THE INTERACTION OF ELECTRON AND POSITIVE ION SPACE CHARGES IN CATHODE SHEATHS

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Abstract

Effect of positive ions generated at a plane anode upon the space charge limitation of electron currents from a parallel cathode.—Mathematical analysis shows that single ions emitted with negligible velocity permit 0.378 $(m_p/m_e)^{1/2}$ additional electrons to pass; but with an unlimited supply of ions the electron current approaches a limiting value 1.860 times that which flows when no ions are present, and the electron current is then $(m_p/m_e)^{1/2}$ times the ion current, both currents thus being limited by space charge and the electric field being symmetrically distributed between the electrodes. Single ions introduced into a pure electron discharge at a point 4/9ths of the distance from cathode to anode produce a maximum effect, 0.582 $(m_p/m_e)^{1/2}$, in increasing the electron current. These conditions apply to a cathode emitting a surplus of electrons surrounded by ionized gas. The cathode sheath is then a double layer with an inner negative space charge and an equal outer positive charge, the field being zero at the cathode and at the sheath edge. The electron current is thus limited to $(m_p/m_e)^{1/2}$ times the rate at which ions reach the sheath edge. If ions are generated without initial velocities uniformly throughout the space between two plane electrodes, a parabolic potential distribution results. If the total ion generation exceeds 2.86 times the ion current that could flow from the more positive to the more negative electrode, a potential maximum develops in the space. Electrons produced by ionization are trapped within this region and their accumulation modifies the potential distribution yielding a region (named plasma) in which only weak fields exist and where the space charge is nearly zero. The *potential distribution in the plasma*, given by the Boltzmann equation from the electron temperature and the electron concentrations, *determines the motions of the ions* and thus fixes the rate at which the ions arrive at the cathode sheath. The *anode sheath* is usually also a positive ion sheath, but with anodes of small size a detached double-sheath may exist at the boundary of the anode glow. In discharges from *hot cathodes in gases* where the current is limited by resistance in series with the anode, the electron current is space-charge-limited, being fixed by the rate of arrival of ions at the cathode sheath. Thus the cathode drop is fixed by the necessity of supplying the requisite number of ions to the cathode. The *effect of the initial velocities of the ions and electrons* that enter a double-sheath from the gas is to decrease the electron current by an amount that varies with the voltage drop in the sheath. A nearly complete theory of this effect is worked out for plane electrodes. A detailed study is made of the *potential distribution in the plasma and near the sheath edge* for a particular case and the conclusion is drawn that the velocities of the ions that enter the sheath can be calculated from the electron temperature if the geometry of the source of ionization is given.

Experiments with double sheaths.—With large cathodes coated with barium oxide in low pressure mercury vapor, simultaneous measurements showed that the electron current density was independent of the cathode temperature and was from 140 to 200 times the ion current density, this ratio being independent of the intensity of ionization and of the gas pressure but varying slowly with the voltage drop in the cathode sheath, in good accord with the theory. The observed ratio, however, was about 40 percent of that calculated, this discrepancy being probably due to non-uniformity in the cathode coating. Similar results were obtained with double sheaths on wire type cathodes, the ratio of the electron current to the ion current through the sheath ranging from 450:1 at high current densities to 2000:1 and more at very low currents, this variation being in agreement with the approximate theory developed for cylindrical sheaths. In these experiments *two cathodes* were used; one at rather large negative voltage to produce any desired intensity of ionization, while from the volt-ampere characteristics of the other cathode the space-charge-limited electron currents were measured. The ion currents were measured either by cooling the test cathode so that it emitted no electrons, or by the use of an auxiliary ion collector.

THE maximum electron current that can flow from a given hot cathode to an anode in high vacuum is limited by the space charge of the electrons.[1] If even a small amount of gas is present and the applied anode voltage is appreciably higher than the ionizing potential, the positive ions formed tend to neutralize the electron space charge and thus allow the current to increase until, with sufficient gas, the current becomes limited only by the electron emission from the cathode as determined by its temperature.

Since in a given electric field the ions move hundreds of times slower than the electrons, the rate at which the ions need to be produced in order to neutralize the space charge is usually less than one percent of the rate at which the electrons flow from the cathode. In formulating a quantitative theory for calculating the increase in electron current produced by a given amount of ionization we meet the difficulty that the ions are produced at different points within the gas and therefore are not all moving with the same velocity. Then, too, the probability of ionization as a function of the electron velocity must be known. The problem thus becomes so complicated that it seems hardly worth while to attempt a general solution.

About 13 years ago the writer derived the equations, given in the present paper, for the space charge problem between parallel planes where the cathode emits a surplus of electrons and the anode emits positive ions without initial velocities. The results, although interesting, did not seem to be applicable directly to experimental conditions and therefore the results were not published. Some years later,[2] however, by the discovery that all caesium atoms which strike a tungsten surface at 1300°K are converted into ions, it became practicable to generate positive ions at the anode in any desired number, and thus obtain the conditions which were assumed in the theory.

Still more recently in a study of gaseous discharges at low pressures[3] it was found that the space charge equations could be applied to the positive ion currents flowing to negatively charged collectors.

If we consider a negatively charged hot collector or in fact any hot cathode in a gas we see that there are present in the positive ion sheath electrons as well as ions. In the present paper it will be shown that the theory which was developed 13 years ago is now applicable to the calculation of the properties of these double sheaths.

[1] Langmuir, Phys. Rev. **2**, 450 (1913); Phys. Zeits. **15**, 348, 516 (1914).

[2] Langmuir and Kingdon, Proc. Royal Soc. **A107**, 61 (1925).

[3] Langmuir and Mott-Smith, Gen. Elec. Rev. **27**, 449, 538, 616, 762, 810 (1924), and Phys. Rev. **28**, 727 (1926).

Jaffe[4] has attempted to develop a theory of the effect of small amounts of gas ionization on currents limited by space charge in gases at low pressures. However, he based his entire treatment upon the inadmissible assumption that as many ions recombine in each element of volume as are produced by ionization within that volume. We now know that even with the high current densities in a mercury vapor arc carrying amperes, recombination of ions in the gas is negligible, compared to the removal of the ions by diffusion to the walls and to the electrodes. Thus the equations which Jaffe derived are not even approximately correct.

Theory of the Effect of Ions on Space Charge Currents Between Parallel Planes

Consider an infinite plane cathode C at zero potential, and a similar parallel plane anode A at the potential V_A and at a distance a from C. Let the cathode emit a surplus of electrons without appreciable initial velocities. There will thus be an infinite concentration of electrons at the cathode surface and the potential gradient will be zero, but a finite electron current, I_0 per unit area, will flow to the anode, this current limited by space charge being given by the equation[1]

$$I_0 = \frac{(2)^{1/2}}{9\pi}\left(\frac{e}{m_e}\right)^{1/2}\frac{V_A{}^{3/2}}{a^2} \tag{1}$$

where e is the charge and m_e the mass of the electrons.

