ON THE ANALYSIS OF ELECTRONIC VELOCITIES BY ELECTROSTATIC MEANS

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Abstract

In magnetic analysis of electronic velocities the plane containing the receiving slit is placed at the angle $\Phi_m = 180°$ to the plane containing the entrance slit. The "resolution," at Φ_m, between two electrons of velocities V and v, both entering normally to the plane of the entrance slit, is $d_m = 2\beta r_0$, where $\beta = (V-v)/V$. The "departure from perfect re-focussing," at Φ_m of two electrons of velocity V, one entering normally to the plane of the entrance list and the other at an angle α to the normal, is $s_m = \alpha^2 r_0$. Since in some experiments the use of a magnetic field is a disadvantage, it is of interest to investigate the analyzing possibilities of electrostatic fields. In the present paper the radial, inverse first power, electrostatic field is considered in this connection. It is found that by a proper construction of the apparatus it should be possible to analyze velocities almost as well as by magnetic means. The plane of the receiving slit should be placed at the angle $\Phi_e = 127°17'$ to the plane of the entrance slit. The resolution at this angle is given by $d_e = (2\beta - 4\beta^2) r_0$, and the "departure from perfect re-focussing" by $s_e = 4\alpha^2 r_0/3$. It is also found that the 180° and 90° positions of the plane of the receiving slit are highly unsatisfactory. These results are obtained from approximate solutions of the equation of the electronic orbits, and are checked by numerical integration.

I. Introduction

It is well known that an electron, projected into a uniform magnetic field in a direction perpendicular to the lines of magnetic force, describes a circle whose radius depends on the velocity of the electron and on the strength of the magnetic field. If electrons enter the field in directions making but small angles with QP (see Fig. 1, the plane of which is perpendicular to the lines of force), it is the geometrical property of their circular paths that good re-focussing of the electrons of a given velocity occurs at the angle $\phi = 180°$. This property has been taken advantage of in many investigations, notably in the case of β-rays, as it enables one to compensate for the feebleness of a source of electrons by using a beam spread over an appreciable angle.

In certain types of experiments it is more convenient to use electrostatic rather than magnetic fields. A type of electrostatic field which, in its effect on an electron path, closely resembles a magnetic field, is the radial, inverse first power, electrostatic field. Such a field has occasionally been used in experimental work, but inasmuch as the conditions for best re-focussing and best resolution do not appear to have been worked out, it has always been necessary to limit the beam of electrons entering the field to a very small angle in order to secure sufficient resolution between electrons of different velocities.

The present investigation shows that in a radial, inverse first power, electrostatic field, good re-focussing and good resolution are to be found at the angle $\Phi = 127° \; 17'$, which, in fact, plays the same part for this type of an electric field as does the 180° position for the magnetic field. The problem

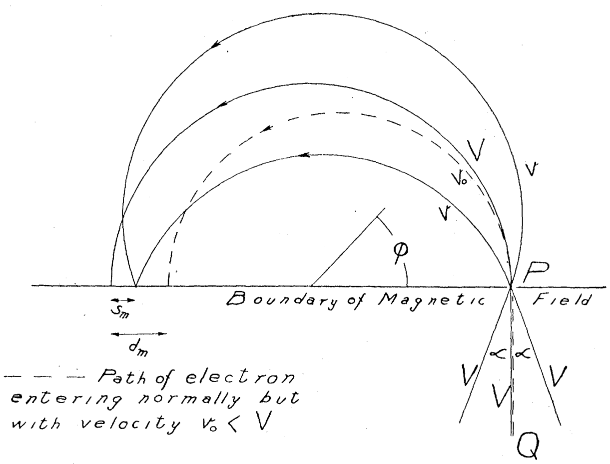

Fig. 1.

was solved by obtaining approximate solutions of the equation of the orbit. The results were checked for particular cases by numerical methods of solving the equation of the path without any approximations other than arithmetical ones.

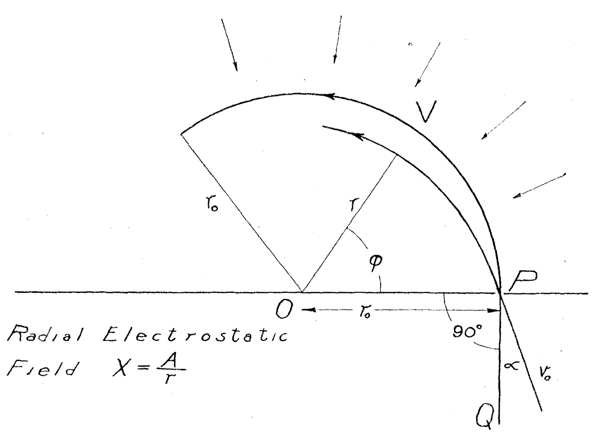

Fig. 2.

