu}\right)\prod_{\mu=1}^{k-1}\left(1+\frac{ia_2}{\mu}\right)\right]^{-1}\left\{F(k+1+ia_1,k+1-ia_2,1-i(a_2-a_1),1-xy)-\frac{1}{y}F(k+ia_1,k+1-ia_2,1-i(a_2-a_1),1-xy)\right\}$$

$$=\left[\prod_{\mu=1}^{k-1}\left(1-\frac{ia_1}{\mu}\right)\prod_{\mu=1}^{k}\left(1+\frac{ia_2}{\mu}\right)\right]^{-1}\left\{\frac{k-ia_2}{k-ia_1}F(k+1+ia_1,k+1-ia_2,1-i(a_2-a_1),1-xy)-\frac{1}{y}F(k+1+ia_1,k-ia_2,1-i(a_2-a_1),1-xy)\right\}.$$

In these expressions (19) and (20) k means the smaller values of k_1 and k_2. The hypergeometric series to be computed numerically is $F(k+1+ia_1, k+1-ia_2, 1-i(a_2-a_1), 1-xy)$ and $F(k+1+ia_1, k-ia_2, 1-i(a_2-a_1), 1-xy)$. They converge fairly rapidly in the region in which we are interested (see the next section).

Before entering into the details of the spectral energy distribution, we shall here study the two limiting cases:

i) $x=y=1,\quad E_1=E_2$

ii) $x=2,\quad y=0\quad E_2=0.$

In the first case, since $E_2=E_1-\nu/R$, there is no radiation, but even in this case the transition probability $A^{E_1k_2}_{E_1k_1}$ can exist. From (12) we obtain in this limiting case

$$P=\frac{(k-ia)(k+1-ia)(k+2-ia)}{(2k-1)2k(2k+1)}F(k-1+ia,k+1-ia,2k+2,1),$$

and since

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\quad R(\gamma-\alpha-\beta)>0,$$

$$P=\frac{\Gamma(2k-1)(k-ia)}{\Gamma(k+1-ia)\Gamma(k+1+ia)}\,.$$

Hence

$$P\cdot P^*=\frac{E_1(k^2+a^2)}{(2\pi N)^2}\left(\frac{\Gamma(2k-1)}{(\Gamma(k+1))^2}\right)^2\frac{e^{2\pi a}(1-e^{-2\pi a})^2}{\left[\prod\limits_{\mu=1}^{k}\left(1+\frac{a^2}{\mu^2}\right)\right]^2}\,.$$

Putting this expression in (12), we get

$$J^2{}_{k_2=k_1+1}=\frac{E_1}{\pi^2(\nu/R)^4},\tag{21}$$

which is independent of k and the atomic number N. As the transition probability $A^{E_1k_2}_{E_1k_1}$ we have

$$A^{E_1k_2}_{E_1k_1}=\frac{8\pi^2e^2R^2E_1}{3mc^3\pi^2}\,\frac{k}{2k_1-1},\qquad \begin{matrix} k=k_1 & \text{for} & k_1<k_2,\\ k=k_2 & \text{for} & k_1>k_2.\end{matrix}\tag{22}$$

In the second case, the applied energy E_1 is consumed totally in the emission of the radiation ν/R. By the aid of the following relations:

$$F_1(\alpha,\beta,\beta',\gamma,x,x)=F(\alpha,\beta+\beta',\gamma,x)=F(\beta+\beta',\alpha,\gamma,x)$$

$$F(\alpha+1,\beta,\gamma,x)=\frac{\beta}{\alpha}F(\alpha,\,\beta+1,\,\gamma,\,x)+\frac{\alpha-\beta}{\alpha}F(\alpha,\beta,\gamma,x),$$

$$F(\alpha+1,\beta,\gamma,x)-F(\alpha,\beta,\gamma,x)=\frac{\beta x}{\gamma}F(\alpha+1,\beta+1,\gamma+1,x),$$

$$F(\alpha,\beta,\alpha,x)=(1-x)^{-\beta},$$

we get from (11)

$$\lim_{E_2=0}F_2\left(2k+3,k+\frac{iN}{E_1^{1/2}},\;k+1+\frac{iN}{E_2^{1/2}},\;2k,2k+2,x,y\right)$$

$$=\left[F(2k+3,k+ia,2k,x)\right]_{x=2}$$

$$=\frac{(-1)^{k+1}e^{-\pi a}}{2k(2k+1)(2k+2)}\{(k+ia)(k+1+ia)(k+2+ia)-6(k^2+a^2)(ia)$$
$$-(k-ia)(k+1-ia)(k+2-ia)\}=\frac{4ia(3k+1-a^2)(-1)^{k+1}e^{-\pi a}}{2k(2k+1)(2k+2)}\,.$$