Let us now consider the effect of introducing positive ions without initial velocities, uniformly distributed over a plane B which is at a distance b from C. Between B and C there is thus an ion current I_p per unit area. Because of the partial neutralization of the electron space charge, the electron current from the cathode will increase to a new value, say I_e per unit area. We assume that the ions and electrons do not collide with gas molecules nor with each other and that no appreciable number of ions or electrons is lost by recombination during the passage between the electrodes. Our problem is to determine how the electron current I_e depends on I_p the positive ion emission and on b the location of the source of ions.

Let v_e be the velocity of the electrons at any point P which is at a distance x from the cathode, ρ_e the electron space charge density at P. The corresponding quantities for the positive ions are denoted by the subscript p. The signs of all these quantities will be taken as positive.

Then

$$\rho_e v_e = I_e \qquad \text{and} \qquad \rho_p v_p = I_p \tag{2}$$

and assuming the ions have unit charge

$$\tfrac{1}{2} m_e v_e{}^2 = V_e \qquad \text{and} \qquad \tfrac{1}{2} m_p v_p{}^2 = (V_B - V)e \tag{3}$$

where V is the potential at the point P and V_B the potential at B.

[4] George Jaffe, Ann. d. Physik **63**, 145–174 (1920).

Poisson's equation gives

$$d^2V/dx^2 = 4\pi(\rho_e - \rho_p) \tag{4}$$

We may eliminate ρ and v by Eqs. (2) and (3) and then after substituting

$$\alpha = (I_p/I_e)(m_p/m_e)^{1/2} \quad \text{and} \quad \phi = V/V_A \tag{5}$$

we obtain

$$\frac{d^2\phi}{dx^2} = 2(2)^{1/2}\pi\left(\frac{m_e}{e}\right)^{1/2}\frac{I_e}{V_A{}^{3/2}}[\phi^{-1/2} - \alpha(\phi_B - \phi)^{-1/2}] \tag{6}$$

Combining Eqs. (1) and (6) and substituting

$$\lambda = x/a \tag{7}$$

we have

$$\frac{d^2\phi}{d\lambda^2} = \frac{4}{9}\frac{I_e}{I_0}[\phi^{-1/2} - \alpha(\phi_B - \phi)^{-1/2}] \tag{8}$$

Since the electron current is limited by space charge, we impose the condition $dV/dx = 0$ when $x = 0$, or in other words

$$d\phi/d\lambda = 0 \qquad \text{when} \qquad \lambda = 0 \tag{9}$$

Integration of Eq. (8) then gives

$$d\phi/d\lambda = (4/3)(I_e/I_0)^{1/2}[\phi^{1/2} + \alpha\{(\phi_B - \phi)^{1/2} - \phi_B{}^{1/2}\}]^{1/2} \tag{10}$$

Ions Emitted From Anode, $\phi_B = 1$

According to Eq. (10) the potential gradient at the surface of the anode, ($\phi = \phi_B = 1$), is proportional to $(1-\alpha)^{1/2}$ and so becomes imaginary if $\alpha > 1$. When $\alpha = 1$ the potential gradient at the anode is zero and the positive ion current as well as the electron current is thus limited by space charge.

It appears, therefore, that even an unlimited supply of positive ions available at the anode is not capable of neutralizing the electron space charge, for the positive ion current cannot become more than a definite fraction of the electron current, this fraction (according to Eq. 5, when $\alpha = 1$) being equal to the square root of the ratio of the mass of the electron to that of the ion.

Examination of Eqs. (8) and (10) shows that when $\phi_B = 1$ and $\alpha = 1$ the equations remain unchanged in form if we substitute $1 - \phi$ in place of ϕ. Thus the curve representing the potential distribution between the cathode and anode is symmetrical about its central point ($\lambda = \frac{1}{2}$, $\phi = \frac{1}{2}$). Between the cathode and this central point there is an excess of negative space charge, while from the central point to the anode there is an excess of positive charge.

To calculate the potential distribution we integrate Eq. (10) after placing $\phi_B = 1$

$$\lambda = (3/4)(I_0/I_e)^{1/2} \int_0^\phi [\phi^{1/2} + \alpha\{(1-\phi)^{1/2} - 1\}]^{-1/2} d\phi. \tag{11}$$

The ratio I_e/I_0 is found by observing that $\phi = 1$ when $\lambda = 1$, thus

$$(I_e/I_0)^{1/2} = (3/4) \int_0^1 [\phi^{1/2} + \alpha\{(1-\phi)^{1/2} - 1\}]^{-1/2} d\phi \tag{12}$$

Table I gives values of λ obtained from Eq. (11) by numerical integration. The values of I_e/I_0 which were used in these calculations as found by Eq. (12) are given for various values of α in the next to last horizontal line of the table.[5]

From the values of I_e/I_0 we see that the electron current increases as more positive ions are emitted from the anode until the positive ion current also becomes limited by space charge. When this occurs the electron current and the positive ion current are each 1.860 times as great as the currents of electrons or ions that could flow (with the same applied potentials) if carriers of the opposite sign were absent.

It is interesting to inquire how large is the effect of single positive ions emitted from the anode, in causing an increased electron flow from the cathode. By differentiating Eq. (12) with respect to α and then placing $\alpha = 0$, and $I_e = I_0$ we find in terms of Gamma functions

$$dI_e/d\alpha = [3 - 3\Gamma(1.25)\Gamma(1.5)/\Gamma(1.75)]I_0 = 0.378 I_0 \tag{13}$$

or by Eq. (5)

$$dI_e/dI_p = 0.378(m_p/m_e)^{1/2} \qquad \text{for} \qquad \alpha = 0 \tag{14}$$

A similar calculation for the case $\alpha = 1$ involves a numerical evaluation of the resulting integral giving

$$dI_e/dI_p = 3.455(m_p/m_e)^{1/2} \qquad \text{for} \qquad \alpha = 1$$

A plot of I_e/I_0 as function of α from the data of Table I shows that the slope of the curve increases gradually from 0.378 at $\alpha = 0$ up to 3.455 at $\alpha = 1$. Thus the effectiveness of the ions in raising the electron current increases as the field strength decreases in the region where they originate, but only up to a certain limiting value. Of course when $\alpha = 1$ the further increase in the electron current is stopped by the space charge limitation of the ion current.

The square root of the ratio of the masses of the ions and the electrons is 607 for mercury vapor, 271 for argon, and 60.8 for hydrogen, and there-

[5] In carrying out these calculations it was found convenient to replace ϕ by a new variable σ such that $\phi = \sigma^{4/3}$. By so doing the infinite value of the integrand that occurs when $\phi = 0$ is avoided. When $\alpha = 1$ and $\phi = 1$ a similar difficulty still occurs but the value of λ in the range $\phi = \frac{1}{2}$ to 1 can be obtained from those calculated in the range $\phi = 0$ to $\frac{1}{2}$ by making use of the fact already noted that in this case λ is symmetrical about the point $\phi = \frac{1}{2}$.

fore each positive ion of these gases liberated at the anode will increase the number of electrons that cross the space by 229, 102 or 23 respectively in the case of a pure electron discharge ($\alpha=0$).