2. Approximate Solution of the Equation of the Orbit. Re-focussing and Resolution

Let us consider a two dimensional, radial, electrostatic field of intensity $X = -A/r$, Fig. 2, and investigate the orbit of an electron, of mass m and charge $-e$, projected into the field at the point P with an initial velocity v_0.

The differential equation of the orbit, in polar coordinates r and ϕ, is

$$h^2u^2\left(\frac{d^2u}{d\phi^2}+u\right)=\frac{Ae}{m}u, \tag{1}$$

where $u=1/r$, and h is the angular momentum per unit mass.[1] If α denote the angle between QP and the initial direction of motion, then $h=r_0v_0\cos\alpha$. Introducing a new variable $y=u/u_0$, and a parameter c defined by $c^2=Ae/mv_0^2 \cos^2\alpha$, we now transform (1) into

$$\frac{d^2y}{d\phi^2}+y=\frac{c^2}{y} \tag{2}$$

which must be solved subject to the initial conditions:

$$\text{at } \phi=0,\ \ y=1,\ \text{ and } \frac{dy}{d\phi}=-\tan\alpha.$$

As (2) cannot be solved in terms of the elementary functions[2], we can either seek an approximate solution in terms of the parameters c and α, or evaluate particular solutions by numerical methods. Both lines of attack are carried out below for the cases of interest in the application of this theory to experimental devices.

Before proceeding further it is advantageous to evaluate the velocity which an electron entering at P must have in order that it shall describe a circular path. It is seen that a circular orbit is possible only if the initial direction of motion is along QP, i.e., normal to OP, and if the velocity, which we denote by V, satisfies the relation $mV^2/r_0=-Xe=Ae/r_0$, or

$$mV^2=Ae.$$

This relation shows, first, that by varying the constant A of the field, it is possible to make any velocity to be that appropriate to circular orbits; secondly, that V is independent of r_0; and, thirdly, that the parameter c in (2) is given by $V/v_0\cos\alpha$.

For the practical problem of ascertaining the position of best re-focussing and the degree of resolution between different velocities, we need to know the paths of electrons entering the radial electrostatic field at but a small angle to QP, and having a velocity deviating but slightly from the velocity V appropriate to a circular path. Let us consider two orbits, which we may call twin orbits, deviating from perpendicular entrance into the field by angles $+\alpha$ and $-\alpha$, respectively, and in which the velocities at entrance are, in each case, v_0. Applying a method of successive approximations described by Routh,[3] we now find that, to a first approximation, the equations of the respective twin orbits are:

[1] See, for example, A. G. Webster, The Dynamics of Particles, and Rigid, Elastic, and Fluid Bodies, (Teubner, 1912), p. 39.

[2] E. T. Whittaker, Analytical Dynamics, (Cambridge, 1927), p. 81.

[3] E. J. Routh, A Treatise on Dynamics of a Particle, (Cambridge, 1898), p. 236.

$$y_1=c+(1-c)\cos(2)^{1/2}\phi-(\tan\alpha\sin(2)^{1/2}\phi)/(2)^{1/2},$$
$$y_2=c+(1-c)\cos(2)^{1/2}\phi+(\tan\alpha\sin(2)^{1/2}\phi)/(2)^{1/2}. \qquad (4)$$

The twin orbits, therefore, cross each other whenever $y_1=y_2$, i.e., whenever the last term in each expression vanishes. Hence they cross for the first time after entrance when $(2)^{1/2}\phi=\pi$. Denoting the corresponding value of ϕ by Φ_e, we have

$$\Phi_e=\pi/(2)^{1/2}, \text{ or } \Phi_e=127°17'. \qquad (5)$$

As this angle is independent of α, it denotes, to the first approximation, the position of best re-focussing.

To investigate the *resolution*, i.e., the separation between the paths of two electrons, both entering normally at P but with slightly *different* velocities, we put $\alpha=0$ in (4), differentiate with respect to ϕ, and obtain: $y_1'=y_2'=-(2)^{1/2}(1-c)\sin(2)^{1/2}\phi$. This vanishes at $\phi=\Phi_e$, and it therefore follows that at this angle the resolution is at a maximum. Thus Φ_e besides being the angle at which the best re-focussing occurs, is also the angle at which the resolution is best. From (4) it follows that, at Φ_e,

