PACKING OF HOMOGENEOUS SPHERES

By W. O. Smith, Paul D. Foote and P. F. Busang
Mellon Institute of Industrial Research*

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Abstract

Although the actual packing of shot is of a very irregular and distorted pattern, for certain statistical purposes the arrangement may be treated as a mixture of close-hexagonal and simple-cubical pilings in the proportion required to yield the observed porosity. The average number of contacts per sphere was determined by a simple method for several porosities and was found to agree with the value computed on the above assumption. Statistical distribution curves for contact numbers are also included. For higher porosities the distribution is Gaussian in general character.

IN THE regular arrangements, uniform shot may be piled in close hexagonal array with the minimum porosity of 0.26, and in simple cubical array with the maximum porosity 0.48. Such ideal arrangements however are not often realized in practice. Ordinarily, shot pack to a porosity in the neighborhood of 0.40, but with special care the porosity may be varied over the range 0.35–0.45. The question naturally arises therefore as to the type of packing yielding observed porosities. A general regularity is found to be absent. In places the shot are packed in close hexagonal array and at a short distance from these points the arrangements may be cubical. The patterns are in general distorted and spheres are frequently missing with a consequent closing-in and bridging of the pore cell. An investigation of the average number of contacts per sphere appeared to offer possibilities for a statistical consideration of the problem.

Lead shot of radius 3.78 mm were poured into a large beaker and a 20 percent solution of acetic acid was slowly introduced from the bottom until the beaker was filled. The acid was then carefully drained and at each contact between grains a small ring of liquid is retained by capillarity. On standing several hours this forms basic lead acetate clearly indicating the contact by a white circular deposit. The contacts may be readily counted as the shot are individually removed from the beaker. From 1,200 to 2,400 shot were used in beakers of 8 to 13 cm diameter but the count of contacts was confined to the interior 900 to 1,600 shot in order to reduce the effect of boundary conditions. The mean porosity for the interior shot was determined in the following manner. The volume of a cylinder passing through the centers of the spheres adjacent to the side walls, and cut by planes through the centers of the top and bottom layers of spheres, was computed, and from this was subtracted the actual volume of the shot and portions of shot lying within the space, as determined by count. The ratio of this final quantity to the total volume of the space considered is the porosity P.

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Table I summarizes the experimental data. The third column gives the average number of contacts n per sphere for a given observed porosity and the succeeding columns show the statistical distribution. The distributions on a percentage basis are plotted in Fig. 1. The curves for the larger porosities resemble Gaussian distributions and a pronounced shift toward higher contacts per sphere occurs as the porosity is decreased. For the lower porosities the closer packing throws more spheres into the hexagonal array thus increasing the number of 10–12 point contacts. Usually the various porosities were obtained by shaking and tapping the beaker but for curves A and B small amounts of shot were successively added and shaken, and for curve B the shot in addition were tamped with a piston. Under this mode of packing the porosity in general decreases considerably with depth while the number of contacts per sphere correspondingly increases. These variations however were carefully investigated and found consistent with the variation of the average number of contacts and the average porosity for the entire group of interior shot; hence it has not been necessary to consider the various zones or layers of shot in further detail.

TABLE I. *Summary of data.*

Counted shot	P	n	Distribution of contacts per sphere								
			4	5	6	7	8	9	10	11	12
1562	0.359	9.14	1	13	77	245	322	310	208	194	192
1494	.372	9.51	0	14	86	192	233	193	161	226	389
887	.426	8.06	0	14	69	182	316	212	87	7	0
906	.440	7.34	3	54	173	309	233	118	14	2	0
905	.447	6.92	6	78	243	328	200	48	2	0	0