TABLE I. *Potential distribution between plane cathode emitting surplus of electrons and parallel plane anode which emits given numbers of ions. Table of values of λ, the fraction of the distance to the anode, at which the potential is a given fraction ϕ of the anode potential (zero potential at cathode).*

	$\alpha=(I_p/I_e)\ (m_p/m_e)^{1/2};\ \lambda=x/a$						
ϕ	$\alpha=0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
0	0	0	0	0	0	0	0
0.02	0.0532	0.0513	0.0491	0.0467	0.0438	0.0419	0.0396
0.05	.1057	.1022	.0981	.0934	.0879	.0842	.0798
0.1	.1778	.1723	.1661	.1588	.1498	.1437	.1367
0.2	.2991	.2911	.2823	.2714	.2573	.2477	.2363
0.3	.4054	.3962	.3855	.3721	.3546	.3423	.3274
0.4	.5030	.4932	.4815	.4667	.4467	.4324	.4146
0.5	.5946	.5847	.5731	.5580	.5371	.5218	.5000
0.6	.6817	.6723	.6612	.6461	.6245	.6080	.5854
0.7	.7653	.7570	.7471	.7332	.7123	.6958	.6726
0.8	.8459	.8395	.8314	.8198	.8016	.7861	.7637
0.9	.9240	.9201	.9149	.9074	.8940	.8813	.8633
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
I_e/I_0	1.0000	1.0839	1.1872	1.3237	1.5186	1.6644	1.8605
a/a_0	1.0000	1.0411	1.0896	1.1505	1.2323	1.2901	1.3640

THE SOURCE OF IONS IS AT A PLANE BETWEEN CATHODE AND ANODE $\phi_B<1$.

Let us consider the effect produced by ions that start from a plane B which lies between C and A. In the region between C and B, Eqs. (6) and (10) are applicable and thus the value of $d\phi/d\lambda$ at B is found by putting $\phi=\phi_B$ in Eq. (10). Equation (6) is applicable also in the region between B and A but here $\alpha=0$ for there is no ion current. Thus by integration we obtain the two equations

$$\left.\begin{aligned}&4(I_e/I_0)^{1/2}\lambda_B=3\phi_B^{3/4}\int_0^1\left[\mu^{1/2}+\alpha\{(1-\mu)^{1/2}-1\}\right]^{-1/2}d\mu\\ \text{and}\quad&4(I_e/I_0)^{1/2}(1-\lambda_B)=3\int_{\phi_B}^1\left[\phi^{1/2}-\alpha\phi_B^{1/2}\right]^{-1/2}d\phi\end{aligned}\right\}\tag{15}$$

The constant of the first integration for the second equation was chosen to make $d\phi/d\lambda$ at $\phi=\phi_B$ the same as for the first equation.

Differentiating Eqs. (15) with respect to α, putting $\alpha=0$, adding the resulting equations and combining with Eq. (5) gives rigorously[6]

$$\frac{dI_e}{dI_p}\left(\frac{m_e}{m_p}\right)^{1/2}=3\phi_B^{1/2}-2.622\phi_B^{3/4}.\tag{16}$$

[6] The coefficient 2.622 is equal to 3 minus the coefficient 0.378 as given in Eq. (14).

This equation, which reduces to Eq. (14) if $\phi_B = 1$, allows us to calculate the number of additional electrons that can flow in a pure electron discharge ($\alpha = 0$) if single positive ions are introduced *at any point* in the space between the electrodes. The values given in Column 3 of Table II were calculated by Eq. (16); the second column represents λ_B the fraction of the distance from cathode to anode at which the ions originate. From Eq. (1) we see that $\lambda_B = \phi_B^{3/4}$.

TABLE II. *The increase in electron current caused by ions originating at various positions between cathode and anode, $\alpha = 0$. Initial velocities neglected.*

ϕ_B	λ_B	$\frac{dI_e}{dI_p} \cdot \left(\frac{m_e}{m_p}\right)^{1/2}$
0.00	0.00	0.0
0.001	0.0056	0.080
0.01	0.0316	0.217
0.1	0.177	0.483
0.2	0.300	0.554
0.338	0.444	0.582 max.
0.5	0.595	0.561
0.6	0.683	0.534
0.7	0.765	0.502
0.8	0.846	0.464
0.9	0.925	0.419
1.0	1.000	0.378

The ions have a maximum effect in increasing the electron current when they are introduced at a point which is 4/9 of the distance from cathode to anode. If a trace of gas is present and the voltage is so high that we may assume the probability of ionization per cm of electron path is uniform, we find readily from Eq. (16) (integrating with respect to λ) that the average value of dI_e/dI_p is 0.489 $(m_p/m_e)^{1/2}$.

If the probability of ionization is greater near the end of the path, as in the case of low anode voltages, the coefficient will lie between 0.489 and 0.378.

Low Pressure Discharges With Hot Cathode

Let us consider for a moment the phenomena that are observed as we pass from a pure electron discharge to a discharge at low gas pressure in which there is abundant gas ionization. When the ions originate at the anode or at a definite plane between the electrodes we have been able to follow through the effects produced by even an unlimited supply of ions. But when the ions are produced throughout all or a large part of the space between the electrodes, we have been able to analyze only the effects produced by a very small total number of ions ($\alpha = 0$). To understand the typical characteristics of gas discharges, however, we must devise methods of treating the problem involving a large intensity of ionization throughout a volume. The nature of the problem will best be realized by considering briefly some experimental observations.

Suppose for example, we have a hot tungsten cathode (at zero volts) capable of emitting 50 ma in a bulb containing mercury vapor saturated at room temperature. With an anode at 10 volts the current is limited by electron space charge and is practically the same as in the absence of mercury vapor. Beginning at the ionization voltage (10.4 volts) the current increases with the anode voltage more rapidly than in good vacuum until at a voltage of 15 to 25 volts depending on the vapor pressure, and the geometry of the tube, the current rises abruptly to the saturation value corresponding to the cathode temperature.

In this second state of the discharge practically the whole of the voltage difference from cathode to anode is concentrated in a cathode sheath in which there is a positive ion space charge; the rest of the volume is nearly field-free, the space charge of the positive ions being neutralized by low-velocity or "ultimate electrons" which accumulate in this space until their concentration is hundreds or thousands of times greater than that of the primary electrons from the cathode.

By means of a sufficiently large resistance in series with the anode it is possible to observe points on the negative resistance part of the current voltage curve that lies between the parts corresponding to the two regions we have just considered. In this transition region the current is limited by the electron and ion space charges in a double layer or double sheath on the cathode.

