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THE EFFECT OF RETARDATION ON THE INTERACTION OF TWO ELECTRONS

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ABSTRACT

An approximate wave-equation (6) is set up which takes into account terms of the order $(v/c)^2$ in the interaction of two electrons. This equation (6) is reduced to a form (48) which can be interpreted in terms of electronic spins. Disregarding the effect of retardation the two electrons are described by formula (10). This is reduced to a form (36) which can also be described in terms of spins. It is shown that the retarded equation (48) differs from the non-retarded (36) by terms which affect the fine structur of orthohelium and which have not been known so far.

The derivation of the wave-equation (6) is made first in configuration space and later by the Heisenberg-Pauli theory of wave-fields. The latter method is used only to terms of the first order in the Coulomb interaction. It is shown from the consideration of (36) that it cannot be the correct equation and that the modifications due to retardation introduced in (6) and (48) are necessary. These modifications are appreciable only for spectra of elements with low atomic number.

1. INTRODUCTION

DIRAC'S equation for the electron enables one to derive the electronic spin and to assign values both to its angular momentum and magnetic moment. In applying his equation to spectroscopic problems it is usually sufficient to express the two small components ψ_1, ψ_2 in terms of the large components ψ_3, ψ_4 and to carry through the calculation to within terms in $(v/c)^2$. This has been done by Darwin. The present paper is an attempt to set up an equation similar to Dirac's for two electrons. The equation set up is only an approximation to within terms in $(v/c)^2$ for the energy. One's first guess in setting up such an equation is (10) below. The interaction-energy is constructed analogously to the classical

$$\frac{e^2}{r}\left[1-\frac{v^{\mathrm{I}}v^{\mathrm{II}}}{c^2}\right]$$

the first term referring to the electrostatic energy and the second to the magnetic.[1]

[1] This equation has been suggested independently by Pauli and Heisenberg. Heisenberg arrived at the result by interpreting Dirac's α's as expressions for the velocity—(v/c) while

Since in the classical theory the difference between retarded and non-retarded potentials makes itself felt in this approximation it becomes necessary to consider this difference in quantum theory as well. As a starting point Darwin's classical Hamiltonian is used. Dirac's matrices α_μ are introduced in such a way as to leave the canonical equations unaltered with the same interpretation of the α_μ as in Dirac's equation. This forms the subject of section 2. In sections 3 and 4 the retarded and non-retarded equations (6) and (10) are reduced to equations in the larger ψ components and expressed in terms of Pauli's two-row spin matrices. It is then possible to discuss the magnetic interaction of the electrons in terms of their spins and to compare them with the form which Heisenberg used in discussing the two-electron problem.

Using the new Heisenberg-Pauli theory of wave-fields it is possible to derive the retarded equation (6) to within terms of the first order in the Coulomb interaction. This derivation is made in section 5. Although all that can be maintained at present is that (6) is correct to within terms of the first order in the Coulomb interaction it is likely, since (6) has been also derived from considerations in the configuration space, that it is the correct result in the approximation $(v/c)^2$.

It can be said with certainty that (10) is not the correct equation to within this order and that therefore Gaunt's work on orthohelium will have to be extended to take into account the last term in the result of reducing (6) [See (48)].[2]

2. Derivation of Retarded and Nonretarded Equations in the Configuration Space

Dirac's equation for a single electron enables one to derive not only the proper values of the energy but also the equations of motion of the electron. This equation is

$$\left(p_0+\sum_{k=1,2,3}\alpha_k p_k+\alpha_4 mc\right)\psi=0,\quad p_0=-\frac{h}{2\pi i}\,\frac{\partial}{c\partial t}+(e/c)A_0,\qquad p_k=\frac{h}{2\pi i}\,\frac{\partial}{\partial x_k}+(e/c)A_k \tag{1}$$

Pauli got it by considerations forming an extension to those of Jordan and Klein with quantizing wave-amplitudes. Essentially the same equation has been used by Eddington and particularly Gaunt who applied it to the fine structure of orthohelium. Professor Pauli kindly suggested to the writer to reduce (10) in a manner similar to that used by Darwin in reducing Dirac's equation. The writer is very grateful to him for his advice and criticism. It is also a pleasure to express his thanks to Professor Heisenberg for discussions about the new Heisenberg-Pauli theory of wave-fields and his encouraging interest in this work.

[2] Estimates indicate that Gaunt's results are improved by this modification. On account of not having sufficiently certain expressions for the proper functions of the 2^3P state of He it is not possible to speak definitely of agreement with experiment.

The results (6) and (48) are simply extended to the case of several electrons by summation over electron-pairs. Also obvious extensions can be made for particles of unequal mass.

the Cartesian coordinates of the electron being x_1, x_2, x_3 the time t; $A_0(A_1, A_2, A_3)$ the scalar and the vector potential respectively, the charge of the electron $-e$, and c the velocity of light. Solving for the Hamiltonian E we have $E/c = -(e/c)A_0 - \sum_{k=1,2,3} \alpha_k(p_k + (e/c)A_k) - \alpha_4 mc$. We obtain with this E.

$$\frac{dx_k}{cdt} = \frac{2\pi i}{h}[(E/c), x_k] = \frac{2\pi i}{h}(Ex_k - x_k E) = -\alpha_k \tag{2}$$

and also

$$\frac{dp_1}{dt} = \frac{\partial\left(\frac{e}{c}A_1\right)}{\partial t} + \frac{2\pi i}{h}[E, p_1] = e\left(\frac{\partial A_0}{\partial x_1} + \frac{1}{c}\frac{\partial A_1}{\partial t}\right) + e(\alpha_2 H_3 - \alpha_3 H_2) \tag{3}$$

where $\boldsymbol{H} = \text{curl } \boldsymbol{A}$. It is seen that in (2) Dirac's matrices α_k are representations of $-\dot{x}_k/c$ and that (3) gives the momentum force equation, the left side being the momentum and the right the electric plus the magnetic Lorentz force.[3]

For two electrons it should be similarly required that the matrix expression for the energy should give consistent equations of motion just as Dirac's gives (2) and (3). The approximate Lagrangian and Hamiltonian functions taking into account the retardation of potentials have been derived on the classical theory by Darwin.[4] These are respectively

$$L = -m_{\rm I}c^2(1-\beta_{\rm I}^2)^{1/2} - m_{\rm II}c^2(1-\beta_{\rm II}^2)^{1/2} - e_{\rm I}e/_{\rm II}r$$
$$+ (e_{\rm I}e_{\rm II}/2c^2)\left[(\dot{\boldsymbol{r}}_{\rm I}\dot{\boldsymbol{r}}_{\rm II})r^{-1} + (\dot{\boldsymbol{r}}_{\rm I}\boldsymbol{r})(\dot{\boldsymbol{r}}_{\rm II}\boldsymbol{r})r^{-3}\right] \tag{4}$$

$$H = p_{\rm I}^2/2m_{\rm I} + p_{\rm II}^2/2m_{\rm II} - p_{\rm I}^4/8c^2m_{\rm I}^3 - p_{\rm II}^4/8c^2m_{\rm II}^3$$
$$- e_{\rm I}A_0^{\rm I} - e_{\rm II}A_0^{\rm II} + (e_{\rm I}/cm_{\rm I})(\boldsymbol{p}_{\rm I}\boldsymbol{A}^{\rm I}) + (e_{\rm II}/cm_{\rm II})(\boldsymbol{p}_{\rm II}\boldsymbol{A}^{\rm II})$$
$$+ e_{\rm I}e_{\rm II}/r - (e_{\rm I}e_{\rm II}/2c^2m_{\rm I}m_{\rm II})\left[(\boldsymbol{p}_{\rm I}\boldsymbol{p}_{\rm II})/r + (\boldsymbol{p}_{\rm I}\boldsymbol{r})(\boldsymbol{p}_{\rm II}\boldsymbol{r})/r^3\right] \tag{5}$$

Here the two electrons are distinguished by roman numerals I, II. Darwin's notation is somewhat changed, the electronic charge being denoted by $-e$ and the vector $\boldsymbol{r}$ and distance r without subscripts refer to the vector from electron I to II. The form of (5) suggests to try for two electrons

$$\Big(p_0 + \sum_{k=1,2,3}(\alpha_k^{\rm I}p_k^{\rm I} + \alpha_k^{\rm II}p_k^{\rm II}) + (\alpha_4^{\rm I} + \alpha_4^{\rm II})mc$$
$$+ (e^2/2c)\Big(\sum_k \alpha_k^{\rm I}\alpha_k^{\rm II}r^{-1} + (\boldsymbol{\alpha}^{\rm I}\boldsymbol{r})(\boldsymbol{\alpha}^{\rm II}\boldsymbol{r})r^{-2}\Big)\Big)\psi = 0. \tag{6}$$

where now

$$p_0 = -\frac{h}{2\pi i}\frac{\partial}{c\partial t} + (e/c)(A_0^{\rm I} + A_0^{\rm II}) - e^2/cr$$

[3] Expression (2) has been communicated to the writer by letter by Dr. P. A. M. Dirac in the summer of 1928. Essentially the same point of view involving also (3) has been subsequently published by Eddington (Proc. Roy. Soc. **A122,** 358 (1929)).

[4] Darwin, Phil. Mag. **39,** 537 (1920).

$$p_k^{\mathrm{I}}=\frac{h}{2\pi i}\frac{\partial}{\partial x_k^{\mathrm{I}}}+(e/c)A_k^{\mathrm{I}};\mathbf{a}^{\mathrm{I}}=(\alpha_1^{\mathrm{I}},\alpha_2^{\mathrm{I}},\alpha_3^{\mathrm{I}})$$

$$p_k^{\mathrm{II}}=\frac{h}{2\pi i}\frac{\partial}{\partial x_k^{\mathrm{II}}}+(e/c)A_k^{\mathrm{II}};\mathbf{a}^{\mathrm{II}}=(\alpha_1^{\mathrm{II}},\alpha_2^{\mathrm{II}},\alpha_3^{\mathrm{II}}).$$

A^{I}, A^{II} are respectively the potentials of electrons I, II disregarding their interaction and the wave function ψ has sixteen components which will be written in what follows in the form ψ_{mn} ($m, n=1, 2, 3, 4$) the first index m referring to electron I and the second n to II. The matrices α_k^{I}, α_k^{II} operate respectively on the upper and lower index. Thus by definition if a^{I} and b^{II} be any of these matrices

$$(a^{\mathrm{I}}b^{\mathrm{II}}\psi)_{m,n}=\sum_{k,l}a_{mk}^{\mathrm{I}}b_{nl}^{\mathrm{II}}\psi_{kl} \tag{7}$$

