CONDUCTIVITY OF IONS IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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Abstract

The effect of collisions of ions moving through crossed electric and magnetic fields is investigated. (*a*) Collisions of ions with one another are shown to be without effect, the transverse current at right angles to the fields having the same value *neu*, where u is the velocity of progression, as would exist in the absence of collisions. (*b*) Collisions of free ions with neutral particles are investigated for the case where u is small compared to the speed v of thermal agitation. As the mean free path is increased the current parallel to the electric field increases to a maximum and then falls asymptotically to zero, the transverse current parallel to u rising from zero to the limiting value *neu* for infinite mean free path. Calculation of the Hall coefficient on the present theory, which differs from the usual theory in that it takes account of long free paths, shows that the coefficient increases with increasing magnetic field.

IN AN earlier paper[1] it was shown that ions in constant electric and magnetic fields progress at right angles to the plane of the fields with the constant velocity

$$\boldsymbol{u}=c[\boldsymbol{E}\times\boldsymbol{H}]/H^2 \tag{1}$$

the projection of the path of each ion on the plane perpendicular to $\boldsymbol{H}$ being a prolate, common or curtate cycloid in accordance with the magnitude and direction of its initial velocity. In the investigation referred to, the ions were supposed to suffer no collisions. The object of the present paper is to examine the effect of collisions on the paths of the ions and to calculate the current densities and conductivities both in the direction of the electric field and in the direction of the velocity of progression $\boldsymbol{u}$. We shall limit ourselves to the case where the electric and magnetic fields are at right angles, since nothing essentially new is to be gained by including a component of the electric intensity parallel to the magnetic lines of force. Then the velocity of progression at right angles to the fields is

$$u=cE/H. \tag{2}$$

We shall consider separately (1) the case where the only collisions are between one ion and another, and (2) the case where the collisions are between ions and neutral particles, the number of collisions of one ion with another being negligible in comparison.

(1) *All collisions between ions.* This case is easily disposed of. We transform to a set of axes XYZ moving with the velocity $\boldsymbol{u}$ given by Eq. (1) rela-

[1] L. Page, Phys. Rev. **33**, 553 (1929).

tive to the observer's inertial system. By so doing we eliminate the electric field, as was shown in the paper[1] to which reference has already been made. Therefore we have a magnetic field alone in the system XYZ. Obviously this field can give rise to no general drift of the ions. Whether they collide with one another or not no current can exist relative to XYZ. Consequently the ions drift relative to the observer's inertial system with the same velocity of progression $\boldsymbol{u}$ at right angles to the fields whether they suffer collisions with one another or not. If n is the number of ions per unit volume and e the charge on each, the current density is simply

$$j = neu = necE/H \tag{3}$$

in a direction at right angles to the plane of $\boldsymbol{E}$ and $\boldsymbol{H}$.

Of course the above reasoning is valid only for $u<c$ and therefore $E<H$.

(2) *Collisions between ions and neutral particles.* Now we shall treat the case where the collisions are between free ions and particles which are unaffected by the fields, such as neutral particles in the case of gaseous conduction or fixed positive ions in the case of electronic conduction in a metal. We shall assume that the collision of one free ion with another is too infrequent to require consideration. In order to handle the problem analytically it will be necessary to disregard the Maxwellian distribution of velocities and follow Clausius' method of treating the particles as if they all had the mean speed of thermal agitation. This approximation, however, can hardly do more than introduce a small error in the numerical coefficients involved in the expressions for the conductivities. In order to avoid consideration of the drag which would be exerted on the free ions by a stream of neutral particles drifting relative to the observer, all calculations will be made relative to the inertial system in which the neutral particles, as a whole, are at rest. Finally, we shall confine ourselves to cases where the normal velocity of progression u relative to this inertial system is small compared to the velocity v of thermal agitation. This condition is undoubtedly satisfied in the upper atmosphere for electrons and protons if not for heavier ions, as the electric field in that region is small compared to the magnetic field.[2]