In order to understand the formation of this double sheath, the final neutralization of space charge and the accumulation of the ultimate electrons, we will first consider the following problem.

Potential Distribution and Current Flow Resulting from the Production of Ions Uniformly throughout the Volume Between Two Planes

Consider two parallel plane electrodes A and C separated by the distance a and let V_c be the potential of C taking that of A to be zero. We assume that S ions of charge e are generated per unit time in each unit volume, and for the present will assume that no electrons are generated by this ionization.

We will first consider the case in which there is no maximum potential in the space between A and C. We will take V_c negative so that the ions move towards C. Then the ion current density I at any plane at a distance x from A is

$$I = Sex \tag{17}$$

but this current is composed of ions having widely different velocities depending on the potentials of their points of origin. The space charge ρ at any point at a distance x from A where the potential is V_1 is thus given by the integral $\int(1/v)dI$ where v, the velocity of an ion which originated at a point of potential V, is found from

$$\tfrac{1}{2}m_p v^2 = (V - V_1)e \tag{18}$$

Thus

$$\rho = S(em_p/2)^{1/2} \int_0^{x_1} (V - V_1)^{-1/2} dx. \tag{19}$$

The problem is now to find V as a function of x which satisfies this integral equation simultaneously with Poisson's equation. By trial the following is found to be a particular solution

$$V = V_c x^2/a^2 \tag{20}$$

which we may now prove as follows.
Substituting this value of V and the corresponding value of V_1 in Eq. (19) We obtain

$$\rho = \tfrac{1}{2}\pi I_a [m_p/(-2eV_c)]^{1/2} \tag{21}$$

where

$$I_a = S_1 ea \tag{22}$$

is the ion current which reaches C, and S_1 is the particular value of S which corresponds to the potential distribution assumed in Eq. (20).
Differentiation of Eq. (20) gives

$$dV/dx = 2V_c x/a^2 \tag{23}$$

and a second differentiation and combination with Poisson's equation gives

$$d^2V/dx^2 = 2V_c/a^2 = -4\pi\rho \tag{24}$$

whence

$$\rho = -V_c/2\pi a^2 \tag{25}$$

We see that both Eqs. (21) and (25) give ρ independent of x or a uniform space charge between the planes. Equating the two expressions for ρ and solving the resulting equation for I_a gives

$$I_a = (2e/m_p)^{1/2}(-V)^{3/2}/\pi^2 a^2. \tag{26}$$

If, as before, I_0 is the current calculated by the ordinary space charge equation, Eq. (1), we see that

$$I_a = (9/\pi)I_0 = 2.865 I_0 \tag{27}$$

We now recognize that the parabolic potential distribution assumed in Eq. (20) is a solution of our problem only when the rate of ionization S has the particular value S_1 given by Eq. (22) and where I_a is given by Eq. (26). Curve 1 in Fig. 1 illustrates this parabolic distribution. It will be noted by Eq. (23) that the potential gradient is zero at A, i.e. at $x = 0$.

If $S < S_1$ the potential distribution curve will lie between the straight line 0 and Curve 1 in Fig. 1, but will not be parabolic. On the other hand, if $S > S_1$ it is evident that there will be a potential maximum between A and C and that Eq. (26) can then be applied separately to the two branches of

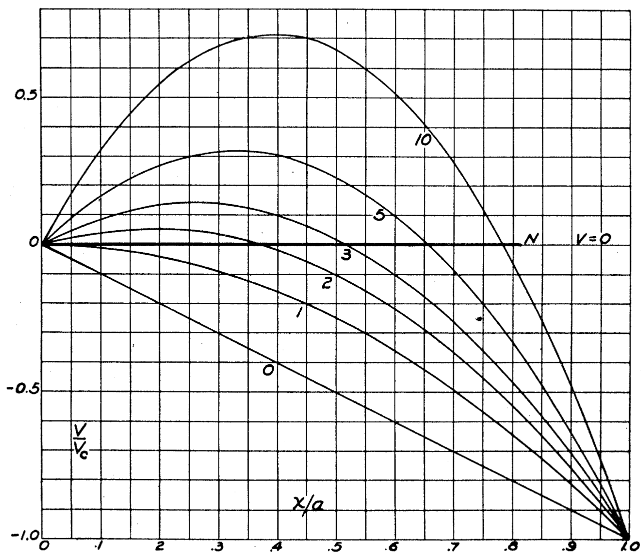

Fig. 1. Potential distributions between plane electrodes when ions are generated uniformly between them.

the curve on the opposite sides of the maximum. The potential distribution curve is thus still a parabola but the origin is no longer at A. The curves marked, 2, 3, 5 and 10 in Fig. 1 have been calculated for values of S equal respectively to 2, 3, 5 and 10 times S_1. The equation of these parabolas is

$$\frac{V}{V_c}=\frac{x}{a}\left[1-\left(\frac{S}{S_1}\right)^{2/3}\left(1-\frac{x}{a}\right)\right]. \tag{28}$$

Effect of the Electrons Generated by Ionization

If the positive ions in the foregoing problem are produced by the ionization of a gas an equal number of electrons will be generated simultaneously. If $S < S_1$ there will be no potential maximum between the electrodes so that these electrons will flow to the electrode A without having appreciable effect on the space charge, for it would take an electron current hundreds of times greater than I_a to neutralize the positive ion space charge due to I_a.

The situation is very different, however, if $S > S_1$ for there is then a tendency to develop a potential maximum as illustrated in Fig. 1. In any region at a potential higher than that of both electrodes the low velocity electrons produced by ionization will accumulate until they nearly neutralize the positive space charge. The potential of the region which would otherwise be above anode potential (above line ON in Fig. 1) is thus lowered to a value at which the electrons, in virtue of their initial velocities, can just escape to the anode as fast as they are produced. The accumulation of the ultimate or low velocity electrons is greatly favored by the smallness of the field available for drawing away the positive ions. In general there will still be a maximum potential in the space but this usually exceeds the anode potential by not more than a volt or so, and thus the ions flow in nearly equal numbers to anode and cathode, while the electrons go to the anode only. Any calculation of the exact potential distribution must involve some knowledge of the velocity distribution of the electrons and ions.

We see from Fig. 1 that when S/S_1 is as great as 10, the fields at the cathode and anode which are necessary to draw these ion currents, are very large.

These regions of strong field due to space charge which cover the electrodes will be referred to as the *sheaths*. The relatively field-free regions between the sheaths where the positive and negative space charges are nearly balanced will be called the *plasma*. We shall find in general that these two regions are rather distinct and have very different properties. Let us first consider some of the characteristics of the plasma.