Equation (6) is thus equivalent to 16 equations involving 16 variables ψ_{mn}. It is clear by performing a calculation similar to that used in deriving (2) from (1) that using (6)

$$\frac{dx_k^{\mathrm{II}}}{dt}=-c\alpha_k^{\mathrm{I}} \qquad \frac{dx_k^{\mathrm{II}}}{dt}=-c\alpha_k^{\mathrm{II}} \tag{8}$$

so that the matrices representing velocities are the same as for one electron. Similarly we find imitating (3) that

$$\frac{dp_k^{\mathrm{I}}}{dt}=e\left(\frac{\partial A_0^{\mathrm{I}}}{\partial x_k^{\mathrm{I}}}+\frac{\partial A_k^{\mathrm{I}}}{c\partial t}\right)+\sum e\alpha_j^{\mathrm{I}}\left(\frac{\partial A_j^{\mathrm{I}}}{\partial x_k^{\mathrm{I}}}-\frac{\partial A_k^{\mathrm{I}}}{\partial x_j^{\mathrm{I}}}\right)$$
$$+\frac{\partial}{\partial x_k^{\mathrm{I}}}\left[-e^2/r+(e^2/2cr)((\mathbf{a}^{\mathrm{I}}\mathbf{a}^{\mathrm{II}})+(\mathbf{a}^{\mathrm{I}}\boldsymbol{r})(\alpha^{\mathrm{II}}\boldsymbol{r})/r^2)\right] \tag{9}$$

and that the same equation follows from (5) on the classical theory provided it is remembered that the p_μ in (5) do not include the potentials as they do in (6). The values for the energy from (5) and (6) are also found to agree to within terms in $(v/c)^2$ provided it is remembered that α_4 should be replaced by $(1-\beta^2)^{1/2}$.[5]

If instead of using (5) the effect of retardation is neglected and the attempt is made to take into account only the magnetic interaction between the electrons corresponding to an interaction energy $(e^2/r)(1-v^{\mathrm{I}}v^{\mathrm{II}}/c^2)$ we obtain instead of (6)

$$\left(p_0+\sum_k(\alpha_k^{\mathrm{I}}p_k^{\mathrm{I}}+\alpha_k^{\mathrm{II}}p_k^{\mathrm{II}})+(\alpha_4^{\mathrm{I}}+\alpha_4^{\mathrm{II}})mc+(e^2/cr)(\mathbf{a}^{\mathrm{I}}\mathbf{a}^{\mathrm{II}})\right)\psi=0. \tag{10}$$

[5] If instead of writing (6) we try to use in the interaction term expressions of the form $(1/2m)\sum_k p_k+\alpha_k\ p_k\ )$ then it is found that the simple form of (8) is no longer true and that it also becomes difficult to get simultaneously the correct value of H and the equations of motion. If the matrices represent the velocity components of a single electron, it would be surprising if for two electrons the velocity components were represented in part by the differential operators p.

This is the equation used by Gaunt and Eddington[6], and claimed by them to be correct. As we shall see the consequences of (10) are in some respects unreasonable. Further (6) will be derived to within first order terms in the interaction from the general theory of Heisenberg and Pauli. The work involved in reducing (10) to a form convenient for spectroscopic applications can be conveniently applied to (6). We, therefore reduce (10) first.

3. Reduction of Nonretarded Equation to Larger Components

We follow here a method used by Darwin in discussing Dirac's equation. It makes it possible to reduce equation (10) which involves 16 components (see (7)) to an approximate equation with 4 components expressed in terms of Pauli's spin matrices. The approximation desired is that needed in most spectroscopic applications and includes terms in $(v/c)^2$. The matrices α are taken to be

$$\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\quad \alpha_2=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\quad \alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$$
$$\alpha_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{11}$$

Writing

$$a^{\mathrm{I}}=p_1{}^{\mathrm{I}}+ip_2{}^{\mathrm{I}},\ b^{\mathrm{I}}=p_1{}^{\mathrm{I}}-ip_1{}^{\mathrm{I}},\ c^{\mathrm{I}}=p_3{}^{\mathrm{I}} \tag{12}$$

the (16) equations (10) involve the following types

$$(p_0+2mc)\psi_{1,1}+b^{\mathrm{I}}\psi_{4,1}+c^{\mathrm{I}}\psi_{3,1}+b^{\mathrm{II}}\psi_{1,4}+c^{\mathrm{II}}\psi_{1,3}+\frac{e^2}{cr}\psi_{3,3}=0 \tag{13.1}$$

$$(p_0+2mc)\psi_{2,2}+a^{\mathrm{I}}\psi_{3,2}-c^{\mathrm{I}}\psi_{4,2}+a^{\mathrm{II}}\psi_{2,3}-c^{\mathrm{II}}\psi_{2,4}+\frac{e^2}{cr}\psi_{4,4}=0 \tag{13.2}$$

$$(p_0+2mc)\psi_{1,2}+b^{\mathrm{I}}\psi_{4,2}+c^{\mathrm{I}}\psi_{3,2}+a^{\mathrm{II}}\psi_{1,3}-c^{\mathrm{II}}\psi_{1,4}+\frac{e^2}{cr}(2\psi_{4,3}-\psi_{3,4})=0 \tag{13.3}$$

$$p_0\psi_{1,3}+b^{\mathrm{I}}\psi_{4,3}+c^{\mathrm{I}}\psi_{3,3}+b^{\mathrm{II}}\psi_{1,2}+c^{\mathrm{II}}\psi_{1,1}+\frac{e^2}{er}\psi_{3,1}=0 \tag{14.1}$$

$$p_0\psi_{1,4}+b^{\mathrm{I}}\psi_{4,4}+c^{\mathrm{I}}\psi_{3,4}+a^{\mathrm{II}}\psi_{1,1}-c^{\mathrm{II}}\psi_{1,2}+\frac{e^2}{cr}(2\psi_{4,1}-\psi_{3,2})=0 \tag{14.2}$$

$$p_0\psi_{2,3}+a^{\mathrm{I}}\psi_{3,3}-c^{\mathrm{I}}\psi_{4,3}+b^{\mathrm{II}}\psi_{2,2}+c^{\mathrm{II}}\psi_{2,1}+\frac{e^2}{cr}(2\psi_{3,2}-\psi_{4,1})=0 \tag{14.3}$$

[6] Eddington, reference 3; Gaunt Phil. Trans. Roy. Soc. **228,** 151–196 Pamphlet **A662** (1929); Proc. Roy. Soc. **A122,** 153 (1929).

$$p_0\psi_{2,4}+a^{\mathrm{I}}\psi_{3,4}-c^{\mathrm{I}}\psi_{4,4}+a^{\mathrm{II}}\psi_{2,1}-c^{\mathrm{II}}\psi_{2,2}+\frac{e^2}{cr}\psi_{4,2}=0 \tag{14.4}$$

$$(p_0-2mc)\psi_{3,3}+b^{\mathrm{I}}\psi_{2,3}+c^{\mathrm{I}}\psi_{1,3}+b^{\mathrm{II}}\psi_{3,2}+c^{\mathrm{II}}\psi_{3,1}+\frac{e^2}{cr}\psi_{1,1}=0 \tag{15.1}$$

$$(p_0-2mc)\psi_{4,4}+a^{\mathrm{I}}\psi_{1,4}-c^{\mathrm{I}}\psi_{2,4}+a^{\mathrm{II}}\psi_{4,1}-c^{\mathrm{II}}\psi_{4,2}+\frac{e^2}{cr}\psi_{2,2}=0 \tag{15.2}$$

$$(p_0-2mc)\psi_{4,3}+b^{\mathrm{I}}\psi_{2,4}+c^{\mathrm{I}}\psi_{1,4}+a^{\mathrm{II}}\psi_{3,1}-c^{\mathrm{II}}\psi_{3,2}+\frac{e^2}{cr}(2\psi_{2,1}-\psi_{1,2})=0 \tag{15.3}$$

It is not necessary to write out the remaining 6 equations since they can be obtained by interchanging I and II. Since $p_0\cong 2mc$ we see that $\psi_{4,4}$, $\psi_{3,3}$, $\psi_{3,4}$, $\psi_{4,3}$ are large while the remaining $\psi_{n,m}$ are small. Using equations (13), (14) we let to start with $p_0=2mc$ and solve for $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{1,3}$ in terms of the four large ψ's neglecting for the present all terms of order higher than v/c. Equations (14) determine the order of magnitude of $\psi_{1,3}$, $\psi_{1,4}$, $\psi_{2,3}$, $\psi_{2,4}$ as v/c. Using these values in (13) the order of $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{2,2}$ is $(v/c)^2$ and to the first approximation these may be neglected. The results for the first approximation may be expressed most conveniently by means of Pauli's matrices with two rows and columns which represent the spin. We let

$$\sigma_1=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{16}$$

[These are the negative of Pauli's matrices. It is for the present purpose somewhat more convenient to have the sign changed as done here.] The two row matrices are used as operators on the suffixes 3, 4 only.

Using the relations:

$$\begin{aligned}&((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{I}})\psi)_{3,3}=-(b\psi_{4,3}+c\psi_{3,3})\ ;\ ((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{I}})\psi)_{3,4}=-(b\psi_{4,4}+c\psi_{3,4})\ ;\ ((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{I}})\psi)_{4,3}\\ &\qquad=-(a\psi_{3,3}-c\psi_{4,3})\ ;\ ((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{I}})\psi)_{4,4}=-(a\psi_{3,4}-c\psi_{4,4})\\ &((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{3,3}=-(b\psi_{3,4}+c\psi_{3,3})\ ;\ ((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{3,4}=-(a\psi_{3,3}-c\psi_{3,4})\ ;\ ((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{4,3}\\ &\qquad=-(b\psi_{4,4}+c\psi_{4,3})\ ;\ ((\boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{4,4}=-(a\psi_{4,3}-c\psi_{4,4})\end{aligned} \tag{17}$$

$$\begin{aligned}&((\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{3,3}=\psi_{3,3}\ ;\ ((\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{3,4}=2\psi_{4,3}-\psi_{3,4}\ ;\ ((\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{4,3}\\ &\qquad=2\psi_{3,4}-\psi_{4,3}\ ;\ ((\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})\psi)_{4,4}=\psi_{4,4}\end{aligned} \tag{18}$$

and letting

$$\chi^{\mathrm{I}}=(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})\psi,\quad \chi^{\mathrm{II}}=(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\psi \tag{19}$$

we have to a first approximation

$$p_0=2mc\ ;\ \psi_{1,1}=\psi_{1,2}=\psi_{2,1}=\psi_{2,2}=0$$

$$\begin{aligned}&\psi_{1,3}=(2mc)^{-1}\overset{\mathrm{I}}{\chi}_{3,3}\ ;\ \psi_{1,4}=(2mc)^{-1}\overset{\mathrm{I}}{\chi}_{3,4}\ ;\ \psi_{2,3}=(2mc)^{-1}\overset{\mathrm{I}}{\chi}_{4,3}\ ;\ \psi_{2,4}=(2mc)^{-1}\overset{\mathrm{I}}{\chi}_{4,4}\\ &\psi_{3,1}=(2mc)^{-1}\overset{\mathrm{II}}{\chi}_{3,3}\ ;\psi_{4,1}=(2mc)^{-1}\overset{\mathrm{II}}{\chi}_{4,3}\ ;\psi_{3,2}=(2mc)^{-1}\overset{\mathrm{II}}{\chi}_{3,4}\ ;\psi_{4,2}=(2mc)^{-1}\overset{\mathrm{II}}{\chi}_{4,4}\end{aligned} \tag{20}$$

Substituting these values in (13.1), (13.2), (13.3) we derive values for $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{2,2}$, $\psi_{2,1}$ to the order $(v/c)^2$ which on using (17) and (18) again reduce to

$$\psi_{1,1}=\eta_{3,3}\ ;\ \psi_{2,2}=\eta_{4,4}\ ;\ \psi_{1,2}=\eta_{3,4}\ ;\ \psi_{2,1}=\eta_{4,3} \tag{21}$$

where

$$\eta=-\frac{e^2}{4mc^2}\ \frac{1}{r}(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})\psi+\frac{(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})}{4m^2c^2}\psi\ . \tag{22}$$