Therefore from (9)

$$\lim_{E_2=0} J^2{}_{k_2=k_1+1} = (-1)^{k+1}(2i)^{2k+2}N^{k+2}\frac{\Gamma(k)e^{-\pi N/E_1^{1/2}}(3k+1-2N^2/E_1)}{\Gamma(2k+2)(E_1^{1/2})^{k+5}} \cdot \left[\frac{\prod_{\mu=1}^{k-1}\left(1+\frac{N^2}{\mu^2 E_1}\right)}{1-e^{-2\pi N/E_1^{1/2}}}\right]^{1/2}$$

from which

$$\lim_{E_2=0} J^2{}_{k_2=k_1+1} = 2^{4k+4}N^{2k+4}\left(\frac{\Gamma(k)}{\Gamma(2k+2)}\right)^2 \frac{e^{-2\pi N/E_1^{1/2}}(3k+1-2N^2/E_1)^2 \prod_{\mu=1}^{k-1}(1+N^2/\mu^2 E_1)}{(\nu/R)^4 E_1^{k+1}(1-e^{-2\pi N/E_1^{1/2}})}$$

Similarly starting from (10), in the case where $k_1=k_2+1$,

$$\lim_{E_2=0} F_2(2k+3, k+1+ia_1, k+ia_2, 2k+2, 2k, x, y)$$

$$= \lim_{E_2=0}\frac{\partial^3}{\partial x^3}F_2(2k, k-2+ia_1, k+ia_2, 2k-1, 2k, x, y)\frac{(2k-1,3)}{(2k,3)(k-2+ia_1,3)}$$

$$= \lim_{E_2=0}\frac{\partial^3}{\partial x^3}F_1\left(k-2+ia_1, k-ia_2, k+ia_2, 2k-1, x, \frac{x}{1-y}\right)\frac{(1-y)^{-k-ia_2}}{(2k+2)(k-2+ia_1,3)}$$

$$= \frac{1}{(2k+2)(k-2+ia_1,3)}\left[\frac{\partial^3}{\partial x^3}F(2k, k-2+ia_1, 2k-1, x)\right]_{x=2}$$

$$= \frac{1}{(2k+2)(k-2+ia_1,3)}\left[\frac{\partial^3}{\partial x^3}\{(k-2+ia_1)(1-x)^{-k+1-ia_1} + (k+1-ia_1)(1-x)^{-k+2-ia_1}\}\right]_{x=2}$$

$$= \frac{(-1)^{k+2}\,2ia_1 e^{-\pi a_1}}{2k+2}.$$

Putting this expression in (9) and multiplying the complex conjugate, we get

$$\lim_{E_2=0} J^2{}_{k_1=k_2+1} = 2^{4k+2}N^{2k+2}\left(\frac{\Gamma(k+1)}{\Gamma(2k)}\right)^2\frac{e^{-2\pi N/E_1^{1/2}}\prod_{\mu=1}^{k}(1+N^2/\mu^2 E_1)}{(\nu/R)^4 E_1^{k}(1-e^{-2\pi N/E_1^{1/2}})}. \quad (24)$$

In the region intermediate between these limiting cases i) and ii), it is not easy to get a simple formula for the square of matrix amplitudes. We will evaluate the values of J^2 for each special ν/R by taking some definite numerical values of E_1, E_2, k_1, k_2 and special elements for the target, by the aid of the expressions (19) and (20).

