NOTE ON THE THEORY OF ACOUSTIC WAVE FILTERS

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Abstract

Comparison is made of the lumped impedance theory of Stewart and the more recent transmission theory of Mason. It is found that for the low frequency pass and single band types the former theory gives to a very close approximation the same frequency limits for the transmission region as the latter. The weak point of the former is the high frequency pass type, where Stewart's formulae are semi-empirical. The connection between these and the corresponding formulae of Mason is investigated.

THE development of the acoustic wave filter has been due mainly to G. W. Stewart[1], who first presented the theory and constructed actual filters. His theory given in the first of the papers noted below (hereafter referred to as *loc. cit.*) assumes lumped acoustic impedances in the main line and branch lines postulating that the air in each section of the filter moves as a whole and that the length of each is short compared with one wavelength of the sound. More recently Mason[2] has worked out a more general theory of the acoustic filter and has applied it to several special cases. It does not appear, however, that anyone has clearly pointed out the relation between the two theories to account in particular for the generally excellent agreement with the experimental data of Stewart's admittedly approximate theory. It is the aim of the present article to make this clear.

Considering the accompanying schematic diagram (Fig. 1) of an infinite filter in which the branches (represented here for simplicity as simple orifices) are assumed arbitrary in nature, the lumped impedance theory, neglecting viscosity damping, (loc. cit. p. 531) deduces the relation

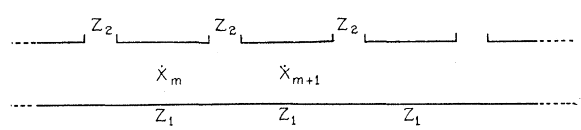

Fig. 1. Diagram of acoustic filter.

$$\dot{X}_{m+1}/\dot{X}_m = e^Y \tag{1}$$

where

$$\cosh Y = 1 + \tfrac{1}{2} Z_1/Z_2. \tag{2}$$

[1] G. W. Stewart, Phys. Rev. **20**, 528 (1922); **23**, 520 (1924); **25**, 90 (1925).
[2] W. P. Mason, Bell System Technical Journal **6**, 258 (1927).

In these equations $\dot{X}_m$ denotes the volume current at the beginning of the m^{th} section, while Z_1 is the acoustic impedance of one section of the main conduit and Z_2 that of one of the recurrent branches. Both Z_1 and Z_2 are here supposed to be pure reactances, so that their ratio is real. Transmission without attenuation occurs only when Y is a pure imaginary, i.e. $-1 < \cosh Y < +1$. Hence the limits of the transmission region will be given by the equations

$$Z_1/Z_2 = 0, \; Z_1/Z_2 = -4. \tag{3}$$

In the theory of Mason, which may perhaps be called the branch transmission theory since it is most simply developed by considering transmission through a conduit with a side branch[3] it is found that the relation corresponding to equation (1) above is

$$\dot{X}_{m+1}/\dot{X}_m = e^{-iW} \tag{4}$$

where

$$\cos W = \cos 2kl + (iZ/2Z_2) \sin 2kl. \tag{5}$$

There is a similar relation for the pressures[4]. In the Eq. (5) $2l$ = length of one section i.e. the distance between successive branches, Z_2 is the branch impedance as before, but Z is *not* the same as Z_1. Rather we have $Z = \rho_0 c/S$, where ρ_0 is the equilibrium density of the air, c the velocity of sound and S is the area of cross section of the main conduit. It is seen that Z is the acoustic resistance of the plane wave in the conduit. As usual, $k = \omega/c = 2\pi\nu/c$, where ν is the frequency. Transmission without attenuation occurs only when W is real, i.e. for $1 > \cos W > -1$. Hence on this theory the limits of the transmission region are given by the transcendental equations (Mason, reference 2, p. 267)

$$(iZ/2Z_2) = -\cot kl, \quad (iZ/2Z_2) = \tan kl. \tag{6}$$

We have now to investigate the relation between Eqs. (3) and (6). Suppose first that kl is very small, corresponding to short sections and low frequencies. Neglecting higher powers of kl than the first, Eqs. (6) become

$$2iklZ/Z_2 = -4, \quad 2iklZ/Z_2 = 0 \tag{7}$$

From the value of Z, it is seen that these take the form of Eqs. (3) provided we set $Z_1 = 2ikl\rho_0 c/S$. But this is precisely $i\omega M_1$ where M_1 = (mass of air per section)/S^2 is the inertance of one section of the conduit. Now for the low frequency pass and the single band type filters Stewart (loc. cit. p. 541 ff) assumes Z_1 to be just of this form, and hence to the above mentioned approximation the two theories give identical results.