Also substituting (21) in (14.1), (14.2), (14.3), (14.4) and again using (17) and (18) we get to the order $(v/c)^3$

$$\begin{aligned}
&\psi_{1,3}=p_0^{-1}(\chi^{\mathrm{I}}+\xi^{\mathrm{I}})_{3,3}\ ;\quad \psi_{1,4}=p_0^{-1}(\chi^{\mathrm{I}}+\xi^{\mathrm{I}})_{3,4}\ ;\quad \psi_{2,3}=p_0^{-1}(\chi^{\mathrm{I}}+\xi^{\mathrm{I}})_{4,3}\ ;\\
&\qquad\qquad \psi_{2,4}=p_0^{-1}(\chi^{\mathrm{I}}+\xi^{\mathrm{I}})_{4,4}\\
&\psi_{3,1}=p_0^{-1}(\chi^{\mathrm{II}}+\xi^{\mathrm{II}})_{3,3}\ ;\ \psi_{4,1}=p_0^{-1}(\chi^{\mathrm{II}}+\xi^{\mathrm{II}})_{4,3}\ ;\ \psi_{3,2}=p_0^{-1}(\chi^{\mathrm{II}}+\xi^{\mathrm{II}})_{3,4}\ ;\\
&\qquad\qquad \psi_{4,2}=p_0^{-1}(\chi^{\mathrm{II}}+\xi^{\mathrm{II}})_{4,4}
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\xi^{\mathrm{I}}&=-\frac{e^2}{4mc^2}\{(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(r^{-1}\psi)+2r^{-1}(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\psi\}+\frac{(\boldsymbol{p}^{\mathrm{I}}\sigma^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\sigma^{\mathrm{II}})^2}{4m^2c^2}\psi\\
\xi^{\mathrm{II}}&=-\frac{e^2}{4mc^2}\{(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(r^{-1}\psi)+2r^{-1}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})\psi\}+\frac{(\boldsymbol{p}^{\mathrm{II}}\sigma^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{I}}\sigma^{\mathrm{I}})^2}{4m^2c^2}\psi
\end{aligned} \tag{24}$$

We have now obtained expressions for all the $\psi_{n,m}$ correct to the order $(v/c)^2$ in terms of $\psi_{3,3}$, $\psi_{3,4}$, $\psi_{4,3}$, $\psi_{4,4}$. We substitute these values into (15.1), (15.2), (15.3). The term p_0-2mc is itself of order mv^2/c or $(E-mc^2)/c$. Our problem is to determine $E-mc^2$ correctly to within the third term in an expansion according to powers of (v/c). Hence we should require that on dividing equations (15) by c all the other terms should be known to an accuracy $(v/c)^4$. Since a^{I}/c, b^{I}/c, c^{I}/c are of the order v/c the accuracy $(v/c)^3$ in determining $\psi_{4,1}$, $\psi_{3,1}$ etc. suffices and similarly since e^2/c^2r is of the order $(v/c)^2$ the accuracy $(v/c)^2$ in determining $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{2,2}$, $\psi_{2,1}$ is also satisfactory. Performing the substitution of (21), (23) into (15.1), (15.2), (15.3) we have using (17), (18) once more

$$\left\{(p_0-2mc)\psi-(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})p_0^{-1}(\chi^{\mathrm{I}}+\xi^{\mathrm{I}})-(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})p_0^{-1}(\chi^{\mathrm{II}}+\xi^{\mathrm{II}})+\frac{e^2}{cr}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})\eta)\right\}_{\alpha,\beta}=0 \tag{25}$$

where

$$\alpha,\beta=3\quad\text{or}\quad 4.$$

This is the result of eliminating in (10) the smaller components and expressing them in terms of the larger ones. It now remains to substitute into (25) equations (19), (22), (24) and to make the result linear in σ^{I}, σ^{II}. We do this by means of the formulas

$$\begin{aligned}
&\sigma_1\sigma_2=-i\sigma_3,\ \ \sigma_2\sigma_3=-i\sigma_1,\ \ \sigma_3\sigma_1=-i\sigma_2\\
&\sigma_1^2=\sigma_2^2=\sigma_3^2=1,\ \ \sigma_i\sigma_k+\sigma_k\sigma_i=0\quad(i\neq k)\,.
\end{aligned} \tag{26}$$

Performing the substitution and collecting terms we have:

$$\left\{(p_0-2mc)-(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})p_0^{-1}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})-(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})p_0^{-1}(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\right.$$
$$+\frac{e^2}{4m^2c^3}[(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}+(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})r^{-1}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})$$
$$+(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})+r^{-1}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})]$$
$$\left.-\frac{(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})^2(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})^2}{4m^3c^3}-\frac{e^4}{4mc^3}\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})^2}{r^2}\right\}\psi=0. \tag{27}$$

The term $e^2/4m^2c^3[\;]$ consisting of four members may be transformed conveniently by using a symbol $\langle\;\rangle$ to indicate that the differential operators p^{I}, p^{II} apply only within the $\langle\;\rangle$. We have then, remembering that $\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}}r^{-1}\psi = \langle\boldsymbol{p}^{\mathrm{I}}r^{-1}\rangle\boldsymbol{p}^{\mathrm{II}}\psi+\langle\boldsymbol{p}^{\mathrm{II}}r^{-1}\rangle\boldsymbol{p}^{\mathrm{I}}\psi+\langle\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}}r^{-1}\rangle\psi+r^{-1}\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}}\psi$ understanding p and σ as vectors

$$(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(r^{-1}\psi)+(p^{\mathrm{I}}\sigma^{\mathrm{I}})r^{-1}(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})\psi+(p^{\mathrm{II}}\sigma^{\mathrm{II}})r^{-1}(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{I}}\sigma^{\mathrm{I}})\psi$$
$$+r^{-1}(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})\psi=\langle(p^{\mathrm{I}}\sigma^{\mathrm{I}})r^{-1}\rangle((p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})+(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{II}}\sigma^{\mathrm{II}}))\psi$$
$$+r^{-1}((p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})+(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}}))\psi+\langle(p^{\mathrm{II}}\sigma^{\mathrm{II}})r^{-1}\rangle((p^{\mathrm{I}}\sigma^{\mathrm{I}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})$$
$$+(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{I}}\sigma^{\mathrm{I}}))\psi+\langle(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})r^{-1}\rangle\psi+r^{-1}((p^{\mathrm{I}}\sigma^{\mathrm{I}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})$$
$$+(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{I}}\sigma^{\mathrm{I}}))\psi=2\langle(p^{\mathrm{I}}\sigma^{\mathrm{I}})r^{-1}\rangle(p^{\mathrm{I}}\sigma^{\mathrm{I}})\psi+2\langle(p^{\mathrm{II}}\sigma^{\mathrm{II}})r^{-1}\rangle(p^{\mathrm{I}}\sigma^{\mathrm{II}})\psi$$
$$+\langle(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})r^{-1}\rangle\psi+2r^{-1}((p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{I}})+(p^{\mathrm{II}}\sigma^{\mathrm{I}})(p^{\mathrm{I}}\sigma^{\mathrm{I}}))\psi=4r^{-1}(p^{\mathrm{I}}p^{\mathrm{II}})\psi$$
$$+\langle(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})r^{-1}\rangle\psi+2(\langle p^{\mathrm{I}}r^{-1}\rangle p^{\mathrm{II}})+(\langle p^{\mathrm{II}}r^{-1}\rangle p^{\mathrm{I}})$$
$$-i[\langle p^{\mathrm{I}}r^{-1}\rangle\times p^{\mathrm{II}}]\sigma^{\mathrm{I}}-i[\langle p^{\mathrm{II}}r^{-1}\rangle\times p^{\mathrm{I}}]\sigma^{\mathrm{II}})\psi. \tag{28}$$

Here the $p_k{}^{\mathrm{I}}$ and $p_k{}^{\mathrm{II}}$ were treated as ordinary differential operators because on division by c the whole term is of the order $(v/c)^4$ so that the higher order terms arising from the noncommutability of the p's may be neglected. Use has been made in the above of formulas following from (26) *viz.* for any two commuting vectors $\boldsymbol{A}$, $\boldsymbol{B}$

$$(\boldsymbol{A}\boldsymbol{\sigma})(\boldsymbol{B}\boldsymbol{\sigma})=(\boldsymbol{A}\boldsymbol{B})-i[\boldsymbol{A}\times\boldsymbol{B}]\boldsymbol{\sigma} \tag{29}$$

where $[\boldsymbol{A}\times\boldsymbol{B}]$ is the vector product of $\boldsymbol{A}$ and $\boldsymbol{B}$. Since the last term in (28) is the result of applying (29) to $2\langle(p^{\mathrm{I}}\sigma^{\mathrm{I}})r^{-1}\rangle(p^{\mathrm{II}}\sigma^{\mathrm{I}})+2\langle(p^{\mathrm{II}}\sigma^{\mathrm{II}})r^{-1}\rangle(p^{\mathrm{I}}\sigma^{\mathrm{II}})$ only one of p's operates on ψ while the other operates on r^{-1}. Again using (29) we have

$$(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})=(p^{\mathrm{I}}\sigma^{\mathrm{I}})\{(p^{\mathrm{II}}\sigma^{\mathrm{I}})+i[p^{\mathrm{II}}\times\sigma^{\mathrm{II}}]\sigma^{\mathrm{I}}\}$$
$$=(p^{\mathrm{I}}p^{\mathrm{II}})-i[p^{\mathrm{I}}\times p^{\mathrm{II}}]\sigma^{\mathrm{I}}+i\{p^{\mathrm{I}}[p^{\mathrm{II}}\times\sigma^{\mathrm{II}}]-i[p^{\mathrm{I}}\times[p^{\mathrm{II}}\times\sigma^{\mathrm{II}}]]\sigma^{\mathrm{I}}\} \tag{30}$$
$$=(p^{\mathrm{I}}p^{\mathrm{II}})-i[p^{\mathrm{I}}\times p^{\mathrm{II}}](\sigma^{\mathrm{I}}-\sigma^{\mathrm{II}})+(p^{\mathrm{II}}\sigma^{\mathrm{I}})(p^{\mathrm{I}}\sigma^{\mathrm{II}})-(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})(p^{\mathrm{I}}p^{\mathrm{II}}).$$

If this operates on $1/r$ it is simply

$$\langle(p^{\mathrm{I}}\sigma^{\mathrm{I}})(p^{\mathrm{II}}\sigma^{\mathrm{II}})(\sigma^{\mathrm{I}}\sigma^{\mathrm{II}})r^{-1}\rangle=\langle(p^{\mathrm{II}}\sigma^{\mathrm{I}})(p^{\mathrm{I}}\sigma^{\mathrm{II}})r^{-1}\rangle. \tag{30'}$$

We have therefore finally

$$\begin{aligned}\frac{e^2}{4m^2c^3}&[(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\mathbf{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}\psi+(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})r^{-1}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\mathbf{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\psi\\ &+(\mathbf{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})\psi+r^{-1}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(p^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\psi]\\ &=\frac{e^2}{m^2c^3r}(\mathbf{p}^{\mathrm{I}}\mathbf{p}^{\mathrm{II}})\psi+\frac{e^2}{4m^2c^3}\langle(\mathbf{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{I}})(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}\rangle\\ &+\frac{e^2}{2m^2c^3}((\langle\mathbf{p}^{\mathrm{I}}\mathbf{r}^{-1}\rangle p^{\mathrm{II}})+(\langle\mathbf{p}^{\mathrm{II}}r^{-1}\rangle\mathbf{p}^{\mathrm{I}})-i[\langle\mathbf{p}^{\mathrm{I}}r^{-1}\rangle\times\mathbf{p}^{\mathrm{II}}]\boldsymbol{\sigma}^{\mathrm{I}}-i[\langle\mathbf{p}^{\mathrm{II}}r^{-1}\rangle\times\mathbf{p}^{\mathrm{I}}]\boldsymbol{\sigma}^{\mathrm{II}})\psi\end{aligned}\tag{31}$$