$$y=2c-1. \qquad (6)$$

We now define the *resolving power*, d_e, to be r_0-r, where r_0 is the radius of the circular path in which an electron of velocity V will travel, while r is the radius vector (at $\phi=\Phi_e$) of the path of an electron entering along QP with velocity v_0. We recall the definitions of y and c (in which we set $\alpha=0$), introduce a new symbol $\beta=(V-v_0)/V$, and, neglecting higher powers of β, obtain from (6)

$$d_e=(2\beta-4\beta^2)r_0. \qquad (7)$$

It is also important to evaluate the *departure from perfect re-focussing*, s_e, which is the separation, at Φ_e, of the twin paths of the electrons entering the electric field at angles $\pm\alpha$ with QP from the path of the electron entering along QP, all electrons having one and the same velocity V. As a satisfactory result cannot be obtained by the use of (6), it is necessary to use in place of (4) the *second* approximation to the solution of (2). The analysis cannot be summarized effectively, and so only the final result will be given: the departure s_e is toward the inside of the circle of radius r_0, and when α is expressed in radians and its powers higher than the second are neglected,

$$s_e=4\alpha^2 r_0/3. \qquad (8)$$

The *third* approximation to the solution of (2) reveals the result that the angle Φ_e, at which the twin orbits cross, decreases with increasing α, but to an extent which is negligibly small in any case likely to occur in an experimental arrangement.

For purposes of comparison, we give the values of the *resolution*, d_m of the *departure from perfect re-focussing*, s_m, in the familiar magnetic case, in which the best re-focussing occurs at $\Phi_m=180°$. These values are:

$$d_m = 2\beta r_0, \tag{9}$$

$$s_m = \alpha^2 r_0. \tag{10}$$

3. Numerical Integration of the Equation of the Orbit

To check the formulas for s_e and d_e obtained above, numerical determination of several particular orbits was effected. As is well known, numerical calculation of a particular solution of an ordinary differential equation can, except in singular instances, be carried out to any desired degree of approximation even though the equation may not admit of explicit integration. Two different methods were used in this work.

One method was to integrate Eq. (2), subject to the initial conditions, once, getting

$$\frac{1}{2}\left(\frac{dy}{d\phi}\right)^2 = (1-\beta)^2 \sec^2\alpha \log y - \frac{1}{2}y^2 + \frac{1}{2}\sec^2\alpha,$$

from which it follows that

$$\phi = \int_1^y [2(1-\beta)^2 \sec^2\alpha \log y - y^2 + \sec^2\alpha]^{-1/2} dy.$$

Particular numerical values were then assigned to α and β, and the values of the integrand were calculated for closely spaced values of $y (y = 1.01, 1.02,$

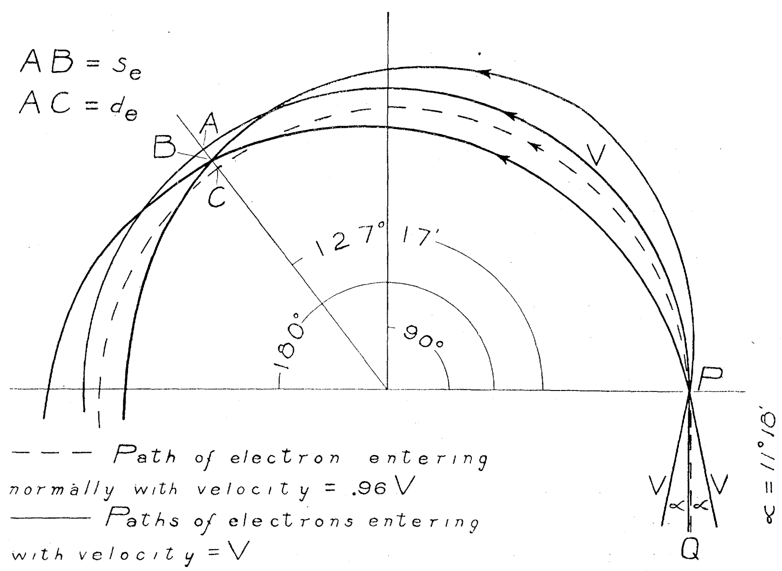

Fig. 3.

$\cdots$, for an inner orbit, $y = 0.99, 0.98, \cdots$, for an outer orbit; near an apse closer spacing was necessary). From these the values of the definite integral, and therefore of ϕ were obtained by *calculating* "the area under the curve."