4. Spectral Energy Distribution of Continuous X-ray Spectrum.

Since the intensity of the radiation which lies between ν and $\nu+d\nu$ is given by (6) or

$$Id\nu=\text{const.}\sum_{k=1}^{\infty}\left\{\frac{k}{2k-1}\left(\frac{\nu}{R}\right)^4 J^2{}_{k_2=k_1+1}+\frac{k}{2k+1}\left(\frac{\nu}{R}\right)^4 J^2{}_{k_1=k_2+1}\right\}d\nu, \quad (25)$$

the effect of the atomic number N of the element of the target and of the velocity of the cathode ray election defined by E_1 on the intensity is so complicated, that we cannot see easily its general features, unless we put numerical values for E_1 and ν/R in (19) and (20) and sum them over all possible k.

As an example, we will take Zn($N=30$) as the target and apply 12.16 kV to the cathode rays. We have then

$$E_1=W_1/Rh=900, \quad \beta_1=0.2145, \quad a_1=N/E_1^{1/2}=1.$$

As shown in the following tables, the convergency of $Id\nu$ with k is fairly rapid. Moreover, since the values of $1-xy$ are so small, it is quite sufficient to take the first five or six terms of the hypergeometric functions in A, B, A' and B', in order to get the sufficient accuracy required here.

Table I. Zn, 12.16 kV.

ν/R	800	675	500	275
$E_2^{1/2}$	10	15	20	25
$a_2=N/E_2^{1/2}$	3	2	1.5	1.2
$1-xy$	1/4	1/9	1/25	1/121
$\sin\Theta$	−.16024	−.35759	−.40927	−.32780
$\cos\Theta$	.98708	.93388	.91237	.94475

Table II

		$k=1$		
A	.35381	.16825	.11025	.04651
B	−.11373	−.13126	−.09289	−.04496
A'	−.09798	−.13681	−.07030	−.03826
B'	−.41525	−.18549	−.10510	−.04588
		$k=2$		
A	.33742	.17768	.09421	.03813
B	−.06302	−.08070	−.05231	−.02149
A'	−.11979	−.08540	−.05254	−.02866
B'	−.08138	−.03570	−.02469	−.01406
		$k=3$		
A	.57443	.24856	.10559	.03730
B	−.09502	−.09238	−.05092	−.01664
A'	−.10757	−.06408	−.04197	−.02383
B'	−.01422	−.00017	−.00458	−.00561
		$k=4$		
A	1.29837	.4055	.1312	.0388
B	−.21124	−.1561	−.06031	−.01545
A'	−.10464	−.06175	−.03717	−.02112
B'	.01027	.01216	.00392	−.00183

By the expansion formulae of θ_σ (17) we can calculate the values of Θ for given ν/R and E_1, which are given in Table I. By the aid of (19) and (20) we obtain the values of A, B, A' and B' for given k, x, y, which are given in Table II for $k=1, 2, 3, 4$. The following table gives the values of

$$(xy)^{2k}\prod_{\mu=1}^{k-1}\left(1+\frac{a_1^2}{\mu^2}\right)\left(1+\frac{a_2^2}{\mu^2}\right)\left\{\frac{k}{2k-1}\left(1+\frac{a_2^2}{k^2}\right)(A\sin\Theta-B\cos\Theta)^2+\frac{k}{2k+1}\left(1+\frac{a_1^2}{k^2}\right)(A'\sin\Theta-B'\cos\Theta)^2\right\}$$

for different ν/R and k.

TABLE III

k	$\nu/R=800$	675	500	275
1	0.0856	0.0414	0.0142	0.00383
2	.0324	.0139	.0058	.00147
3	.0053	.0033	.0022	.00071
4	.0004	.0009	.0008	.00038
5	.0001	.0003	.0003	.00019
6		.0001	.0001	.00010
7				.00005
$k>7$				.00006
Σ_k	.1238	.0599	.0234	.00679

Multiplying

$$\frac{NE_1E_2}{\left(\dfrac{\nu}{R}\right)(E_1^{1/2}+E_2^{1/2})(1-e^{-2\pi(a_2-a_1)})}$$

by these values (Table III), we get the intensity measures for different frequencies:

TABLE IV. *Intensity measures for different frequencies.*

ν/R	900	800	675	500	275
$\frac{\pi(\nu/R)^4}{2^7}\sum_{k=1}^{\infty}\left(\frac{k}{2k-1}J^2{}_{k_2=k_1+1}+\frac{k}{2k+1}J^2{}_{k_1=k_2+1}\right)$	10.442	10.449	10.455	10.565	10.589

In Table IV the value for $\nu/R=900$ is computed by (23) and (24). As shown in the table we get a spectral energy distribution which is nearly independent of the frequency ν of the radiation. This result agrees with the classical quantum theoretical result of Kramers[8] for an infinitely thin target.