In computing $(p^{\mathrm{I}}\sigma^{\mathrm{I}})p_0^{-1}(p^{\mathrm{I}}\sigma^{\mathrm{I}})$ we may not disregard the noncommutability of the p's because this term on division by c is of the order $(v/c)^2$. We have:

$$(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})p_0^{-1}(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})=(2mc)^{-1}(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})[1-(2mc^{2-1}(E-2mc^2+eV)](\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})$$

where

$$V=A_0{}^{\mathrm{I}}+A_0{}^{\mathrm{II}}-\frac{e^2}{r}\tag{32}$$

or

$$\begin{aligned}(\mathbf{p}^{\mathrm{I}}\sigma^{\mathrm{I}})p_0^{-1}(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})&=(2mc)^{-1}\sum\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_k{}^{\mathrm{I}}}+\frac{e}{c}A_k{}^{\mathrm{I}}\right)\sigma_k{}^{\mathrm{I}}[1-(2mc^2)^{-1}(E-2mc^2\\ &\qquad+eV)]\sum\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_k{}^{\mathrm{I}}}+\frac{e}{c}A_k{}^{\mathrm{I}}\right)\sigma_k{}^{\mathrm{I}}\\ &=(2mc^{-1}\left(\sum\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_k{}^{\mathrm{I}}}+\frac{e}{c}A_k{}^{\mathrm{I}}\right)\sigma_k{}^{\mathrm{I}}\right)^2\\ &\qquad-(2mc)^{-1}(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(2mc^2)^{-1}(E-2mc^2+eV)(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})\end{aligned}$$

where in the second part of the expression the difference between $p_k{}^{\mathrm{I}}$ and $(h/2\pi i)\partial/\partial x_k{}^{\mathrm{I}}$ is not essential since that part on division by c is of the order $(v/c)^4$. We have therefore

$$\begin{aligned}(p^{\mathrm{I}}\sigma^{\mathrm{I}})p_0^{-1}(p^{\mathrm{I}}\sigma^{\mathrm{I}})&=(2mc)^{-1}\sum(p_k{}^{\mathrm{I}})^2-\frac{eh}{4\pi mc^2}(\mathbf{H}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})-\frac{E-2mc^2+eV}{4m^2c^3}\sum(p_k{}^{\mathrm{I}})^2\\ &\qquad-\frac{ehi}{8\pi m^2c^3}(\mathcal{E}^{\mathrm{I}}\mathbf{p}^{\mathrm{I}})-\frac{eh}{8\pi mc^3}\left[\mathcal{E}^{\mathrm{I}}\times\frac{\mathbf{p}^{\mathrm{I}}}{m}\right]\cdot\boldsymbol{\sigma}^{\mathrm{I}}\end{aligned}\tag{33}$$

where

$$\begin{aligned}&\mathbf{H}^{\mathrm{I}}=\operatorname{curl}\mathbf{A}^{\mathrm{I}}\\ &\mathcal{E}^{\mathrm{I}}=-\operatorname{grad}^{\mathrm{I}}V=-\left(\frac{\partial}{\partial x_1{}^{\mathrm{I}}},\frac{\partial}{\partial x_2{}^{\mathrm{I}}},\frac{\partial}{\partial x_3{}^{\mathrm{I}}}\right)V.\end{aligned}\tag{33'}$$

The term $(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})^2(p^{\mathrm{II}}\sigma^{\mathrm{II}})^2$, entering as $(v/c)^4$, we neglect in it A_k and obtain

$$(\mathbf{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})^2(\mathbf{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})^2=\sum(p_k{}^{\mathrm{I}})^2\sum(p_k{}^{\mathrm{II}})^2=(\mathbf{p}^{\mathrm{I}})^2(\mathbf{p}^{\mathrm{II}})^2\tag{34}$$

Using (26)

$$(\sigma^{I}\sigma^{II})^2 = 3 - 2(\sigma^{I}\sigma^{II}) \tag{35}$$

Substituting (31), (33), (34), (35) into (27) we have

$$\left\{E-2mc^2+eV-\frac{1}{2m}\left(1-\frac{E-2mc^2+eV}{2mc^2}\right)\sum((p_k{}^{I})^2+(p_k{}^{II})^2)\right.$$
$$+\frac{he}{4\pi mc}[(H^{I}\sigma^{I})+(H^{II}\sigma^{II})]+\frac{ehi}{8\pi m^2c^2}[(\mathcal{E}^{I}p^{I})+(\mathcal{E}^{II}p^{II})]$$
$$+\frac{he}{8\pi mc^2}\left(\left[\mathcal{E}^{I}\times\frac{p^{I}}{m}\right]\sigma^{I}+\left[\mathcal{E}^{II}\times\frac{p^{II}}{m}\right]\sigma^{II}\right)+\frac{e^2}{c^2r}\left(\frac{p^{I}}{m}\cdot\frac{p^{II}}{m}\right)$$
$$-\left(\frac{eh}{4\pi mc}\right)^2\langle(\nabla^{II}\sigma^{I})(\nabla^{I}\sigma^{II})r^{-1}\rangle+\frac{e^2hi}{4\pi m^2c^2}\left[\left(\frac{r^{II}-r^{I}}{r^3}\cdot p^{I}\right)+\left(\frac{r^{I}-r^{II}}{r^3}\cdot p^{II}\right)\right]$$
$$-\frac{e^2h}{4\pi mc^2}\left(\left[\frac{r^{II}-r^{I}}{r^3}\times\frac{p^{II}}{m}\right]\sigma^{I}+\left[\frac{r^{I}-r^{II}}{r^3}\times\frac{p^{I}}{m}\right]\sigma^{II}\right)-\frac{(p^{I})^2(p^{II})^2}{4m^3c^2}$$
$$\left.-\frac{e^4}{4mc^2}\,\frac{3-2(\sigma^{I}\sigma^{II})}{r^2}\right\}\psi=0.$$

Solving for $E-2mc^2+eV$

$$\left\{E-2mc^2+eV-\frac{1}{2m}((p^{I})^2+(p^{II})^2)+\frac{(p^{I})^4+(p^{II})^4}{8m^3c^2}+\frac{e^2}{r}\left(\frac{p^{I}}{mc}\cdot\frac{p^{II}}{mc}\right)\right.$$
$$+\frac{he}{4\pi mc}[(H^{I}\sigma^{I})+(H^{II}\sigma^{II})]+\frac{ehi}{8\pi m^2c^2}\left[(\mathcal{E}^{I}p^{I})+(\xi^{II}p^{II})+2e\left(\frac{r^{II}-r^{I}}{r^3}\cdot p^{I}\right)\right.$$
$$\left.+2e\left(\frac{r^{I}-r^{II}}{r^3}\cdot p^{II}\right)\right]+\frac{he}{8\pi mc^2}\left(\left[\mathcal{E}^{I}\times\frac{p^{I}}{m}\right]\sigma^{I}+\left[\mathcal{E}^{II}\times\frac{p^{II}}{m}\right]\sigma^{II}\right.$$
$$\left.+2e\left[\frac{r^{I}-r^{II}}{r^3}\times\frac{p^{II}}{m}\right]\sigma^{I}+2e\left[\frac{r^{II}-r^{I}}{r^3}\times\frac{p^{I}}{m}\right]\sigma^{II}\right)$$
$$\left.-\left(\frac{eh}{4\pi mc}\right)^2\langle(\nabla^{II}\sigma^{I})(\nabla^{I}\sigma^{II})r^{-1}\rangle-\frac{e^4}{4mc^2}(3-2(\sigma^{I}\sigma^{II}))r^{-2}\right\}\psi=0. \tag{36}$$

This is the result of reducing equation (10) with 16 components to equations in the 4 larger components. Comparing this with equation (24) of Heisenberg[7] we see that all of these terms are included in his (24) with the exception of

$$\frac{(p^{I})^4+(p^{II})^4}{8m^3c^2}+\frac{e^2}{r}\left(\frac{p^{I}}{mc}\cdot\frac{p^{II}}{mc}\right)+\frac{ehi}{8\pi m^2c^2}\left[(\mathcal{E}^{I}p^{I})+(\mathcal{E}^{II}p^{II})+2e\left(\frac{r^{II}-r^{I}}{r^3}\cdot p^{I}\right)\right.$$
$$\left.+2e\left(\frac{r^{I}-r^{II}}{r^3}\cdot p^{II}\right)\right]-\frac{e^1}{4mc^2}(3-2(\sigma^{I}\sigma^{II})r^{-2}.$$

[7] Heisenberg, Zeits. f. Physik 39, 499 (1926).

The first of these may be called the relativity correction to the kinetic energy, the second is the magnetic interaction of the orbits, the third is characteristic of using Dirac's equation, has been discussed by Darwin,[8] for a single electron and will be discussed in more detail below; the fourth term may be called a higher order interaction of the spins but as we shall see later enters here with a wrong factor on account of having neglected retardation. In particular we observe that the fine structure of helium should be exactly the same according to (36) and (10) as according to Heisenberg's (24). In making the comparison with Heisenberg it is necessary to remember that our σ_k^{I}, σ_k^{II} are just the negative of Pauli's and that Heisenberg's last term representing the dipole interaction is given in (36) as $-(eh/4\pi mc)^2\langle(\nabla^{\mathrm{II}}\sigma^{\mathrm{I}})(\nabla^{\mathrm{I}}\sigma^{\mathrm{II}})r^{-1}\rangle$.