The situation is different if l is no longer small and if high frequencies are considered, as is the case with the high pass filter, the simplest type of which is a tube with simple orifices as branches. Here since $Z_2 = i\omega M_2$

[3] G. W. Stewart, Phys. Rev. **26**, 688 (1925).

[4] See the above noted article of Mason, p. 266, and allow for change in notation.

(inertance only in the orifice) the lumped impedance theory can not define Z_1 as above. In this case Stewart (loc. cit. p. 538) assumes that Z_1 is due to a combination of inertance M_1 and capacitance C_1 in parallel, that is

$$Z_1 = i\omega M_1/(1-M_1C_1\omega^2) \tag{8}$$

where M_1 is as given above and $C_1 = V/\rho_0c^2 = 2lS/\rho_0c^2$. As an alternative choice, he also tried

$$Z_1 = -i/C_1\omega \tag{9}$$

corresponding to $M_1 = \infty$ in (8). There is little justification for either of these, except on empirical grounds, as was pointed out by Stewart. There is, indeed, some argument in favor of attributing capacitance to the conduit in this case, but no one knows *a priori* how much. However, the choice (8) led to rather good agreement with experiment, and we wish to see why. Substitution and reduction of (8) yield

$$Z_1 = i\rho_0c/S\cdot 2kl/(1-4k^2l^2)$$

and if $Z_2 = i\omega M_2 = i\omega\rho_0/c_0$, where c_0 is the acoustic conductivity of the orifice, the low frequency limit is given by

$$c_0/2kS = -(1-4k^2l^2)/kl. \tag{10}$$

For the same limit the second of Eqs. (6) gives

$$c_0/2kS = \tan kl \tag{11}$$

whence there is agreement between the two theories for this case only if

$$\tan kl = -(1-4k^2l^2)/kl$$

which is not an identity but is approximately satisfied for kl in the immediate neighborhood of $(1/3)^{1/2}$. As a matter of fact this condition is met approximately in most of Stewart's data on the high pass filter (loc. cit. p. 548). Recalculation, using the limit given by Eq. (11) gives somewhat better agreement with the experimental results, which are, of course, themselves not too precise.

It is perhaps worthy of comment that the assumption which attributes capacitance only to the conduit (see Eq. (9)) gives low frequency limits about *half* as large as the experimental values. Thus if we assume[5] $Z_1 = -4i/C_1\omega$, the second of Eqs. (3) gives for the low limit

$$c_0/2kS = kl \tag{12}$$

instead of (11). If $kl < 0.6$, the difference between kl and $\tan kl$ is less than 13 percent and the calculated frequencies differ in percentage by roughly half this. This explains why Stewart's F' values (loc. cit. p. 547 and Table III p. 548) are about half the right order of magnitude.

[5] This fact has been called to my attention by Professor Stewart.

The conclusion we reach then is that as far as the low pass and single band type filters are concerned the lumped impedance theory is as good in practice as Mason's branch transmission theory.[6] The weak point of the former lies in the high pass case, but the above discussion shows that even here it is possible to give it a reasonably satisfactory interpretation. Its greater simplicity will undoubtedly continue to render it preferable to the branch transmission theory in the practical construction of filters. Reference may here be made to a forthcoming article by G. W. Stewart and C. W. Sharp[7] in which further comparison between the two points of view is made on the basis of more recent experimental data. The measurements here referred to are of the so called *characteristic* or *mid-series* impedance, i.e. the impedance with which it is necessary to match a *finite* filter terminated half way between two successive branches in order that it may act as an *infinite* filter. On the lumped impedance theory this is $(Z_1Z_2+Z_1^2/4)^{1/2}$, while the branch transmission theory (Mason, loc. cit. p. 265) yields for the same quantity

$$p_m/\dot{X}_m=Z_0=Z\left(\frac{1+iZ/2Z_2\cdot\tan kl}{1-iZ/2Z_2\cdot\cot kl}\right)^{1/2}. \qquad (13)$$

For kl small, substitution and reduction show that to the approximation already employed in establishing Eqs. (7) these two impedances are identical.

I am indebted to Professor Stewart for discussion of the material in this article.

[6] This statement neglects the presence of the additional bands found in these types of filter, which are perhaps more easily interpreted on Mason's theory. But see Phys. Rev. **25**, 90, (1925) for Stewart's interpretation.

[7] Stewart and Sharp, Journal of the Opt. Soc. Amer. **19**, 17 (1929).