If we orient axes so that the y axis is parallel to $\boldsymbol{E}$, the z axis to $\boldsymbol{H}$, and therefore the x axis to $\boldsymbol{u}$, and put

$$\omega \equiv -eH/mc\ ;$$
$$v_x = v \sin\theta \cos\phi = V\cos\alpha + u,$$
$$v_y = v \sin\theta \sin\phi = V \sin\alpha,$$
$$v_z = v\cos\theta,$$

the integrated equations of motion[3] take the form

$$x = \frac{V}{\omega}[\sin(\omega t + \alpha) - \sin\alpha] + ut, \tag{4}$$

[2] L. Page, Phys. Rev. **33,** 823, (1929).
[3] L. Page, Phys. Rev. **24,** 284 (1924), Eqs. (1), (2), (3).

$$y=\frac{V}{\omega}[-\cos(\omega t+\alpha)+\cos\alpha], \tag{5}$$

$$z=v\cos\theta\cdot t. \tag{6}$$

An element of path is therefore

$$ds=vdt[1+2(uV/v^2)\{\cos(\omega t+\alpha)-\cos\alpha\}]^{1/2}.$$

As u is small compared to v by hypothesis we can expand the radical as a power series in u/v, retaining only the first order term.[4] Integrating we have

$$s=vt+(uV/\omega v)\{\sin(\omega t+\alpha)-\sin\alpha\}-(uV/v)\cos\alpha\cdot t,$$

and solving for t we get

$$t=s/v[1-(uV/\beta v^2)\{\sin(\beta+\alpha)-\sin\alpha-\beta\cos\alpha\}],$$

where β has been put for $\omega s/v$. In the term containing u we can put $v\sin\theta$ for V and ϕ for α. Hence

$$t=\frac{\beta}{\omega}\left[1-\frac{u}{\beta v}\sin\theta\{\sin(\beta+\phi)-\sin\phi-\beta\cos\phi\}\right]. \tag{7}$$

Substituting this value of t in the expressions (4), (5), (6) for x, y, z we get

$$x=\frac{v}{\omega}\sin\theta[\sin(\beta+\phi)-\sin\phi\ ]$$
$$+\frac{u}{\omega}[\beta-\sin\beta-\sin^2\theta\cos(\beta+\phi)\{\sin(\beta+\phi)-\sin\phi-\beta\cos\phi\}], \tag{8}$$

$$y=\frac{v}{\omega}\sin\theta[-\cos(\beta+\phi)+\cos\phi]$$
$$+\frac{u}{\omega}[-(1-\cos\beta)-\sin^2\theta\sin(\beta+\phi)\{\sin(\beta+\phi)-\sin\phi-\beta\cos\phi\}] \tag{9}$$

$$z=\frac{v}{\omega}\beta\cos\theta-\frac{u}{\omega}\sin\theta\cos\theta\{\sin(\beta+\phi)-\sin\phi-\beta\cos\phi\}. \tag{10}$$

Equations (8), (9) and (10) give the components of displacement of an ion as functions of the direction of its initial velocity—specified by the angles θ and ϕ—and of the length of arc traversed, $s=\beta v/\omega$.

In order to find the x component of the current density we must calculate the number of ions passing through a unit cross-section perpendicular to the x axis. Considering x, y and z as functions of s, θ and ϕ we find that if x is kept constant

[4] This approximation is also valid for small magnetic fields even if u is large compared to v, provided the electric field is small enough so that its square can be neglected. For if H is small the radical becomes: $1+(e/mv)Et\sin\theta\sin\phi+$terms in H and E^2 and higher powers.

$$[dydz]_x = \frac{J}{\frac{\partial x}{\partial s}} d\theta\, d\phi,$$

where J is the Jacobian

$$J = \frac{\partial(x, y, z)}{\partial(s, \theta, \phi)} \cdot$$

Therefore the value of $d\theta d\phi$ corresponding to a unit cross-section AB (Fig. 1) perpendicular to the x axis is

$$d\theta d\phi = \frac{\partial x/\partial s}{J} \cdot$$

If N represents the number of collisions made by the ions per unit volume per unit time and if as many ions start off in one direction as in any other after suffering a collision, the number of ions starting off on new free paths

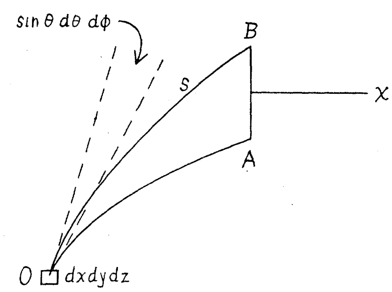

Fig. 1.