Random Currents and Potential Distribution in the Plasma

Experiment has shown[3] that the ultimate electrons in the plasma usually have a velocity distribution corresponding closely with that of Maxwell. Thus we may define the electron velocities in terms of an electron temperature T_e. Through any imaginary plane there is a certain current density I_e of electrons passing from one side to the other and a nearly equal current in the reverse direction, this current being related to n_e the number of electrons per unit volume by the equation

$$n_e = (2\pi m_e/kT_e)^{1/2} I_e/e = 4.03 \times 10^{13} I_e/T_e^{1/2} \qquad (29)$$

if I_e is expressed in amperes per cm^2 and n in cm^{-3}, k being the Boltzmann constant 1.372×10^{-16} erg per degree.

The ion velocity distribution is not so easily determined and is far less accurately Maxwellian. Measurements with perforated collectors have shown that the normal components of velocity of the ions that reach the edge of a cathode are roughly that of a Maxwellian distribution[7] corresponding to a temperature T_p which is about half that of the electron, i.e., $T_p = \frac{1}{2} T_e$. In these low pressure discharges the ions probably acquire practically all the kinetic energy they possess from the electric fields within the plasma. The momentum of the electrons is so small that ionizing collisions of electrons with gas molecules cannot impart to the ions appreciable kinetic energy.

[7] Tonks, Mott-Smith and Langmuir, Phys. Rev. **28**, 104 (1926). See pp. 120–123.

All the ions which reach the edge of a cathode sheath pass to the electrodes and thus only half of the ions corresponding to a Maxwellian distribution T_p can be present in the plasma near the sheath edge, so that the directions of the ions can be distributed only over a hemisphere. Thus if I_s is the positive ion current density reaching the sheath edge, and therefore the electrode, we have by analogy with Eq. (29)

$$n_p = \tfrac{1}{2}(2\pi m_p/kT_p)^{1/2} I_s/e = 2.02\times 10^{13} I_s (m_p/m_e T_p)^{1/2} \qquad (29a)$$

The accumulation of ultimate electrons in the plasma will go on until n_e and n_p are very nearly equal and thus from Eqs. (29) and (29a)

$$I_e/I_s = \tfrac{1}{2}(m_p T_e/m_e T_p)^{1/2} \qquad (30)$$

or if

$$T_p = \tfrac{1}{2} T_e \qquad (30a$$

$$I_e/I_s = (m_p/2m_e)^{1/2}$$

For mercury vapor this gives

$$I_e = 429 I_s \qquad (30b)$$

The electron current density I_e which enters this equation is practically the same as that in the plasma at a small distance from the sheath edge for since no electron current flows to the collector the electrons are in a state of equilibrium and have a full Maxwellian velocity distribution. The values of n_e and n_p at the sheath edge are thus the same as in the plasma.

To measure I_s we need merely to determine the current density of the ions flowing to a negatively charged plane collector. By then making the collector positive it is frequently possible in a similar way to obtain an *electron current* independent of applied voltage and thus to measure the rate I_e' at which electrons reach the edge of the electron sheath. Although now equilibrium of the electrons is disturbed, so that only half of the full Maxwellian distribution can exist at the sheath edge, we may assume with reasonable accuracy that the observed electron current density I_e' at the sheath edge is the same as that in neighboring regions of the plasma where the distribution has more nearly spherical symmetry. Thus the values of n_e and n_p at the sheath edge are half of those in the interior of the plasma. This non-uniform distribution n_e and n_p is made possible by the fact that the positive ions contain no intrinsic energy by which they could force the electron concentration to become uniform. We can therefore conclude that the measured I_e' is the same as the I_e in Eq. (30b).

Direct measurements[8] of I_e' and I_s in mercury arcs have given for the ratio I_e/I_s the values 405 ± 25 for 14 experiments, 423 ± 30 for 8 experiments

[8] These data are the averages of the values given as I_e/I_p in Tables III, XIII and XV on pages 544, 765 and 769 of Vol. 27 General Electric Rev. Reference 3. At that time the necessity for introducing the factor 1/2 in Eq. (29a) was not realized and therefore it was thought that I_e/I_p should be equal to the square root of the ratio of the masses, viz. 607. The difference between this and the observed value of about 410 was attributed to the possible presence of negative ions.

and 407 ± 15 for 3 other experiments or for the average of all 25 exps. 411 ± 17 which is in good agreement with the value 429 from Eq. (30a).[9] Returning to our consideration of a plasma discharge between parallel plane electrodes we see that, since the current of ultimate electrons flowing to the anode cannot exceed the positive ion current which is simultaneously produced by ionization, there must be an anode sheath with a field sufficient to repel all but this small fraction of the electrons that move towards the anode, but the ions are not repelled.

To obtain clearer conceptions of the typical distribution of ions and electrons in the plasma and of the resulting plasma fields, we may consider, as before, that there is a uniform rate of ionization throughout the plasma and we may neglect for the present the currents carried by primary electrons.

If a is the distance betweeen the anode and cathode sheaths then the current density I_s of ions flowing to each sheath is approximately

$$I_s = \tfrac{1}{2} Sea \tag{31}$$

and to the anode the ultimate electron current density is 2 I_s since no electrons pass to the cathode.

At somewhat higher gas pressures than we have been considering the mean free paths λ_p of the ions may be small compared to a and then the ions and electrons that are formed in the interior of the plasma must move to the sheaths by diffusion. Since the electrons tend to diffuse more rapidly they leave the interior of the plasma positively charged and thus there is a field which aids in moving the ions. The general differential equations for this "ambipolar diffusion" of ions and electrons have been given by Schottky and v. Issendorf.[10] The "ambipolar diffusion coefficient" is

$$D = \frac{M_p M_e k(T_e + T_p)}{(M_p + M_e)e} \tag{32}$$

where M_p and M_e are the mobilities of the ions and electrons (velocity per unit field). The effect of the field produced by the electrons, in accelerating the ions, has been taken into account in this derivation and is represented by the term T_e in the equation.

The current density I of ions (drift current, not random current) at any point in a direction normal to the sheaths is then

$$I = -De(dn/dx) \tag{33}$$

n being the number of ions, or electrons per unit volume. At any point at a distance x from the central plane of the plasma the current will be Sex and therefore from Eq. (33) by integration we find that the ion concentration at x is

$$n = n_M - Sx^2/2D \tag{34}$$

[9] Dr. Tonks is developing the more rigorous theory of the positive ion velocity distribution assuming that the energies are acquired from the plasma fields.