Reduction of Retarded Equation to Larger Components

We can now perform a similar calculation for the retarded equation (6). The first approximation and therefore (20) remains the same as before. Using (20), (17), (18) in equations which now correspond to (13.1), (13.2), (13.3)

$$\psi_{1,1}=\eta'_{3,3},\ \psi_{2,2}=\eta'_{4,4},\ \psi_{1,2}=\eta'_{3,4},\ \psi_{2,1}=\eta'_{4,3} \tag{37}$$

where

$$\eta'=-(e^2/8mc^2)((\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}+(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})r^{-3})\psi+(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\psi/4m^2c^2 \tag{38}$$

use being made of equations such as

$$((\boldsymbol{\alpha}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\alpha}^{\mathrm{II}}\boldsymbol{r})\psi)_{1,3}=\frac{1}{2mc}((\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})\chi^{\mathrm{II}})_{3,3} \tag{39}$$

which follow on substituting (20) as well as

$$((\boldsymbol{\alpha}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\alpha}^{\mathrm{II}}\boldsymbol{r})\psi)_{n,m}=((\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})\psi)_{n,m}\ ;\ (n,m=3 \text{ or } 4). \tag{40}$$

Using these values for (37) the quantities $\psi_{1,3}$, $\psi_{2,3}$ etc. are evaluated to a higher approximation than (20) and analogous to (23). This again is obtained from the equations which now take place of (14) as

$$\psi_{1,3}=p_0^{-1}(\chi^{\mathrm{I}}+\xi'^{\mathrm{I}})_{3,3}\ ;\ \psi_{1,4}=p_0^{-1}(\chi_0^{\mathrm{I}}+\xi'^{\mathrm{I}})_{3,4} \text{ etc.} \tag{41}$$

where

$$\xi'^{\mathrm{I}}=-\frac{e^2}{8mc^2}\left\{(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\left(\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})}{r}+\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}\right)+2\left(\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})}{r}+\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}\right)(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\right\}\psi + \frac{(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})^2}{4m^2c^2}\psi \tag{42}$$

Putting these values into the modified (15) we have

$$\left\{(p_0-2mc)\psi-(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})p_0^{-1}(\chi^{\mathrm{I}}+\xi'^{\mathrm{I}})-(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})p_0^{-1}(\chi^{\mathrm{II}}+\xi'^{\mathrm{II}}) + \frac{e^2}{2c}\left(\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})}{r}+\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}\right)\eta\right\}_{\alpha,\beta}=0 \tag{43}$$

[8] C. C. Darwin, Proc. Roy. Soc. **118,** 654 (1928).

where $\alpha, \beta = 3$ or 4. Substituting into this (19), (38), (42) we obtain

$$\Big\{ p_0 - 2mc - (\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})p_0^{-1}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}}) - (\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})p_0^{-1}(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}) + \frac{e^2}{8m^2c^3}[(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})X + (\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})X(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}) + (\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})X(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}}) + X(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})] - \frac{(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})^2(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})^2}{4m^3c^3} - \frac{e^4}{16mc^3}X^2 \Big\}\psi = 0 \tag{44}$$

where

$$X = \frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})}{r} + \frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}. \tag{44'}$$

In addition to (31), (33), (34) we need also the following expressions:

$$(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3} + (\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}) + (\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}}) + \frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}) = 4\sum \frac{(x_i^{\mathrm{II}} - x_i^{\mathrm{I}})(x_j^{\mathrm{II}} - x_j^{\mathrm{I}})}{r^3}p_i^{\mathrm{I}}p_j^{\mathrm{II}} + 2(\langle \boldsymbol{p}^{\mathrm{II}}r^{-1}\rangle, \boldsymbol{p}^{\mathrm{II}} - \boldsymbol{p}^{\mathrm{I}}) - 2i[\langle \boldsymbol{p}^{\mathrm{II}}r^{-1}\rangle \times \boldsymbol{p}^{\mathrm{I}}]\boldsymbol{\sigma}^{\mathrm{II}} - 2i[\langle \boldsymbol{p}^{\mathrm{I}}r^{-1}\rangle \times \boldsymbol{p}^{\mathrm{II}}]\boldsymbol{\sigma}^{\mathrm{I}} + \langle (\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{I}})r^{-1}\rangle \tag{45}^*$$

which when combined with (31) gives

$$\frac{e^2}{8m^2c^3}[(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})X + (\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})X(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}) + (\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})X(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}}) + X(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})] = \frac{e^2}{2m^2c^3}\left(r^{-1}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}}) + \sum \frac{(x_i^{\mathrm{II}} - x_i^{\mathrm{I}})(x_j^{\mathrm{II}} - x_j^{\mathrm{I}})}{r^3}p_i^{\mathrm{I}}p_j^{\mathrm{II}}\right) + \frac{e^2}{4m^2c^3}\langle (\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}\rangle - \frac{e^2 i}{2m^2c^3}([\langle \boldsymbol{p}^{\mathrm{I}}r^{-1}\rangle \times \boldsymbol{p}^{\mathrm{II}}]\boldsymbol{\sigma}^{\mathrm{I}} + [\langle \boldsymbol{p}^{\mathrm{II}}r^{-1}\rangle \times \boldsymbol{p}^{\mathrm{I}}]\boldsymbol{\sigma}^{\mathrm{II}}). \tag{46}$$

It should be noted here for future reference that $(\langle \mathbf{p}^{\mathrm{II}}r^{-1}\rangle, \mathbf{p}^{\mathrm{II}} - \mathbf{p}^{\mathrm{I}})$ *in* (45) *has been cancelled by a similar term in* (31). Using (29) we also have

$$X^2 = \frac{6 - 4(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})}{r^2} + \frac{2(\boldsymbol{\sigma}^{\mathrm{I}}\mathbf{r})(\boldsymbol{\sigma}^{\mathrm{II}}\mathbf{r})}{r^4}. \tag{47}$$

Substituting (33), (34), (46) into (44) and solving for $E - 2mc^2 + eV$ as has been done in deriving (36) we have

$$\Big\{ E - 2mc^2 + eV - \frac{1}{2m}((\boldsymbol{p}^{\mathrm{I}})^2 + (\boldsymbol{p}^{\mathrm{II}})^2) + \frac{(\boldsymbol{p}^{\mathrm{I}})^4 + (\boldsymbol{p}^{\mathrm{II}})^4}{8m^3c^2} + \frac{e^2}{2m^2c^2}(r^{-1}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}}) + \sum r^{-3}(x_i^{\mathrm{II}} - x_i^{\mathrm{I}})(x_j^{\mathrm{II}} - x_j^{\mathrm{I}})p_i^{\mathrm{I}}p_j^{\mathrm{II}}) + \frac{he}{4\pi mc}[(\boldsymbol{H}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}}) + (\boldsymbol{H}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})]$$

* For derivation see appendix.

$$+\frac{ehi}{8\pi m^2c^2}[(\mathcal{E}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})+(\mathcal{E}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})]+\frac{he}{8\pi mc^2}\left(\left[\mathcal{E}^{\mathrm{I}}\times\frac{\boldsymbol{p}^{\mathrm{I}}}{m}\right]\sigma^{\mathrm{I}}+\left[\mathcal{E}^{\mathrm{II}}\times\frac{\boldsymbol{p}^{\mathrm{II}}}{m}\right]\sigma^{\mathrm{II}}\right.$$

$$\left.+2e\left[\frac{\boldsymbol{r}^{\mathrm{I}}-\boldsymbol{r}^{\mathrm{II}}}{r^3}\times\frac{\boldsymbol{p}^{\mathrm{II}}}{m}\right]\boldsymbol{\sigma}^{\mathrm{I}}+2e\left[\frac{\boldsymbol{r}^{\mathrm{II}}-\boldsymbol{r}^{\mathrm{I}}}{r^3}\times\frac{\boldsymbol{p}^{\mathrm{I}}}{m}\right]\boldsymbol{\sigma}^{\mathrm{II}}\right)-\left(\frac{eh}{4\pi mc}\right)^2\langle(\nabla^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{I}})(\nabla^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})r^{-1}\rangle$$

$$\left.-\frac{e^4}{8mc^2}\left(\frac{3-2(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})}{r^2}+\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\mathbf{r})(\boldsymbol{\sigma}^{\mathrm{II}}\mathbf{r})}{r^4}\right)\right\}\psi=0. \tag{48}$$

This is the result of reducing equation (6) with 16 components to equations in the four larger components just as (36) is the result of reducing equation (10). This equation (48) is presumably the correct one to use since it has been derived using a correction for retardation in (6). It is further seen that it contains terms

$$(e^2/2m^2c^2)\left[r^{-1}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}})+\sum_{i,j}r^{-3}(x_i^{\mathrm{II}}-x_i^{\mathrm{I}})(x_j^{\mathrm{II}}-x_j^{\mathrm{I}})\boldsymbol{p}_i^{\mathrm{I}}\boldsymbol{p}_j^{\mathrm{II}}\right]$$

which are the quantum theory analogon of the classical $(e^2/2c^2)[(\boldsymbol{v}^{\mathrm{I}}\boldsymbol{v}^{\mathrm{II}})r^{-1}+(\boldsymbol{v}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{v}^{\mathrm{II}}\boldsymbol{r})r^{-3}]$. The $e^2hi/4\pi m^2c^2(\boldsymbol{r}(\boldsymbol{p}^{\mathrm{I}}-\boldsymbol{p}^{\mathrm{II}}))$ occurring with $(\mathcal{E}\boldsymbol{p})$ in (36) are absent here. So far as the $(\mathcal{E}\boldsymbol{p})$ combinations go these terms would mean that in an atom with nuclear charge Ze the whole bracket multiplied by $ehi/8\pi m^2c^2$ is

$$((Ze\boldsymbol{r}_{\mathrm{I}}r_{\mathrm{I}}^{-3}-3e^2(\boldsymbol{r}_{\mathrm{I}}-\boldsymbol{r}_{\mathrm{II}}))\boldsymbol{p}^{\mathrm{I}})+((Ze\boldsymbol{r}_{\mathrm{II}}r_{\mathrm{II}}^{-3}-3e^2(\boldsymbol{r}_{\mathrm{II}}-\boldsymbol{r}_{\mathrm{I}})r^{-3})\boldsymbol{p}^{\mathrm{II}})$$

so that if the electron I is far out the effective screened nuclear charge for p^{I} is $Z-3$ instead of $Z-1$. In equation (48) however the effective screened nuclear charge is $Z-1$. This argument is not conclusive because the terms $(\boldsymbol{r}r^{-3}(\boldsymbol{p}^{\mathrm{I}}-\boldsymbol{p}^{\mathrm{II}}))$ can be absorbed in $e^2/rm^2c^2)(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}})$ provided this is changed into $(e^2/2m^2c^2)(p^{\mathrm{I}}r^{-1}p^{\mathrm{II}}+p^{\mathrm{II}}r^{-1}p^{\mathrm{I}})$. The question cannot be decided unless it is possible to know[9] in what order the factors p^{I}, p^{II}, r^{-1} correspond to the classical $v^{\mathrm{I}}v^{\mathrm{II}}/r$. In (48) the order of factors in corresponding expressions is readily verified to be immaterial.

Quite unambiguously we can see, however from the fact that (36) does not contain the combination

$$r^{-1}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{II}})+\sum_{i,j}r^{-3}(x_i^{\mathrm{II}}-x_i^{\mathrm{I}})(x_j^{\mathrm{II}}-x_j^{\mathrm{I}})p_i^{\mathrm{I}}p_j^{\mathrm{II}}$$

that it cannot be a correct result. We must require that for cases where the effects of spins are negligible the result should agree with Darwin's (5). This is the case for (48) and not for (36) and therefore (48) rather than (36) is the correct equation. A generalization to the cases of particles of unequal mass can easily be made. The result is

$$\left\{E-(m_{\mathrm{I}}+m_{\mathrm{II}})c^2+eV-(p_{\mathrm{I}}^2/2m_{\mathrm{I}})-(p_{\mathrm{II}}^2/2m_{\mathrm{II}})+(p_{\mathrm{I}}^4/8m_{\mathrm{I}}^3c^2)+(p_{\mathrm{II}}^4/8m_{\mathrm{II}}^3c^2)\right.$$

[9] According to Jordan and Klein [Zeits. f. Physik **45**, 751 (1927)] the order $\frac{1}{2}(p^{\mathrm{I}}r^{-1}p^{\mathrm{II}}+p^{\mathrm{II}}r^{-1}p^{\mathrm{I}})$ is the correct one to use. This is in agreement with the results below.