inside the conical angle sin $\theta d\theta d\phi$ is $(N/4\pi)$ sin $\theta d\theta d\phi$ per unit volume per unit time. Hence the number originating in a unit volume at O per unit time so directed as to pass through the unit cross-section AB is

$$\frac{N}{4\pi} \frac{\partial x}{\partial s} \frac{\sin\theta}{J},$$

and the number originating in a volume $dx\, dy\, dz$ at O per unit time so directed as to pass through AB is

$$\frac{N}{4\pi} \frac{\partial x}{\partial s} \frac{\sin\theta}{J} dx\, dy\, dz.$$

But

$$dx\, dy\, dz = J ds\, d\theta\, d\phi.$$

Therefore this number is

$$\frac{N}{4\pi} \frac{\partial x}{\partial s} \sin\theta ds d\theta d\phi.$$

If all the ions had exactly the same free path l the x component of the current density would be

$$j_x=\frac{Ne}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{l}\frac{\partial x}{\partial s}\sin\theta\, ds\, d\theta\, d\phi \tag{11}$$

with corresponding expressions for j_y and j_z, the letter e standing for the charge on each ion.

The chance of an ion colliding before reaching AB, however, must be taken into account, giving for j_x the integral

$$j_x=\frac{Ne}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}e^{-s/l}\frac{\partial x}{\partial s}\sin\theta\, ds\, d\theta\, d\phi \tag{12}$$

instead of (11), where l is the mean free path. Corresponding expressions hold for j_y and j_z.

While Eq. (11) does not correspond to the physical conditions actually existing, the current will be calculated on the basis of this equation for the purpose of comparison with the current calculated from Eq. (12).

Differentiating (8), (9) and (10), substituting in (11) and the corresponding equations for j_y and j_z and integrating, we get

$$j_x=neu\left[1+\frac{1}{3}\cos\gamma-\frac{4}{3\gamma}\sin\gamma\right], \tag{13}$$

$$j_y=neu\left[-\frac{1}{3}\sin\gamma+\frac{4}{3\gamma}(1-\cos\gamma)\right], \tag{14}$$

$$j_z=0. \tag{15}$$

where

$$\gamma\equiv-\beta=\frac{elH}{mcv},\qquad n\equiv\frac{Nl}{v}=\text{ number of ions per unit volume.}$$

Furthermore if we denote by σ the conductivity

$$\sigma=\frac{1}{3}\frac{ne^2l}{mv}$$

which would exist in the absence of the magnetic field, and define the conductivities σ_x, σ_y, σ_z in the x, y, z directions in the presence of both fields by the relations

$$\sigma_x=\frac{j_x}{E_y},\qquad \sigma_y=\frac{j_y}{E_y},\qquad \sigma_z=\frac{j_z}{E_y},$$

we find

$$\frac{\sigma_x}{\sigma}=\frac{3}{\gamma}\left[1+\frac{1}{3}\cos\gamma-\frac{4}{3\gamma}\sin\gamma\right], \tag{16}$$

$$\frac{\sigma_y}{\sigma}=\frac{3}{\gamma}\left[-\frac{1}{3}\sin\gamma+\frac{4}{3\gamma}(1-\cos\gamma)\right], \tag{17}$$

$$\frac{\sigma_z}{\sigma}=0. \tag{18}$$

Figure 2 shows j_x and j_y plotted against γ. The curves in this figure show how the currents change with increasing free path, the magnetic field as well

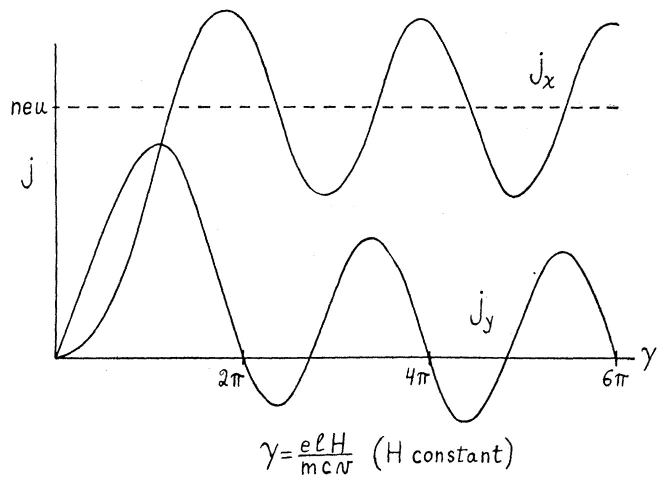

Fig. 2.