When we take into account the thickness of the target, the problem becomes very complicated. We have in this case to consider the stopping of swiftly moving electrons through matter (something like Thomson-Whiddington law), the absorption of emitted rays in the target, the deflection of electrons during its passage in the target and so on. There is, however,

[8] H. A. Kramers, reference 1.

still no quantum mechanical theory on the stopping of swiftly moving electrons through matter, corresponding to the Thomson-Whiddington law. We will not enter here into these complicated problems. Since we have now sufficient experimental data[9] for the continuous x-ray spectrum emitted from a thin target, we will be satisfied with the transition probability between two states with positive energy in an atom.

As shown in the computed results described already, the spectral energy distribution is nearly independent of the frequency of the radiation emitted from an infinitely thin target. We can therefore discuss the dependency of intensity of the continuous radiation on the atomic number and on the velocity of the cathode ray electrons by taking the limiting case, where the applied energy is totally consumed in the emission of the radiation $\nu/R=E_1$. From (23) and (24), taking the expression of the intensity (25), we get.

$$I_\nu d\nu=N_0\frac{8\pi^2e^2hR^3}{3mc^3}\,\frac{4\,N^2e^{-2\pi a}}{1-e^{-2\pi a}}\,\sum_{k=1}^{\infty}k\left(\frac{(4a)^k\Gamma(k)}{\Gamma(2k)}\right)^2\prod_{\mu=1}^{k-1}\left(1+\frac{a^2}{\mu^2}\right)$$
$$\cdot\left\{\frac{1}{2k-1}\left(\frac{a(3k+1-2a^2)}{k(2k+1)}\right)^2+\frac{1}{2k+1}(k^2+a^2)\right\}d\nu, \tag{26}$$

where

$$a=N/E_1^{1/2}$$

In order to find the dependency of the intensity of the continuous x-rays on the applied voltage, and to compare the computed results with those obtained experimentally by Kuhlenkampff[10] we will take Al ($N=13$) as the target. In the case of the alminum target, the values of I_ν/CN^2, where $C=N_0\ 32\pi^2e^2hR^3/3mc^3$, corresponding to several values of a, which is related with the applied voltage by the relations: $a=N/E_1^{1/2}$, and $E_1=eV/Rh$, are given in Table V.

TABLE V. Al ($N=13$)

a	I_ν/CN^2	$E_1=eV/Rh$	V (in kV)
1/4	0.1334	2704	36.53
5/19	0.1380	2440.36	32.97
2/7	0.1460	2070.25	27.97
1/3	0.1604	1521	20.55
1/2	0.1856	676	9.13
2/3	0.1749	380.25	5.14
1	0.1184	169	2.28
3/2	0.0661	75.11	1.01
2	0.0381	42.25	

Taking Kuhlenkampff's empirical formula

$$I_\nu=CN(\nu_0-\nu)$$

and the Thomson-Whiddington law, we get the isochromat as hyperbola $I_\nu\sim 1/V$ for the continuous radiation emitted from an infinitely thin target.

[9] H. Kuhlenkampff, reference 2.

[10] H. Kuhlenkampff, Ann. d. Phys. **87**, 597 (1928).

The experiment shows, however, that the steepness of the curve $I_\nu(V)$ depends on the direction of emission in such a way as the isochromat for 90° is the steepest which is still flatter than the hyperbola $1/V$. In Figure 1, the curve I represents the calculated intensity values I_ν (in certain scale) given in Table V, curve II the observed values (Kuhlenkampff) for the radiation filtered by Zr in the azimuthal angle 90° and curve III the values expected from the $1/V$ law. The sudden fall of the experimental curve at $V \approx 20$ kV is due to the K-absorption of Zr whose head is at $\lambda = 0.687$A or about 18 kV. Although we cannot, from the present calculations, speak of the dependency of the isochromat on the azimuthal angle, the calculated results are, as shown in the figure, better in agreement with the experimental than those derived from the $1/V$ law.