$$+(e^2/2m_{\rm I}m_{\rm II}c^2)\Big[(\boldsymbol{p}^{\rm I}\boldsymbol{p}^{\rm II})r^{-1}+\sum_{i,j}r^{-3}(x_i^{\rm II}-x_i^{\rm I})(x_j^{\rm II}-x_j^{\rm I})p_i^{\rm I}p_j^{\rm II}\Big]$$
$$+(he/4\pi c)\big[(\boldsymbol{H}^{\rm I}\sigma^{\rm I})m_{\rm I}^{-1}+(\boldsymbol{H}^{\rm II}\sigma^{\rm II})m_{\rm II}^{-1}\big]+(ehi/8\pi c^2)\big[(\mathcal{E}^{\rm I}\boldsymbol{p}^{\rm I})m_{\rm I}^{-2}+(\mathcal{E}^{\rm II}\boldsymbol{p}^{\rm II})m_{\rm II}^{-2}$$
$$+(he/8\pi c^2)(m_{\rm I}^{-2}[\mathcal{E}^{\rm I}\times\boldsymbol{p}^{\rm I}]\sigma^{\rm I}+m_{\rm II}^{-2}[\mathcal{E}^{\rm II}\times\boldsymbol{p}^{\rm II}]\sigma^{\rm II}+2(m_{\rm I}m_{\rm II}r^3)^{-1}\{[(\boldsymbol{r}^{\rm I}-\boldsymbol{r}^{\rm II})$$
$$\times\boldsymbol{p}^{\rm II}]\boldsymbol{\delta}^{\rm I}+[(\boldsymbol{r}^{\rm II}-\boldsymbol{r}^{\rm I})\times\boldsymbol{p}^{\rm I}]\boldsymbol{\delta}^{\rm II}\})-(eh/4\pi c)^2(m_{\rm I}m_{\rm II})^{-1}\langle(\nabla^{\rm II}\boldsymbol{\delta}^{\rm I})(\nabla^{\rm I}\boldsymbol{\delta}^{\rm II})r^{-1}\rangle$$
$$-(e^4/4(m_{\rm I}+m_{\rm II})c^2)\big[(3-2(\boldsymbol{\delta}^{\rm I}\boldsymbol{\delta}^{\rm II}))r^{-2}+(\boldsymbol{\delta}^{\rm I}\boldsymbol{r})(\boldsymbol{\delta}^{\rm II}\boldsymbol{r})r^{-4}\big]\big\}\psi=0. \qquad (48')$$

For $m_{\rm II}=\infty$ this degenerates into the equation for a single electron.

Derivation by Heisenberg-Pauli Theory of Wave Fields

We use the results of Heisenberg and Pauli in the form of their equations (109), (110)

$$-E^{(2)}=\sum_{st,r\lambda}{}'e^2h/4\pi)(E_s-E_t+h\nu_{r\lambda})^{-1}(N_s^0+1)N_t^0(d_{st}^{r\lambda}-ic_{st})(d_{ts}^{r\lambda}+ic_{ts}^{r\lambda})$$
$$+\sum_{st,r\lambda}(e^2/4\pi\nu_{r\lambda})N_s^0(d_{ss}^{r\lambda}+ic_{ss}^{r\lambda})N_t^0(d_{tt}^{r\lambda}+ic_{tt}^{r\lambda}) \qquad (109\ H.P.)$$

applying to the case of Einstein-Bose statistics for matter and for our purpose an equivalent expression for the case of matter obeying the exclusion principle. Here E_s, E_t are possible energy values of a single unperturbed electron neglecting therefore its interaction with other electrons. The system consists initially of N_s^0 electrons in state s, N_t^0 electrons in state t and no light quanta. On account of interactions between matter and radiation light quanta appear. Their frequency is $\nu_{r\lambda}$ where r refers to a particular set of vibrations in Jean's cube and $\lambda=1, 2, 3, 4$. For the reasons which make it necessary to introduce the index λ the reader must be referred to the paper of Heisenberg and Pauli. We use the notation

$$c_{st}^{r\lambda}=\int u_\rho^{*s}\alpha_{\rho\sigma}^{i}u_\sigma^{t}v_i^{r\lambda}dV$$

$$d_{st}^{r\lambda}=\int u_\rho^{*s}u_\rho^{t}v_0^{r\lambda}dV \qquad (101\ H.P.)$$

$$\phi_i=Q^{r\lambda}v_i^{r\lambda},\quad \phi_0=P^{r\lambda}v_0^{r\lambda} \qquad (97\ H.P.)$$

meaning by u_ρ^s, the proper functions of Dirac's equation corresponding to E_s. Also letting κ_r, λ_r, μ_r be integers ϕ_0 the scalar potential ϕ_i/c $(i=1, 2, 3)$ the vector potential

$$\phi_1=(8/L^3)^{1/2}q_1^r\cos\pi L^{-1}\kappa_r x\sin\pi L^{-1}\lambda_r y\sin\pi L^{-1}\mu_r z$$
$$\phi_0=(8/L^3)^{1/2}q_0^r\sin\pi L^{-1}\kappa_r x\sin\pi L^{-1}\lambda_r y\sin\pi L^{-1}\mu_r z \qquad (84\ H.P.)$$
$$2L\nu_r=c\nu_r'\ ;\ \nu_r^{12}{}_{,1}=\kappa_r^2+\lambda_r^2+\mu_r^2\ ;\ \nu_{r,0}'^2(\epsilon+\delta)=\epsilon(1-\delta)\nu_{r,1}'^2.$$

In these formulas the integers κ, λ, μ determine the modes of vibration of Jean's cube, ϵ and δ are small constants which are made zero in the limit. They evaluate in their (116)

$$-\sum(2\pi\nu_{r\lambda})^{-1}d_{st}{}^{r\lambda}{}_{nm}=\int r^{-1}{}_{P,P'}u_\rho{}^{*s}(P)u_\rho{}^t(P)u_\sigma{}^{*n}(P')u_\sigma{}^m(P')dVdV'$$

$r_{p,p'}$ being the distance between the point $P=(x,\ y,\ z)$ and $P'=(x',\ y',\ z')$. Whenever the same dummy suffix occurs twice a summation is performed with respect to it. They have also shown that to within the approximation in which $E_s-E_t+h\nu_{r\lambda}$ can be replaced by $h\nu_{r\lambda}$ the cross product terms $d_{st}c_{ts}$ contribute nothing and that the terms $c_{st}c_{ts}$ give a term in

$$\int r^{-1}{}_{PP'}u_\rho{}^{*s}(P)\alpha^i{}_{\rho\sigma}u_\sigma{}^t(P)\cdot u_\mu{}^{*t}(P')\alpha^i{}_{\mu\nu}u_\nu{}^s(P')dVdV'.$$

These two terms due to $d_{st}d_{ts}$ and $c_{st}c_{ts}$ give as a result equation (10) which we have used to deduce (36).

The way to get corrections for retardation is indicated also in their paper. They state that the difference between $(E_s-E_t+h\nu_{r\lambda})^{-1}$ and $(h\nu_{r\lambda})^{-1}$ gives these corrections.

Since the $c_{st}c_{ts}$ terms contribute results involving products of two α's their effects are of terms in $(v/c)^2$. For the purpose of this paper we want only an approximation to this order $(v/c)^2$. It is sufficient therefore to consider in $c_{st}c_{ts}$ the first approximation to $(E_s-E_t+h\nu_{r\lambda})^{-1}=(h\nu_{r\lambda})^{-1}$. The $d_{st}d_{ts}$ terms, however, are of order 1. For them we must expand $(E_s-E_t+h\nu_{r\lambda})^{-1}$. Now their (112) is replaced by

$$\sum_{r\lambda}((h\nu_{r\lambda})^{-1}+(E_t-E_s)(h\nu_{r\lambda})^{-2}+(E_t-E_s)^2(h\nu_{r\lambda})^{-3})d_{st}d_{ts}. \tag{49}$$

The evaluation of the first term of this is performed by them by letting

$$\sum_{r\lambda}\nu_{r\lambda}{}^{-1}v_0{}^{r\lambda}(P)v_0{}^{r\lambda}(P')=G(P,P')$$

and then showing that for $\delta\rightarrow 0$

$$\Delta_P G(P,P')=64\pi^2L^{-3}\sum_r \sin\pi L^{-1}H_rx\sin\pi L^{-1}\lambda_ry\sin\pi L^{-1}\mu_rz$$

$$\sin\pi L^{-1}H_rx'\sin\pi L^{-1}\lambda_ry'\sin\pi L^{-1}\mu_rz'=8\pi^2\delta(P-P')\quad(114)(H.P)$$

whence it is deduced that

$$G(P,P')=-2\pi r^{-1}{}_{PP'}.\qquad(115)(H.P.)$$

For reasons which will be seen presently it is more advisable for us to perform here a direct calculation. We let $L\rightarrow\infty$. In the limit we can then replace summations by integrals. In order to be dealing with free particles we must simultaneously remove them from the walls of the enclosure. Thus we also increase x, y, z keeping $x-L/2$, $y-L/2$, $z-L/2$ constant. Under these conditions the walls of the enclosure recede from the particles to infinity and the coordinates of the particles with respect to the center of the enclosure are kept constant. We have then

$$G(P,P') = -(64/\pi)\int_0^\infty\int_0^\infty\int_0^\infty (\omega_1^2+\omega_2^2+\omega_3^2)^{-1}\sin\omega_1 x\sin\omega_2 y\sin\omega_3 z$$
$$\sin\omega_1 x'\sin\omega_2 y'\sin\omega_3 z' d\omega_1 d\omega_2 d\omega_3. \quad (50)$$

The product of sines in the integral can be combined into a sum of eight products of which $(1/8)\cos\omega_1(x-x')\cos\omega_2(y-y')\cos\omega_3(z-z')$ is the only one of importance. The others contain one or more factors of the type $\cos\omega_1(x+x')$. These considered as functions of ω alternate very rapidly because $x+x' \cong L$. The contribution of these other terms is therefore negligible. We have further

$$4\cos\omega_1 X\cos\omega_2 Y\cos\omega_3 Z = \cos(\omega_1 X+\omega_2 Y+\omega_3 Z)+\cos(-\omega_1 X+\omega_2 Y+\omega_3 Z)$$
$$+\cos(\omega_1 X-\omega_2 Y+\omega_3 Z)+\cos(\omega_1 X+\omega_2 Y-\omega_3 Z). \quad (50')$$

In using integrals such as (50) we are concerned therefore with

$$\int_0^\infty\int_0^\infty\int_0^\infty f(\omega_1^2+\omega_2^2+\omega_3^2)\cos\omega_1 X\cos\omega_2 Y\cos\omega_3 Z d\omega_1 d\omega_2 d\omega_3$$
$$= \frac{1}{8}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\omega_1^2+\omega_2^2+\omega_3^2)\cos(\omega_1 X+\omega_2 Y+\omega_3 Z)d\omega_1 d\omega_2 d\omega_3 \quad (51)$$

by (50′). Since the last integral extends over all the space ω it can be written

$$\frac{1}{8}\int_{-\infty}^{+\infty}\int\int {}^2{}_1\omega)f+\omega_2^2+\omega_3^2)\cos(\omega_1 R)d\omega_1 d\omega_2 d\omega_3 \; ; \; R^2 = X^2+Y^2+Z^2 \quad (51')$$

Substituting these values into (50) we have

$$G(P,P') = -(1/\pi)\int_\omega f(\omega_1^2+\omega_2^2+\omega_3^2)\cos(\omega_1 R)d\omega_1 d\omega_2 d\omega_3. \quad (52)$$