as the electric field remaining constant. The current j_y parallel to the electric field increases from zero to a maximum, decreases to zero and reverses in direction, finally oscillating about the value zero for very long free paths. The

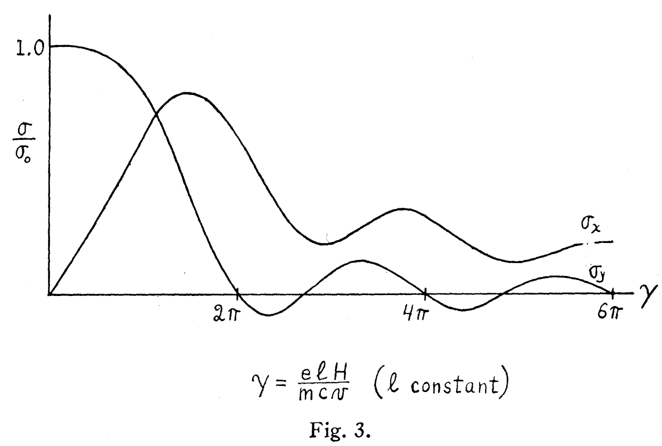

Fig. 3.

transverse current j_x, on the other hand, mounts from zero to a maximum greater than that of the current in the direction of the electric field, and finally oscillates about the normal value *neu* of the current of progression at right angles to the two fields.

In Figure 3 the conductivities σ_x and σ_y are plotted against γ. These curves show how the conductivities change with increasing magnetic field for a constant free path l. It is to be noted that while the transverse conductivity σ_x is less than the conductivity σ_y parallel to the electric field for small magnetic fields, the reverse is true for large magnetic fields. Again currents in the direction opposite to the electric field are found for certain values of the magnetic field. Both conductivities approach zero as the magnetic intensity is indefinitely increased.

Next we will evaluate (12) and the corresponding expressions for j_y and j_z. Obtaining the partial derivatives from (8), (9) and (10) as in the previous case and evaluating the integrals we get

$$j_x = neu\left[1-\frac{1}{1+\gamma^2}-\frac{2}{3}\,\frac{\gamma^2}{(1+\gamma^2)^2}\right], \tag{19}$$

$$j_y = neu\gamma\left[\frac{1}{1+\gamma^2}-\frac{1}{3}\,\frac{1-\gamma^2}{(1+\gamma^2)^2}\right], \tag{20}$$

$$j_z = 0, \tag{21}$$

and denoting by σ the conductivity

$$\sigma = \frac{2}{3}\,\frac{ne^2 l}{mv}$$

which would exist in the absence of the magnetic field in this case,

$$\frac{\sigma_x}{\sigma} = \frac{3}{2\gamma}\left[1-\frac{1}{1+\gamma^2}-\frac{2}{3}\,\frac{\gamma^2}{(1+\gamma^2)^2}\right], \tag{22}$$

$$\frac{\sigma_y}{\sigma} = \frac{3}{2}\left[\frac{1}{1+\gamma^2}-\frac{1}{3}\,\frac{1-\gamma^2}{(1+\gamma^2)^2}\right], \tag{23}$$

$$\frac{\sigma_z}{\sigma} = 0. \tag{24}$$

Figure 4 shows how the currents change with increasing mean free path, the fields being kept constant. The current j_y parallel to the electric field rises to a maximum at a value of γ equal to 1.335 and then falls, approaching zero asymptotically. The transverse current j_x increases more slowly at the start, becoming equal to the current parallel to $\boldsymbol{E}$ for a value of γ of 1.427 and then approaching asymptotically the normal current of progression *neu*. Comparison of Figs. 2 and 4 shows how the curves have been ironed out by taking into account the variations in the free paths of the ions. In this connection it must be remembered that the scale of abscissae is different in the two figures.