[10] W. Schottky and J. v. Issendorf, Zeits. f. Physik **31**, 163 (1925).

or at the sheath edges where $x = \pm a/2$ we have

$$n_S = n_M - Sa^2/8D. \tag{35}$$

The drift ion current I is zero at the center $x = 0$ and increases to I_S at the sheath edge while the random ion current is proportional to n and therefore should decrease from a maximum value I_M at the center to a value $I_M(n_S/n_M)$ at the sheath edge. Although strictly speaking the current at the sheath edge is not a random current for it flows in one directly only we may provisionally[11] identify this calculated value of the random current with the drift current I_S at the sheath edge, so that

$$I_S = I_M(n_S/n_M). \tag{36}$$

The mobility M_p of Eq. (32) is very small compared to M_e and therefore M_e drops out. The ion mobility is

$$M_p = e\lambda_p(2/\pi m_p kT_p)^{1/2} \tag{37}$$

and thus by Eq. (32)

$$D = \lambda_p\left(\frac{2k}{\pi m_p T_p}\right)^{1/2}(T_e + T_p) = 3.15\times 10^5 \lambda_p(T_e + T_p)\left(\frac{m_e}{m_p T_p}\right)^{1/2} \tag{38}$$

if we use c. g. s. units.

Substituting this value of D and the value of S from Eq. (31) in Eq. (34) and combining with Eqs. (36) and (29a)

$$\frac{I_M}{I_S} = \frac{n_M}{n_S} = 1 + \frac{Sa^2}{8n_S D} = 1 + \frac{aT_p}{8\lambda_p(T_e + T_p)} \tag{39}$$

or if we put $T_e = 2T_p$

$$I_M/I_S = 1 + a/24\lambda_p. \tag{40}$$

This simple result indicates that the ion concentration will be nearly uniform throughout the plasma if the free path of the ions is large compared to 1/24th of the distance between the electrodes. With mercury vapor at 1 barye pressure and 20°C, λ_p is of the order of 5 cm and therefore if a, the distance between the electrode sheaths, is 5 cm the pressure may be as great as 24 baryes (saturated vapor at 56°C) before the concentration of ions at the center of the plasma becomes twice as great as at the edges.

The potential distribution within the plasma may now be calculated by means of the Boltzmann equation

$$n/n_M = \exp\left[(V - V_M)e/kT_e\right] \tag{41}$$

[11] Although this assumption isof doubtful accuracy because of the non-Maxwellian character of the ion velocity distribution near the sheath edge, it is far more nearly in accord with the facts than the assumption made by Schottky and Issendorf and others that $n_S = 0$ at the sheath edge.

which will be applicable within the plasma since the electron drift currents are negligible compared to the random currents and therefore the conditions will not depart appreciably from those of equilibrium. The difference of potential between the sheath edge and the center of the plasma is thus by Eqs. (41) and (40), since $e/k=11600$ degrees per volt,

$$V_M - V_S = 1.98\times 10^{-4} T_e \log_{10}(1+a/24\lambda_p) \text{ volts.} \tag{42}$$

These fields are ordinarily very weak, for example if $T_e = 10{,}000°$ in a mercury arc at 24 baryes pressure and with $a=5$ cm we find $V_M - V_S = 0.6$ volts. At a pressure of 240 baryes this voltage difference would be 2.1 volts if T_e remained the same, but actually T_e decreases as the pressure rises so that the potential difference would probably remain approximately constant at about 0.6 volt. Although this field is extremely small compared to that in the sheaths, it must play an essential rôle in gaseous discharges in drawing the positive ions towards the sheath edges. In fact the kinetic energy of the positive ions is probably almost entirely derived from the action of this field.

Anode Sheath

The electron current flowing to the anode is usually dependent on the conditions that exist in other parts of the circuit which supplies the energy for the anode current. For example, it may be fixed by the electron emission from the cathode or by the current that can flow through an external resistance placed in series with the anode. If the anode area is comparable with the cross-section of the plasma the current density I_a of the electron current flowing to the anode is usually far smaller than I_e the electron current density in the plasma near the anode. There will then be a retarding potential V_a (negative anode drop) in the anode sheath whose magnitude can be calculated from the Boltzmann equation which takes the form

$$V_a = 1.98\times 10^{-4} T_e \log_{10}(I_e/I_a). \tag{43}$$

In cases of low pressure plasma discharges with hot cathodes when no measurements of I_e are available, methods of the following type will usually give I_e with sufficient accuracy to give a fairly reliable value of the anode drop.

Each primary electron from the cathode, in falling through the cathode sheath, acquires a high kinetic energy, which it retains until the energy is dissipated in excitation and ionization of the gas molecules. The glass walls of the tube become so strongly negatively charged that they receive no more electrons than positive ions and thus in ordinary cases only a negligible fraction of the primary electrons are lost to the walls. It has been shown[12] under a wide range of conditions of this kind, that the total number of ions formed per primary electron, which we may call β, is independent of current density, of pressure and of the geometry of the tube, but depends only on the nature

[12] I. Langmuir and H. A. Jones. Phys. Rev. **31**, 357–404 (1928). See particularly pp. 402–3.

of the gas and the energy of the primary electrons. Thus 50 volt electrons in mercury vapor give 1.4 ions per primary electron. Neglecting the relatively small ion currents to the cathode and anode, the primary electron current from the cathode is equal to i_a the electron current to the anode. Thus the total rate of production of ions within the tube is βi_a.

Since recombination in the gas is negligible at low pressures, all these βi_a ions will flow to the electrodes and to the walls of the tube. With envelopes of roughly spherical shape, the ion current density will be fairly uniform over the whole surface. Thus if B is the area of the plasma envelope (sheaths over all electrodes and glass surfaces) find that the positive ion current density I_p is given by

$$I_p = \beta i_a / B. \tag{44}$$

Substituting this value of I_p for I_s in Eq. (30) we can thus calculate I_e, the random electron current density and then Eq. (43) gives the negative anode drop:

$$V_a = 1.98 \times 10^{-4} T_e \log_{10} \left[\frac{A\beta}{2B} \left(\frac{m_p T_e}{m_e T_p} \right)^{1/2} \right] \tag{45}$$

where A is the anode area.

Experiments with discharges from hot cathodes in mercury vapor in spherical and cylindrical bulbs and with anodes of various sizes have demonstrated the general usefulness and reasonable accuracy of this equation.

With large anode area A the anode sheath is thus a positive ion sheath but as A decreases a point is reached where the sheath disappears and is then replaced by an electron sheath, the anode drop becoming positive. When the positive anode drop approaches the ionizing voltage an anode glow appears and with sufficiently high gas pressure the rate of positive ion production becomes so great as to break-down the electron space charge, causing a *second plasma* to develop near the anode. This second plasma usually takes the form of a globular luminous protuberance from the anode. Its interior is a typical plasma with high electrical conductivity; its boundary is a double sheath with an inner positive space charge and an outer negative space charge, the potential drop being of the order of the ionizing potential.

The conditions at this double layer are essentially like those that we postulated in the first section of this paper; the potential gradient vanishes at the inner and outer edges and the potential distribution is determined by the charges of moving ions which acquire their kinetic energy mainly from the field within the double layer.