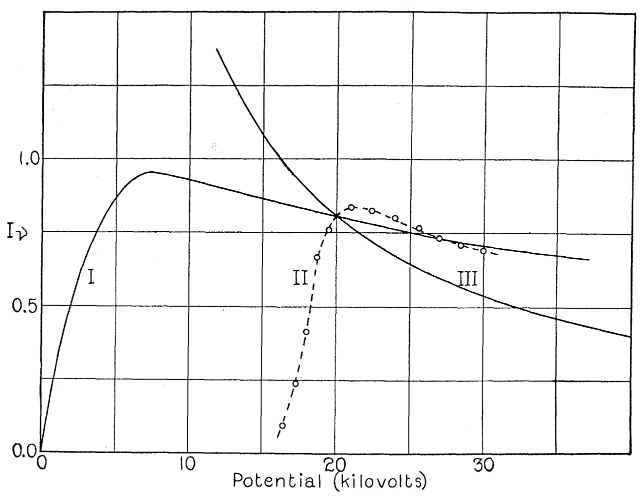

Fig. 1.

The dependency of the intensity on the atomic number N of the element of the target is not simple. Since the number of atoms per unit area N_0 is very roughly proprotional to $1/N$, and since the computation gives $I_\nu/C \sim N^2$ approximately, we find $I_\nu \sim N$.

Finally we shall note that the isochromat for any element of the target can be computed from the values of I_ν/CN^2 given in the second column of Table V, as the function of the applied voltage which is related to a by the formulae: $a = N/E_1^{1/2}$ and $E_1 = eV/Rh$.

5. Polarization and Angular Intensity Distribution of Continuous X-rays

When we assume the cathode ray electron and the nucleus as one system and take the quantum mechanical expression for the so-called hyperbolic orbit, whose angular part is $P_{k-1}^{m}(\cos\theta)e^{im\phi}$, we cannot define the direction of the cathode ray electron. In reality, however, the cathode ray has its definite direction, and at a great distance from the nucleus it ought to be a plane wave which has a definite direction of propagation. So far as we were concerned with the problem of the total energy of radiation emitted from an atom, it was not necessary to consider the direction of the cathode ray electron. The intensity expression obtained in the previous section represents the intensity of the continuous radiation emitted from an atom orientated at random, or in other words the mean integral intensity for the whole space, just as for the intensity of a spectral line in the usual sense. When we want to find out the angular intensity distribution and the polarisation of the radiation, we have to take into account the initial direction of the cathode ray electron.

Taking the initial direction of the cathode ray electron as the z-axis, we have the wave function[11] for the initial state

$$\psi_1 = e^{ik_1 z} G(ia_1, 1, ik_1\eta)$$

$$k_1 = \frac{2\pi m v_1}{h} = \frac{E_1{}^{1/2}}{a_0}, \quad a_1 = \frac{N}{E_1{}^{1/2}}, \quad \eta = r(1-\cos\theta) = r - z, \tag{27}$$

where

$$G(ia_1, 1, ik_1\eta) = \sum_{n=0}^{\infty} \frac{(ia_1, n)}{\{(1, n)\}^2} (ik_1\eta)^n, {}^{12}$$

η being the parabolic coordinate, in which the waves associated with the electron have a symmetry about the z-axis. The asymptotic expression for large r of the wave function is given by

$$\psi_1 \sim \frac{e^{ik_1 z - ia_1 \log k_1(r-z) - \pi a_1/2}}{\Gamma(1-ia_1)} \sum_{n=0}^{\infty} \frac{\{(ia_1, n)\}^2}{(1, n)} \left(\frac{i}{k_1\eta}\right)^n, \tag{28}$$

where $(\lambda, n) = \Gamma(\lambda+n)/\Gamma(\lambda)$. The wave function for the final state, which might be perturbed by the radiation emitted by the encounter of the electron with the nucleus, is to be that corresponding to the modified wave scattered by the bare nucleus and associated with the electron which now has less energy than the initial by an amount $h\nu$. The unmodified scattered wave has the wave function expressed asymptotically by

$$\psi_2 \sim \frac{e^{ik_2 r + ia_2 \log k_2(r-z) - \pi a_2/2}}{ik_2\eta\Gamma(ia_2)} \sum_{n=0}^{\infty} \frac{\{(1-ia_2, n)\}^2}{(1, \eta)} \left(\frac{-i}{k_2\eta}\right)^n. \tag{29}$$