For the three terms in (49) in the case of $\epsilon = -1$ (this procedure is justified at the end of the paper) we need to evaluate this expression with $f = \Omega^{-2}$, Ω^{-3}, Ω^{-4} where $\Omega^2 = \omega_1^2+\omega_2^2+\omega_3^2$. We have

$$\int_\omega \Omega^{-2}\cos(\omega_1 R)d\omega_1 d\omega_2 d\omega_3 = 2\pi^2/R \quad (53)$$

$$\int_\omega \Omega^{-4}[\cos(\omega_1 R)-1]d\omega_1 d\omega_2 d\omega_3 = -\pi^2 R \quad (54)$$

$$\int_\omega \Omega^{-3}[\cos(\omega_1 R)-1]d\omega_1 d\omega_2 d\omega_3 = \text{const.} \quad (55)$$

By means of (53), (50)

$$G(P,P') = -2\pi/R \quad (57)$$

which is (115) of *H. P.*

Next in (52) we replace the summation by the first term corresponding to (57). With this understanding we have:

$$\sum_{s,t,r,\lambda} (e^2h/4\pi)(E_s-E_t)^2 d_{st}^{r\lambda} d_{ts}^{r\lambda}(h\nu_{r\lambda})^{-3}=$$

$$(e^2/4\pi h^2)\int\int u_\rho{}^{*s}(P)u_\rho{}^t(P)u_\sigma{}^{*t}(P')u_\sigma{}^s(P')\sum_{r,\lambda}(\nu_{r\lambda})^{-3}v_0{}^{r\lambda}(P)v_0{}^{r\lambda}(P')(E_s-E_t)^2 dVdV'$$

Writing

$$F(P,P')=\sum_{r,\lambda}\nu_{r\lambda}^{-3}v_0{}^{r\lambda}(P)v_0{}^{r\lambda}(P') \tag{58}$$

we find

$$F(P,P')=-4\pi c^{-2}\int_\omega \Omega^{-4}(\cos(\omega_1 R)-1)d\omega_1 d\omega_2 d\omega_3=4\pi^3 c^{-2}R \tag{59}$$

where (55) has been applied and 1 has been subtracted from cos $\omega_1 R$. This amounts to adding and subtracting in the integrand Ω^{-4} four times. The result is unchanged because as we shall see an additive constant in (59) has no effect. Thus:

$$\sum_{s,t,r,\lambda}(e^2h/4\pi)(h\nu_{r\lambda})^{-3}(E_s-E_t)^2 d_{st}^{r\lambda}d_{ts}^{r\lambda}$$

$$=\pi^2e^2h^{-2}\sum_{s,t}\int\int(E_s-E_t)^2c^{-2}Ru_\rho{}^{*s}u_\rho{}^t(u_\sigma{}^{*t}u_\sigma{}^s)'dVdV'. \tag{60}$$

Using (1) we have:

$$\frac{1}{c}(E_t+eA_0)u_\rho{}^t+\sum_{k=1,2,3}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_k}+\frac{e}{c}A_k\right)\alpha_{\rho\tau}{}^k u_\tau{}^t+mc\alpha_{\rho\tau}{}^4u_\tau{}^t=0$$

$$\frac{1}{c}(E_s+eA_0)u_\rho{}^{*s}+\sum_{k=1,2,3}\left(-\frac{h}{2\pi i}\frac{\partial}{\partial x_k}+\frac{e}{c}A_k\right)u_\tau{}^{*s}\alpha_{\tau\rho}{}^k+mcu_\tau{}^{*s}\alpha_{\tau\rho}{}^4=0.$$

Multiplying the first of these by $u_\rho{}^{*s}$, the second by $u_\rho{}^t$ and subtracting we have:

$$\frac{1}{c}(E_t-E_s)u_\rho{}^{*s}u_\rho{}^t+\sum_k\frac{h}{2\pi i}\left(u_\rho{}^{*s}\alpha_{\rho\tau}{}^k\frac{\partial u_\tau{}^t}{\partial x_k}+\frac{\partial(u_\tau{}^{*s}\alpha_{\tau\rho}{}^k)}{\partial x_k}u_\rho{}^t\right)=0$$

which may be written on summation as

$$\frac{1}{c}(E_t-E_s)u_\rho{}^{*s}u_\rho{}^t+\sum_k\frac{h}{2\pi i}\frac{\partial}{\partial x_k}(u_\rho{}^{*s}\alpha_{\rho\tau}{}^k u_\tau{}^t)=0 \tag{61}$$

analogously to the conservation theorem being in fact that theorem for the part of the current four vector due to cross product terms of states t, s. Formula (61) shows that an additive constant in (59) can contribute nothing since it disappears on partial integration. Similarly it shows using (55) that the second term in (49) contributes nothing. Substituting (61) into (60) we have

$$\sum_{s,t,r,\lambda} \frac{e^2h}{4\pi} \frac{(E_s-E_t)^2}{(h\nu_{r\lambda})^2} d_{st}{}^{r\lambda} d_{ts}{}^{r\lambda} = \frac{e^2}{4} \int r_{pp'} \frac{\partial(u_\rho{}^{*s}\alpha_{\rho\tau}{}^k u_\tau{}^t)}{\partial x_k} \frac{\partial(u_\sigma{}'^{*t}\alpha^l{}_{\sigma\kappa} u_\kappa{}'^s)}{\partial x_1'} dVdV'.$$

Combining this with

$$(e^2h/4\pi) \sum_{r,\lambda} (h\nu_{r\lambda})^{-1} d_{st}{}^{r\lambda} d_{ts}{}^{r\lambda} = -(e^2/2) \int r^{-1}{}_{pp'} u_\rho{}^{*s} u_\rho{}^t u_\sigma{}'^{*t} u_\sigma{}'^s dVdV'$$

and

$$(e^2h/4\pi) \sum_{r,0} (h\nu_{r\lambda})^{-1} c_{st}{}^{r\lambda} c_{ts}{}^{r\lambda} = \frac{e^2}{2} \int r^{-1}{}_{pp'} u_\rho{}^{*s} \alpha_{\rho\sigma}{}^i u_\sigma{}^t (u_\mu{}^{*t} \alpha_{\mu\nu}{}^i u_\nu{}^s)' dVdV'$$

so that (118) and (119) of *H. P.* is still true but corresponds now to

$$\begin{aligned} A_{st,ts} = \int r^{-1}{}_{pp'} \Big\{ & u_\rho{}^{*s} u_\rho{}^t (u_\sigma{}^{*t} u_\sigma{}^s)' - u_\rho{}^{*s} \alpha_{\rho\sigma}{}^i u_\sigma{}^t (u_\mu{}^{*t} \alpha_{\mu\nu}{}^i u_\nu{}^s)' \\ & -(1/2) \sum_{k,l=1,2,3} \frac{\partial(u_\rho{}^{*s} \alpha_{\rho\tau}{}^k u_\tau{}^t)}{\partial x_k} \frac{\partial(u_\mu{}^{*t} \alpha_{\mu\nu}{}^l u_\nu{}^s)'}{\partial x_l'} \Big\} dVdV' \\ = \int \Big\{ & r_{PP'}{}^{-1} u_\rho{}^{*s} u_\rho{}^t (u_\sigma{}^{*t} u_\sigma{}^s)' - \frac{1}{2} \Big(\frac{\delta_{lk}}{r_{PP'}} \\ & + \sum_{k,l} \frac{(x_k - x_k')(x_l - x_l')}{r_{PP'}} \Big) u_\rho{}^{*s} \alpha_{\rho\tau}{}^k u_\tau{}^t (u_\mu{}^{*t} \alpha_{\mu\nu}{}^l u_\nu{}^s)' \Big\} dVdV' \qquad (62) \end{aligned}$$

We have performed here two partial integrations and also used the fact that terms due to retardation contribute nothing to $A_{ss,tt}$. By an accent put over a parenthesis we mean that the parenthesis is to be evaluated at (x', y', z').

This result (62) means that the interaction energy in the configuration space is of the form used in (6) since that form according to ordinary quantum mechanics gives as a first approximation to the interaction energy the expression (62). The method of approximations used gives so far only the first approximation in the Coulomb interaction between the electrons. It does not necessarily follow therefore that (6) is correct to within all terms in $(v/c)^2$. But if we combine this derivation with the arguments given in the first section it seems at least very likely that (6) is the correct equation.

It is satisfying to note that just as Darwin's form (5) takes into account retardation by considering second order corrections to the electrostatic potential so our retarded equation owes its retarded terms to the third member of (49). The first order correction vanished both in Darwin's classical calculation and here.

The equation (6) and its reduced form (48) differs, as stated before, from (10) and (36) by the presence of $-e^4(8mc^2)^{-1}(\boldsymbol{\sigma}^{\rm I}\boldsymbol{r})(\boldsymbol{\sigma}^{\rm II}\boldsymbol{r})r^{-4}$ which should affect the fine structure of orthohelium. In order to see its effect it is convenient to combine it with the term just preceding it which represents dipole interaction. These two terms together contribute to the energy the amount:

$$\mu^2\left[\frac{(\boldsymbol{\sigma}^{\rm I}\boldsymbol{\sigma}^{\rm II})}{r^3}+\left(-\frac{3}{r^5}+\frac{2\pi^2e^2m}{h^2r^4}\right)(\boldsymbol{\sigma}^{\rm I}\boldsymbol{r})(\boldsymbol{\sigma}^{\rm II}\boldsymbol{r})\right] \text{where } \mu=\frac{eh}{4\pi mc}\,.$$

Only the second part of this expression contributes to the fine structure on account of the presence of $(\boldsymbol{\sigma}^{\rm I}\boldsymbol{r})(\boldsymbol{\sigma}^{\rm II}\boldsymbol{r})$. Retardation makes itself felt here through the term in r^{-4}. Its effect is seen to decrease the influence of the ordinary dipole interaction. For helium in the 2^3P state the retardation term is roughly 1/3 of the dipole term.

A preliminary calculation of the orthohelium fine structure using an equation equivalent to (36) has been made by Heisenberg[7]. Recently an attempt to refine Heisenberg's calculations has been made by Gaunt[6]. Gaunt also uses an equation similar to (36). It is clear from Gaunt's calculation that the result is uncertain on account of not knowing a sufficiently good approximation for the proper functions of the 2^3P state of helium. Preliminary calculations of the new term in (48) show it is not in contradiction with experimental values for it improves Gaunt's present result. It is premature, however, to claim agreement with experiment before the proper functions of the 2^3P state are worked out and the perturbation calculation for the fine structure is made with them in such a way as to be sure about the accuracy of the result.