Cathode Sheaths

An electrode which is at a negative potential with respect to the plasma repels all the ultimate electrons which move towards it, except those that have a sufficient component of energy normal to the surface of the electrode, to enable them to move against the retarding field. The positive ions in the plasma that move towards the electrode are collected by it. The fraction of the ultimate electrons that penetrate through this positive ion sheath is given by the Boltzmann equation or may be calculated from the ratio I_a/I_e in Eq. (43). Thus if the temperature of the ultimate electrons is 10000° the current density of these electrons which reach the electrode will be only 1/1000th of that in the plasma if the electrode is 5.9 volts negative with respect to the plasma.

If the electrode is 50 volts or more negative with respect to the plasma the sheath will contain no appreciable number of electrons and the positive ions will enter the sheath with energies negligible compared to those they acquire in the sheath itself. Thus the velocity of the ions is given by

$$\tfrac{1}{2}m_p v_p^2 = -Ve$$

and since there is no appreciable field in the plasma itself we will have at the outer edge of the sheath $dV/dx=0$.

Since these conditions are exactly those that are postulated in the derivation of the ordinary space charge Eq. (1) we may apply this equation to cathode sheaths in gaseous discharges, merely replacing m_e by m_p.

In the usual applications of Eq. (1) the voltage and the distance between the electrodes are known and we wish to calculate the current density that can flow when the current is limited by space charge. In the present case, however, the current density I_p is fixed by conditions within the plasma and since the applied voltage is usually known the equation can be used to calculate only the thickness x of the sheath (a in Eq. (1)). Thus the current density I_p is independent of the applied voltage and x varies in proportion to $V^{3/4}$.

The cathode sheath can usually be seen as a dark space and under favorable conditions its thickness can be measured. In low pressure discharges in mercury vapor the observed sheath thickness agrees well with that calculated by the space charge equations.[13]

Of course, modifications or corrections to this theory may be necessary under special conditions, for example, when the gas pressure is so high that the ions collide with gas molecules within the sheath or when the voltage is so low that the initial velocities of the ions and electrons at the sheath edge can not be neglected.

Sheath on Hot Cathode. Double Sheath.

If the cathode is heated so that it emits electrons these flow out through the positive ion sheath and if this electron current becomes sufficiently great it will neutralize the positive ion space charge in the immediate neighborhood of the cathode. Since the electrons start with negligible velocities from the cathode, the conditions are the same as those postulated when we calculated the effect of ions liberated at the anode upon the electron currents from a hot cathode emitting a surplus of electrons, for example, in our derivation of Eqs. (5), (11) and (12). In the present case, however, it is the

[13] Reference 3, p. 545.

sheath edge, functioning as anode, that emits a surplus of ions (since $dV/dx=0$ at this point) while a limited number of electrons is emitted from the cathode. Thus, in applying our equations to the present problem we need merely interchange the subscripts p and e.

Let us consider the currents of ions and electrons that flow to or from a plane cathode in a plasma having given characteristics, such as I_e, I_p and T_e. Let the cathode, at potential $-V$ with respect to the plasma first be at such low temperature that it emits no electrons. We let a_0 denote the sheath thickness under these conditions, as calculated from the known value of I_p by means of Eq. (1), in which for this purpose we replace I_0 by I_p, V_a by V, m_e by m_p and a by a_0.

Now let the cathode temperature be raised until the current density of the emitted electrons is I_e. To calculate the coefficient α we interchange e and p in Eq. (5) giving

$$\alpha_e=(I_e/I_p)(m_e/m_p)^{1/2} \tag{46}$$

The effect of these electrons in neutralizing the ion space charge cannot cause an increase in I_p for this is fixed by the plasma, but manifests itself by changing the sheath thickness so that this becomes a instead of a_0. In Eq. (12) the ratio I_e/I_0 was calculated on the assumption that the distance between the electrodes was fixed. Evidently for our present problem we must replace I_e in this ratio by I_p and I_0 must be replaced by the positive ion current density that would flow to a cathode of voltage $-V$ if no electrons were emitted and if the sheath thickness were a. This current may be calculated from Eq. (1) by replacing m_e by m_p, and V_a by V. Thus we see that $(I_e/I_0)^{1/2}$ in Eq. (12) must be replaced by a/a_0. The last horizontal line of Table I gives the values of a/a_0 (which are merely the square roots of the numbers in the line above) for various values of α_e. An electron emission for which $\alpha_e=0.2$ would correspond to $I_e=0.2\times607I_p=121I_p$ in a discharge in mercury vapor. The electron current would then be 121 times the positive ion current and yet we see from Table I that this would cause the sheath thickness to increase merely 4.1 percent. The potential distribution functions of Table I are of course applicable to these sheaths on a hot cathode.

As the cathode temperature is raised the electron current density I_e increases and is equal to the electron emission from the cathode until, when α_e becomes equal to unity, the current becomes limited by space charge and a further increase in electron current can not occur. The cathode is then covered by a double layer or double sheath and the ratio of the electron current to the ion current is equal to $(m_p/m_e)^{1/2}$. Even a change in cathode voltage will not cause a change in electron current if the positive ion current I_p remains constant.

These facts are of vital importance in any understanding of discharges from hot cathodes in low pressures of gas. We see that the use of a hot cathode, no matter how high its temperature, does not destroy the cathode drop.

In discharges with a single cathode the intensity of ionization in the plasma depends on the electron current from the cathode and on the cathode drop which gives to these electrons their velocity. Under these conditions, when the total current is restricted by a resistance in series with the anode, and a surplus of electrons is emitted by the cathode, the cathode drop adjusts itself to such a value that I_p in the plasma bears the proper ratio (corresponding to $\alpha=1$) to the electron current that is drawn. For example, with mercury vapor, the cathode drop must be such as to cause an ionization in the plasma that will give a positive ion current equal to 1/607th of the electron current that flows from the cathode.

For very low pressure discharges in which the plasma is very uniform we may calculate the magnitude of the cathode drop and thus learn upon what factors it depends. If C is the effective area of the cathode, then the electron current from it will be CI_e and this will be practically equal to the anode current i_a. Substituting CI_e for i_a in Eq. (44) and then putting for I_e/I_p its value $(m_p/m_e)^{1/2}$ we find

$$\beta=(B/C)(m_e/m_p)^{1/2} \tag{47}$$

where as before, B is the surface of the plasma envelope. The cathode drop will become such as to give to β (which is a function of voltage only) this value. The ratio C/B measures the fraction of the positive ions that are generated that get back to the cathode. The greater this ratio the smaller will be the cathode drop.

At higher gas pressures the intensity of ionization in the plasma will not be uniform but there will be a greater intensity near the cathode; this will greatly decrease the cathode drop. Under the most favorable conditions nearly all the ions formed will pass to the cathode and then β (for mercury vapor) will only need to be 0.0016, a value which will be reached at about 0.1 volt above the ionizing voltage.