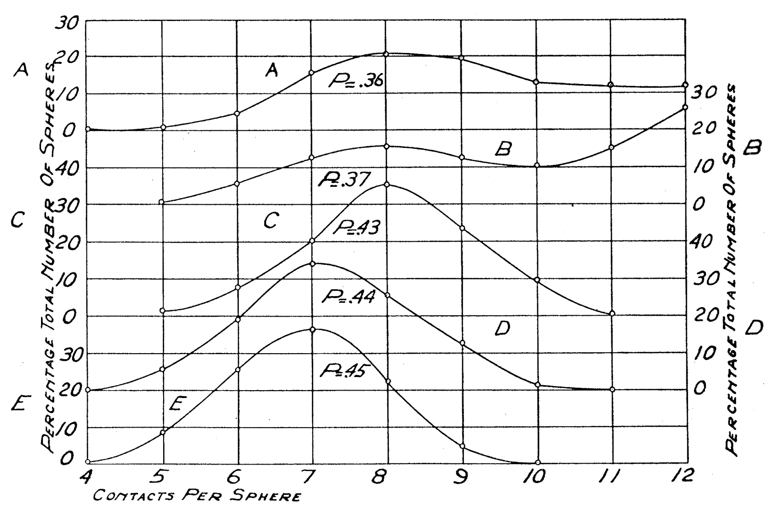

Fig. 1. Distribution of number of contacts per sphere for several porosities.

In Fig. 2 is plotted the average number of contacts per sphere as a function of the porosity. The closest packing possible is hexagonal giving a porosity of 0.2595 and 12 contacts per sphere. It is evident that any theoretical or empirical curve must pass through this point. The most open regular array is simple cubic with porosity 0.4764 and 6 contacts per sphere. It appears doubtful that the presence of bridging could produce higher porosity and lower average contacts per sphere over an extended volume, and confirmation of this statement is amply afforded by the experimental data.

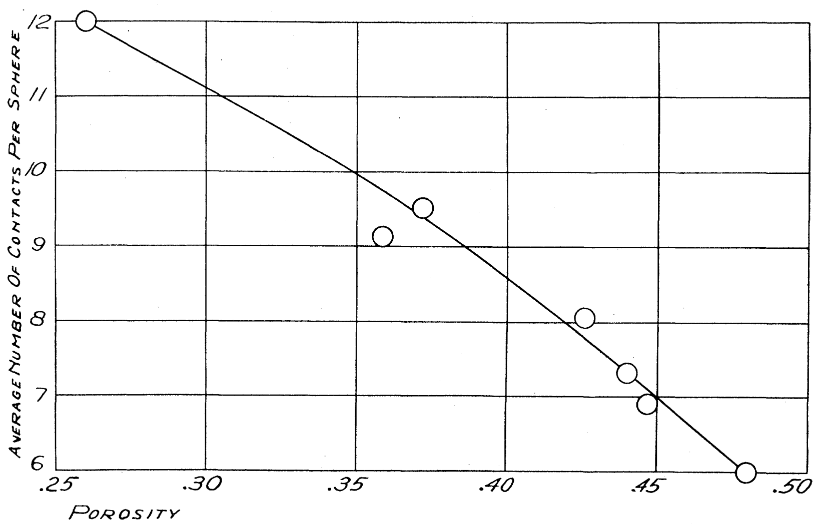

Fig. 2. Average number of contacts per sphere as a function of the porosity.

The fact therefore that the experimental results may be represented by a smooth curve through these two extreme points suggests the possibility of statistically treating the actual packing as an arrangement in separate hexagonal and cubical arrays in the proportion required to yield the observed porosity.

Let x be the fraction of the total volume of shot packed hexagonally $1-x$ the fraction packed cubically. Then

$$P=0.2595x+0.4764(1-x)$$

and

$$x=(0.476-P)/0.217. \tag{1}$$

In hexagonal array the number of spheres in unit volume is $2^{1/2}/8r^3$ and in cubical array $1/8r^3$. Hence for the mixture the average number of contacts per sphere is given by the relation

$$n=\frac{12x2^{1/2}/8r^3+6(1-x)/8r^3}{x2^{1/2}/8r^3+(1-x)/8r^3}=6\frac{1+1.828x}{1+0.414x}\,. \tag{2}$$

The quantity n as a function of P may be therefore obtained from Eqs. (1) and (2). The curve in Fig. 2 was drawn through points computed in the above manner and the agreement with experiment is surprisingly good. It is of interest that the regular body-centered piling with porosity 0.32 and 8 contacts per grain can not be correlated with the curve.

Especial theoretical significance should not be attached to the proposed interpretation of the packing problem. The complete solution must explain how shot acted upon by gravitational and frictional forces are arranged to give the contact distribution curves of Fig. 1. This problem presents extreme difficulty and its solution is not likely to be obtained in the near future. The present work however empirically demonstrates that when spheres are piled irregularly they may be statistically regarded as an arrangement in separate, close-hexagonal and simple-cubical arrays in a proportion to yield the observed porosity.