The fact that an orbit is symmetrical about the apsidal lines was also used in mapping out the orbits. This method was applied to the cases $\alpha = \pm 11°18'$, $\beta = 0$ and $\alpha = \pm 5°42'$, $\beta = 0$ to study re-focussing, and to the cases $\alpha = 0$, $\beta = 0.02$ and $\alpha = 0$, $\beta = 0.04$ to study resolution. Fig. 3 represents some of the orbits calculated in this manner. The full lines *PA* and *PB* correspond to the cases $\alpha = \pm 11°18'$, $\beta = 0$, and the dotted line to the case $\alpha = 0$, $\beta = 0.04$, i.e., when the electron enters perpendicularly with a velocity 0.96 V. It is seen that the "resolution" between electrons of slightly different velocities, but all entering the field perpendicularly, is considerably poorer at 90° and at 180° than at 127°. Moreover, this resolution can be detected easily at 127°, even if the entering beam of electrons is spread over an angle $\pm 11°18'$, but it is hopelessly obliterated at the 90° or 180° position by the wide "spread" of the electron beam at these positions.

Another method of numerical integration used here was that due to Milne.[4] Its technique is quite simple, and in the present instance only the four operations of arithmetic were involved. The cases $\alpha = \pm 5°42'$, $\beta = 0$ and $\alpha = \pm 11°18'$, $\beta = 0$ were investigated in this manner, the calculations being carried out to five places of decimals.

The results of the numerical work lead to the conclusion that the formulas for s_e and d_e are correct, for the ranges considered, to the order to which they were derived, as the discrepancy between (7) and the true value of d_e was found to be of the order of β^3, and that between (8) and the true value of s_e, of the order of α^4.

4. Conclusions

From a practical experimental point of view, it is important to know how good the re-focussing is for the paths of two electrons, one entering the electrostatic field with just the right velocity to describe a circular path, and the other, having the same velocity, but entering at an angle $\pm\alpha$. Hence we plot the values of s_e, the departure from perfect re-focussing at 127°17', as a function of the angle α (Fig. 4). (For the sake of comparison, we plot also the values of s_m, at 180°, the corresponding quantity in the magnetic case.) On the same diagram, we also record the values of d_e and d_m, which measure the electric and magnetic resolutions, respectively, for electrons entering the field *perpendicularly*, but with velocities 0.99 V, 0.98 V and 0.97 V, (which correspond to $\beta = 0.01$, 0.02, and 0.03). As to scale, the values of s and d are given directly in centimeters, when the radius of the circular path, taken by an electron entering perpendicularly with the velocity V is just one centimeter.

So long as the existence of the re-focussing property of an inverse first power, radial, electrostatic field is unsuspected, the position of the receiving slit (or photographic plate) in any particular experimental arrangement in which electrons are deflected by such a field, is largely a matter of experimental convenience. Under such conditions, one would perhaps naturally put the receiving slit (or photographic plate) at 90° or 180°. It is therefore

[4] W. E. Milne, American Math. Monthly **33**, 455 (1926).

instructive to tabulate the spread of the electron beam at each of these positions together with the much smaller departure from perfect re-focussing

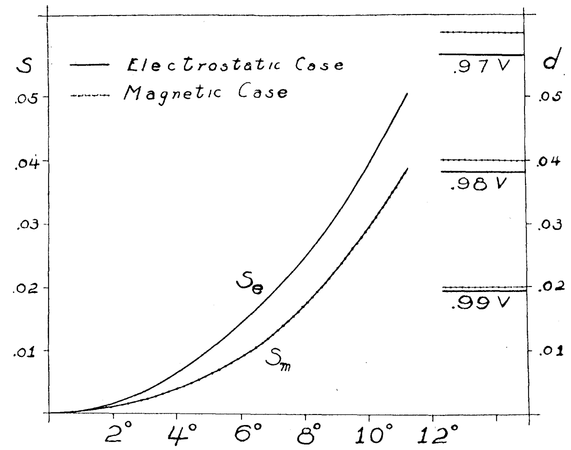

Fig. 4.

at 127°17′ for different values of α. For purposes of comparison, the maximum possible spread is included.

Angle	s_e at 127°17′	Spread at 90°	Spread at 180°	Max. Spread at 63°
11° 18′	0.0501	0.202	0.258	0.270
5° 40′	.0129	.111	.132	.134
2° 51′	.0032	.057	.066	.067

(The last line is an interpolation. Note that while s_e diminishes as the *square* of the angle, the other spreads diminish only as the *first power* of the angle.) It is seen from this table that to place the receiving slit (or photographic plate) in the 90° or 180° position, gives almost the worst departure from perfect re-focussing that the apparatus is capable of. This conclusion also follows from an inspection of Fig. 3.

In the analysis given above, just as in the usual derivation of the re-focussing properties of a magnetic field, some of the complications arising in experiment, such as a finite entrance slit, etc., were disregarded.[5] An account of an experimental test of the conclusions reached here is given in the following paper.

[5] It may be remarked that in the electrostatic case the velocity of a given electron varies with the radius vector.