Even if we could apply (28) and (29) to the problem of the emission of continuous x-rays, after taking the matrix elements by integrating

$$\left.\begin{matrix} X \\ Y \\ Z \end{matrix}\right\}_{E_1, E_2} = \int \begin{matrix} x \\ y \\ z \end{matrix} \psi_1 \psi^*{}_2 d\tau$$

for the whole space, we cannot get the intensity formula for the angular distribution. We may hope to solve the problem of the angular intensity

[11] G. Temple, Roy. Soc. Proc., **A121,** 673 (1928).

[12] The function $G(\alpha, \gamma, x)$ has been studied by Kummer, Jour. reine u. angew. Math. **15,** 139 (1836).

distribution, either by assuming some light emission process in a similar way to the classical theory of Sommerfeld,[13] or by carrying out the perturbation of the scattered electron by the radiation emitted. In the present state of the quantum mechanics, however, we cannot say anything about the light emission process regarding time and space. It is an important point of the quantum mechanics, that we do not need to assume any other light emission process than the matrix elements $q_{nm}=\int q\psi_n\psi_m^*d\tau$ to get a right answer for the intensity problem.[14] In the case of the continuous x-rays also the matrix elements should give a reasonable solution of the angular intensity distribution problem, when we take correct wave functions for the initial and final states.

After some elementary calculations we can get the wave functions which are physically admissible for the initial and the final states in the following integral forms:

$$\psi_1=e^{ik_1z}\frac{(e^{-2\pi a_1}-1)}{2\pi i}\int_0^\infty u^{-1+ia_1}(ik_1\eta+u)^{-ia_1}e^{-\eta}du, \tag{30}$$

$$\psi_2=e^{ik_2r}\frac{(e^{-2\pi a_2}-1)}{2\pi i}\int_0^\infty (-u)^{-ia_2}(ik_2\eta-u)^{-1+ia_2}e^{-u}du \tag{31}$$

which are not normalised. We will not here enter into the detail of the calculation of the matrix elements by (30) and (31), but we can see that the radiation emitted in this way is perfectly polarised. When polarised light is emitted, it is natural to take into account the perturbation of the final state by the polarised radiation emitted, just in the same way as in the case of photo-electrons. We should, therefore, take different wave functions of the final state for the different direction of the observation of radiation, or in other words the expression for the wave function of the final state contains the angle between the initial direction of the cathode rays and that of the observation of the radiation. We may hope that one can get, by this method of calculation, the angular intensity distribution, but it seems to be very complicated to carry out a rigorous calculation.

[13] I should like to express here many thanks to Prof. A. Sommerfeld for his kindness in showing me his paper before publication (Nat. Acad. Washington). In this paper he has taken the classical idea of the stopping process for the light emission, and has taken the arithmetical mean velocity during the emission process. I am afraid, however, that the standpoint of his theory is rather unsatisfactory from the point of view of the quantum mechanics. In spite of this, he has obtained results in a very good agreement with the experimental values of Kuhlenkampff. I hope to see his discussion on this very point, which will be published in the near future in Ann. d. Phys.

[14] When we calculate the retarded coordinate matrix elements, where the direction of cathode rays is specified, we can get a right answer for the angular intensity distribution. Without either the Sommerfeld's classical assumption for the light emission process or the calculation of the pertubation stated here, I could get the angular factor

$$\sin^2\delta \Big/ \left(1-\frac{v_1+v_2}{2c}\cos\delta\right)^6,$$

which is just the same as Sommerfeld's (Proc. Nat. Acad. Sci. **15,** 393 (1929)). (Note added in proof.)