Finally Fig. 5 shows how the conductivities change with increasing magnetic field for a constant mean free path. The conductivity parallel to the electric field falls steadily with increasing H while the transverse conductivity rises to a maximum at γ equal to 1.468. Both conductivities

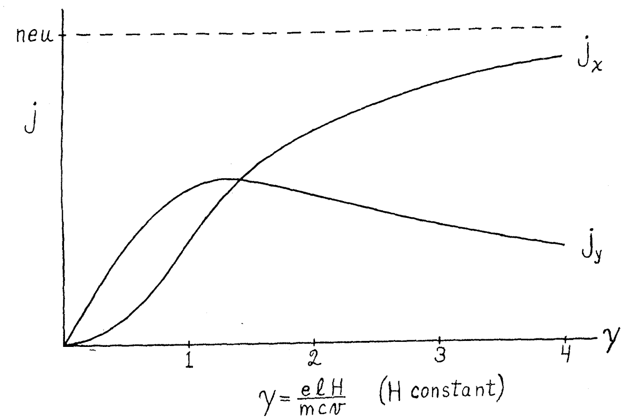

Fig. 4.

approach zero for large magnetic fields. In fact a sufficiently large magnetic field exerts a grip on the ions which makes the electric field impotent to produce appreciable motion in any direction. Again comparison of Figs. 3 and 5 shows the ironing-out effect of variations in the free paths of the ions.

Heretofore we have considered ions of one sign only. If ions of both signs are present, the total current in the direction of the electric field is the *sum*

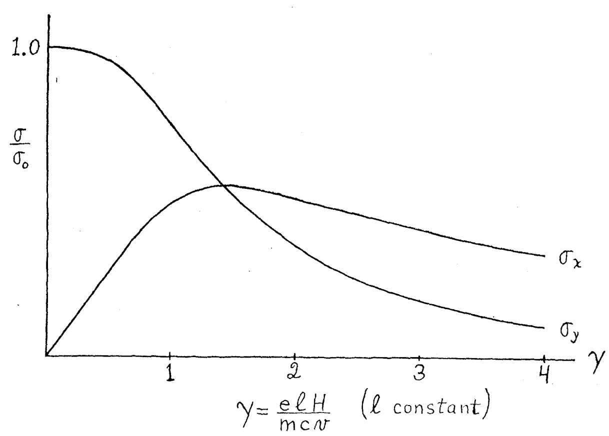

Fig. 5.

of the currents due to the two sets of ions. Ions of both signs, however, progress in the same sense perpendicular to the plane of the field, and therefore the transverse current is the *difference* of the currents due to the two sets. As the quantity denoted by γ may have a different value for the one set from what it has for the other, either on account of a difference in mass or a difference in mean free path or both, the transverse current need not vanish even when equal numbers of ions of each sign are present, except in the case of infinitely large $\boldsymbol{H}$.

If we put σ_E for the conductivity parallel to $\boldsymbol{E}$—our previous σ_y as given by Eq. (23)—and σ_H for the conductivity at right angles to $\boldsymbol{E}$—the σ_x given by Eq. (22)—we can express the current density in vector form by the equation

$$j=\sigma_E\boldsymbol{E}+\sigma_H\frac{H}{c}\boldsymbol{u}.$$

Therefore, for any orientation of the xy axes in the plane perpendicular to $\boldsymbol{H}$,

$$j_x=\sigma_E E_x+\sigma_H E_y,$$
$$j_y=-\sigma_H E_x+\sigma_E E_y.$$

If, now, the current is entirely in the x direction j_y vanishes and

$$E_y=\frac{\sigma_H}{\sigma_E}E_x,$$

$$j_x=\frac{\sigma_E{}^2+\sigma_H{}^2}{\sigma_E}E_x,$$

and the Hall coefficient obtained from the present theory is

$$\pi\equiv\frac{E_y}{Hj_x}=\frac{\sigma_H}{H(\sigma_E{}^2+\sigma_H{}^2)}\,. \tag{25}$$

Putting in the values of the conductivities given by Eqs. (22) and (23) we find

$$\pi=\frac{3}{4nec}\,\frac{1+3\gamma^2}{1+\dfrac{9}{4}\gamma^2}, \tag{26}$$

indicating that the Hall coefficient increases with the field, approaching for very large magnetic intensities a value 4/3 of its value for zero field.

In contradiction to observation the computed resistance shows a slight decrease with increasing magnetic field, approaching a limiting value 8/9 of its value for zero field as the magnetic intensity is indefinitely increased.