All the terms in (48) with the exception of the last one due to retardation in the action of the electrostatic potential can be interpreted physically in terms of the spin model and in fact have been written down and used by Heisenberg before the discovery of the Dirac equation. The effect of retardation is to introduce interactions which have no simple explanation in terms of the spin. Since Dirac's equation is based on much more solid ground than the spin model the new terms have in all probability a physical existence. In the case of electrons they would be expected to be capable of experimental detection only in the spectra of elements of low atomic number. There is also a possibility of testing the formula on the fine structure of band spectra[10]. If Dirac's equation applies to protons it may also be possible to test the existence of the new terms in such cases as radiation probabilities of ortho and parahydrogen. If we are interested in higher approximations than $(v/c)^2$ it is more convenient not to use the expansion (49) but to consider pairs of terms

$$N_t^0N_s^0(E_s-E_t+h\nu_{r\lambda})^{-1}(d_{st}^{r\lambda}-ic_{st}^{r\lambda})(d_{ts}^{r\lambda}+ic_{ts}^{r\lambda})$$
$$+N_s^0N_t^0(E_t-E_s+h\nu_{r\lambda})^{-1}(d_{ts}^{r\lambda}-ic_{ts}^{r\lambda})(d_{st}^{r\lambda}+ic_{st}^{r\lambda})\,.$$

The evaluation of this expression is laborious and need not be given in detail here. For the limit $\delta=0$ we find for instance

$$-(e^2h/4\pi)\sum_{r\lambda}N_s^0N_t^0\left[d_{st}^{r\lambda}d_{ts}^{s\lambda}(E_s-E_t+h\nu_{r\lambda})^{-1}+d_{ts}^{r\lambda}d_{st}^{r\lambda}(E_t-E_s+h\nu_{r\lambda})^{-1}\right]$$
$$=e^2\int u^{*s}u^t\{\cos(2\pi r_{PP'}/\lambda_{st})$$
$$-(1/2)(1+\epsilon^{-1})(2\pi r_{PP'}/\lambda_{st})\sin(2\pi r_{PP'}/\lambda_{st})(1/r_{PP'})(u^{*t}u^s)'dVdV'$$

[10] H. A. Kramers, Zeits. f. Physik **53**, 422 (1929).

where

$$\lambda_{st}=ch/\mid E_s-E_t\mid .$$

Applying the conservation of charge (61) we get for the second part of the integral and similar expressions for the $d_{st}^{r\lambda}c_{ts}^{r\lambda}-d_{ts}^{r\lambda}c_{st}^{r\lambda}$ as well as $c_{st}^{r\lambda}c_{ts}^{r\lambda}$ contribu-

$$-(e^2/2)(1+\epsilon^{-1})\int\frac{\partial(u^{*s}\alpha^{it}u^t)}{\partial x^i}\left(\sin\frac{2\pi r_{PP'}}{\lambda_{st}}\right)\left(\frac{2\pi r_{PP'}}{\lambda_{st}}\right)^{-1}\frac{\partial(u^{*t}\alpha^k u^s)'}{\partial x_k'}dVdV'$$

tions to the interaction energy. The first of these contributes only terms in $1+\epsilon^{-1}$ while the last gives $(u^{*s}\alpha^i u^t)(1/r_{pp'})\cos(2\pi r_{pp'}/\lambda_{st})(u^{*t}\alpha^i u^s)'$ combinations as well. On adding all the terms it is found that $1+\epsilon^{-1}$ disappears on using (61) and partial integration. The terms independent of ϵ combine into

$$A_{st,ts}=\int(u_\rho{}^{*s}u_s{}^t\{1/r_{PP'}\}(u_\sigma{}^{*t}u_\sigma{}^s)' - u_\rho{}^{*s}\alpha_{\rho\sigma}{}^i u_\sigma{}^t\{1/r_{PP'}\}(u_\mu{}^{*t}\alpha_{\mu\nu}{}^i u_\nu{}^s)')dVdV' \tag{63}$$

where

$$\{1/r_{PP'})\}=(1/r_{PP'})\cos(2\pi r_{PP'}/\lambda_{st})$$

may be said to be the retarded value of $1/r_{pp'}$ for the transition (st).

The calculations can be shortened by observing to start with that all terms in $1+\epsilon^{-1}$ must disappear since otherwise the interaction energy would become infinite for $\epsilon=0$. This means that at this stage of the calculation we may work with $\epsilon=-1$. The calculation then becomes quite short.

This short cut is a close equivalent of the procedure followed in classical electrodynamics in deriving the retarded potentials. The non quantized field equations of Heisenberg and Pauli can be solved as a sum of two parts provided conservation of charge holds. The first part is given by the usual retarded potentials in the form of the well known integrals. On account of the conservation of charge these integrals cause the coefficient of $1+\epsilon^{-1}$ to disappear and can therefore be calculated as though ϵ were -1. It is these retarded potentials that determine the interaction between particles. The other part of the general solution depends on ϵ and is obtained by solving the field equations as though the current s_α were 0. In the classical theory this part of the solution has nothing to do with the interaction. The disappearance of ϵ is thus entirely due to the conservation of charge both in the quantized and classical field equations.

In (63) the phase under the cosine is that which would exist for light waves of wave-length λ_{st}

$$\lambda_{st}=ch/\mid E_s-E_t\mid \tag{64}$$

corresponding to emissions and absorptions between states s and t.

The correction which we have used in (6) is from this point of view due to the second term in the expansion $\cos x=1-x^2/2$ of the phase factor. We expect the correction to apply therefore only for small values of $r_{pp'}/\lambda_{st}$.

APPENDIX. DERIVATION OF (45)

We let $y_i = (x_i^{\mathrm{II}} - x^{\mathrm{I}}_i)/r^{3/2}$ and expand as follows:

$$\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}) + (\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}$$

$$+(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}) + (\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}})\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r^3}(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}})$$

$$= \sum(\sigma_i^{\mathrm{I}}\sigma_j^{\mathrm{II}}\sigma_k^{\mathrm{I}}\sigma_l^{\mathrm{II}}y_iy_jp_k^{\mathrm{I}}p_l^{\mathrm{II}} + \sigma_k^{\mathrm{I}}\sigma_l^{\mathrm{II}}\sigma_i^{\mathrm{I}}\sigma_j^{\mathrm{II}}p_k^{\mathrm{I}}p_l^{\mathrm{II}}y_iy_j + \sigma_k^{\mathrm{I}}\sigma_i^{\mathrm{I}}\sigma_j^{\mathrm{II}}\sigma_l^{\mathrm{II}}p_k^{\mathrm{I}}y_iy_jp_l^{\mathrm{II}}$$
$$+\sigma_l^{\mathrm{II}}\sigma_i^{\mathrm{I}}\sigma_j^{\mathrm{II}}\sigma_k^{\mathrm{I}}p_l^{\mathrm{II}}y_iy_jp_k^{\mathrm{I}}) = A+B+C+D \text{ where using (26)}$$

$$A = \sum_{i=k,j=l} \quad \sum_{i,j}(y_iy_jp_i^{\mathrm{I}}p_j^{\mathrm{II}} + p_i^{\mathrm{I}}p_j^{\mathrm{II}}y_iy_j + p_i^{\mathrm{I}}y_iy_jp_j^{\mathrm{II}} + p_j^{\mathrm{II}}y_iy_jp_i^{\mathrm{I}})$$
$$B = \sum_{i=k,j\neq l} \quad \sum_{i,j\neq l}\sigma_j^{\mathrm{II}}\sigma_l^{\mathrm{II}}(y_iy_jp_i^{\mathrm{I}}p_l^{\mathrm{II}} - p_i^{\mathrm{I}}p_l^{\mathrm{II}}y_iy_j + p_i^{\mathrm{I}}y_iy_jp_l^{\mathrm{II}} - p_l^{\mathrm{II}}y_iy_jp_i^{\mathrm{I}})$$
$$C = \sum_{i\neq k,j=l} = \sum_{j,i\neq k}\sigma_i^{\mathrm{I}}\sigma_k^{\mathrm{I}}(y_iy_jp_k^{\mathrm{I}}p_j^{\mathrm{II}} - p_k^{\mathrm{I}}p_j^{\mathrm{II}}y_iy_j - p_k^{\mathrm{I}}y_iy_jp_j^{\mathrm{II}} + p_j^{\mathrm{II}}y_iy_jp_k^{\mathrm{I}})$$
$$D = \sum_{i\neq k,j\neq l} = \sum_{i\neq k,j\neq l}\sigma_i^{\mathrm{I}}\sigma_k^{\mathrm{I}}\sigma_j^{\mathrm{II}}\sigma_l^{\mathrm{II}}(y_iy_jp_k^{\mathrm{I}}p_l^{\mathrm{II}} + p_k^{\mathrm{I}}p_l^{\mathrm{II}}y_iy_j - p_k^{\mathrm{I}}y_iy_jp_l^{\mathrm{II}} - p_l^{\mathrm{II}}y_iy_jp_k^{\mathrm{I}})$$

On carrying out the differentiations $p_i^{\mathrm{I}}p_j^{\mathrm{II}}$

$$A = \sum_{i,j}(4y_iy_jp_i^{\mathrm{I}}p_j^{\mathrm{II}} + 2\{p_i^{\mathrm{I}}y_iy_j\}p_j^{\mathrm{II}} + 2\{p_j^{\mathrm{II}}y_iy_j\}p_i^{\mathrm{I}} + \{p_i^{\mathrm{I}}p_j^{\mathrm{II}}y_iy_j\})$$
$$B = -\sum_{i,j\neq l}\sigma_j^{\mathrm{II}}\sigma_l^{\mathrm{II}}(2\{p_l^{\mathrm{II}}y_iy_j\}p_i^{\mathrm{I}} + \{p_i^{\mathrm{I}}p_l^{\mathrm{II}}y_iy_j\})$$
$$C = -\sum_{j,i\neq k}\sigma_i^{\mathrm{I}}\sigma_k^{\mathrm{I}}(2\{p_k^{\mathrm{I}}y_iy_j\}p_j^{\mathrm{II}} + \{p_k^{\mathrm{I}}p_j^{\mathrm{II}}y_iy_j\})$$
$$D = \sum_{i\neq k,j\neq l}\sigma_i^{\mathrm{I}}\sigma_k^{\mathrm{I}}\sigma_j^{\mathrm{II}}\sigma_l^{\mathrm{II}}\{p_k^{\mathrm{I}}p_l^{\mathrm{II}}y_iy_j\}$$

Using the value $y_i = (x_i^{\mathrm{II}} - x_i^{\mathrm{I}})/r^{3/2}$ we have

$$\sum_{i,j}\{p_i^{\mathrm{I}}p_j^{\mathrm{II}}y_iy_j\} = \sum_{i,j\neq l}\{p_i^{\mathrm{I}}p_l^{\mathrm{II}}y_iy_j\}\sigma_j^{\mathrm{II}}\sigma_l^{\mathrm{II}} = \sum_{j,i\neq k}\{p_k^{\mathrm{I}}p_j^{\mathrm{II}}y_iy_j\}\sigma_i^{\mathrm{I}}\sigma_k^{\mathrm{I}} = 0$$

$$\underset{i,j+l}{D} = \{(\boldsymbol{p}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}})(\boldsymbol{p}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{I}})r^{-1}\} - 2\sum_{i,j\neq l}\sigma_j^{\mathrm{II}}\sigma_l^{\mathrm{II}}\{p_l^{\mathrm{II}}y_iy_j\}p_i^{\mathrm{I}} = -2i[\{\boldsymbol{p}^{\mathrm{II}}r^{-1}\}\times\boldsymbol{p}^{\mathrm{I}}]\boldsymbol{\sigma}^{\mathrm{II}} - 2\sum_{j,i\neq k}\sigma_i^{\mathrm{I}}\sigma_k^{\mathrm{I}}\{p_k^{\mathrm{I}}y_iy_j\}p_j^{\mathrm{II}}$$
$$= -2i[\{\boldsymbol{p}^{\mathrm{I}}r^{-1}\}\times\boldsymbol{p}^{\mathrm{II}}]\boldsymbol{\sigma}^{\mathrm{I}}$$

$$2\sum_{i,j}(\{p_i^{\mathrm{I}}y_iy_j\}p_j^{\mathrm{II}} + \{p_j^{\mathrm{II}}y_iy_j\}p_i^{\mathrm{I}}) = 2(\{\boldsymbol{p}^{\mathrm{II}}r^{-}\}\cdot(\boldsymbol{p}^{\mathrm{II}} - \boldsymbol{p}^{\mathrm{I}})).$$

Substituting these values into A, B, C, D and forming $A+B+C+D$ we get (45).