Theory of Double-Sheath Considering Initial Velocities

The foregoing theory of the cathode double sheath, simplified by neglecting the effects of the initial velocities of the ions and electrons, has given us a useful picture of the essential character of the phenomena at a hot cathode in gas. We have seen, however, that its main field of application will be to cases involving cathode drops of 10 or 20 volts so that we are not wholly justified in ignoring these initial velocities. Fortunately there is no great difficulty in calculating the ratio I_e/I_p even when these velocities are taken into account.

Let us consider a sheath having parallel plane boundaries or "edges." Any group of electrons or ions entering the sheath with a given velocity distribution will contribute to the space charge at any point P within the sheath the amount $\int(1/v)dI$, where dI denotes the current density, normal to the plane of the sheath, of an elementary group which has the normal component of velocity v. Since with a given initial distribution, both dI

and v are functions solely of the potential at the point P, it follows that the total space charge ρ at any point depends only on the potential at that point.

Multiplying both members of the Poisson equation (for plane case) by dV and integrating we obtain

$$(dV/dx)^2 = -8\pi \int \rho dV + \text{const}. \tag{48}$$

In the case of the double sheath where the electron and ion currents are both limited by space charge, $dV/dx=0$ at both the inner and outer edges so that we have

$$\int_{V_M}^{V_S} \rho dv = 0 \tag{49}$$

where V_M and V_S are the potentials at these two edges, or more specifically at the places where dV/dx is zero, one of these being at the minimum potential region very close to the cathode and the other S, the outer edge of the sheath.

Later we shall find it necessary to analyze more closely the nature of this sheath edge S, but for the present we will define it as a place where $dV/dx=0$ and $\rho=0$ (as in the plasma) so that according to the Poisson equation d^2V/dx^2 is also zero.

Within the sheath there are three groups of carriers to be considered;—the positive ions from the plasma, the ultimate electrons from the plasma and the electrons emitted by the hot cathode. For each of these groups we shall express its space charge in terms of the potential.

1. *Plasma ions.* We have already seen that the velocity distribution at S cannot be Maxwellian and we shall see that it cannot even have hemispherical symmetry. Assuming that the velocities of the ions originate from the weak plasma fields which we now wish to ignore, the velocity must in general be nearly normal to the plane of the sheath. Although the velocities of all the ions cannot be equal, we shall make no great error (especially since the ions are accelerated into the sheath) if we replace the actual distribution by a homogenous swarm of ions moving normal to the plane of the sheath with a kinetic energy corresponding to the potential E_p. If ρ_{pS} is the space charge of these ions at S then their space charge ρ_p at a point of potential V is

$$\rho_p = \rho_{ps} \{ E_p/(V_s - V + E_p) \}^{1/2} \tag{50}$$

We shall wish to calculate the integral of Eq. (49). If we let H_p be the part contributed to this integral by the ions, we have

$$H_p = \int_{V_M}^{V_S} \rho_p dV = 2\rho_{pS} E_p^{1/2} \{ (V_{SM} + E_p)^{1/2} - E_p^{1/2} \} \tag{51}$$

where V_{SM} is the difference of potential between S and M, ie.: $V_S - V_M$.

The space charge ρ_{pS} at the sheath edge can be determined from the positive ion current density I_p flowing to the cathode from the equation

$$\rho_{pS} = I_p(m_p/2E_p e)^{1/2}. \tag{52}$$

2. *Plasma electrons.* We assume that the cathode is sufficiently negative with respect to the plasma so that no appreciable fraction of the ultimate electrons reaches the cathode and thus the velocity distribution of these electrons at S will be a complete Maxwellian distribution corresponding to a temperature T_e and a random electron current density I_{eS}. Then according to the Boltzmann equation the space charge at any point due to these electrons is

$$\rho_e = \rho_{eS} \exp \left\{ (V - V_s)/E_e \right\} \tag{53}$$

where E_e is the potential which corresponds to the energy kT_e that is, $E_e = T_e/11600$ volts. The integral H_e is

$$H_e = \int_{V_M}^{V_S} \rho_e dV = E_e \rho_{eS} \tag{54}$$

Finally ρ_{eS} may be expressed in terms of I_e:

$$\rho_{eS} = I_{eS}(2\pi m_e/E_e e)^{1/2} \tag{55}$$

3. *Electrons from cathode.* If the cathode emits more electrons than can flow across the sheath there will be a potential minimum V_M at a short distance from the cathode surface.

If V_c is the potential of the cathode then V_M may be calculated [14] from the equation

$$I_{eM} = I_c \exp \left\{ (V_M - V_c)/E_c \right\} \tag{56}$$

where I_c is the current density of the saturation emission from the cathode corresponding to its temperature T_c and I_{eM} is that which gets past the potential minimum and is thus the electron current density passing through the sheath; E_c is the potential corresponding to kT_c, that is $T_c/11600$ volts. The space charge of the electrons at any point in the sheath, between M and S is thus rigorously

$$\rho_c = \rho_{cM} \exp \left\{ (V - V_M)/E_c \right\} \operatorname{erf} \left\{ (V - V_M)/E_c \right\}^{1/2} \tag{57}$$

where erf denotes the error function defined by

$$\operatorname{erf}(x) = 2\pi^{-1/2} \int_x^\infty \exp(-y^2) dy$$

When x is greater than about 2 a very close approximation is

$$\exp x^2 \operatorname{erf} x = 1/\pi^{1/2}(x + 1/2x). \tag{58}$$

[14] Eq. (56) and (57) have been derived from equations in a paper by I. Langmuir, Phys. Rev. **21**, 419 (1923).

The quantity E_c is only 0.1 or 0.2 volts and if $V-V_c$ is more than 4 times as great this approximation is justified and Eq. (57) thus becomes

$$\rho_c=\rho_{cM}\{E_c/\pi(V-V_M+E_c)\}^{1/2} \tag{59}$$

Here ρ_{cM} is the space charge density of the emitted electrons at the point of minimum potential.

Integration of Eq. (57) gives rigorously

$$H_c=\int_{V_M}^{V_S}\rho_c dV=E_c[\rho_{cS}-\rho_{cM}+2\rho_{cM}(V_{SM}/\pi E_c)^{1/2}]. \tag{60}$$

Putting $V=V_S$ in Eq. (59) we can express ρ_{cM} in terms of ρ_{cS}. Thus Eq. (60), expanded as a power series in E_c/V_{SM}, becomes

$$H_c=2\rho_{cS}V_{SM}[1-(\pi E_c/4V_{SM})^{1/2}+E_c/V_{SM}\cdots]. \tag{61}$$

The space charge ρ_{cS} expressed as a function of I_{eM} is

$$\rho_{cS}=I_{eM}[m_e/2(V_{SM}+E_c)e]^{1/2} \tag{62}$$

When the current I_M is limited by space charge Eq. (49) can now be used to determine the relation between I_M and I_p. The equation takes the form

$$H_p=H_e+H_c. \tag{63}$$

The values of these quantities from Eqs. (51), (54) and (61) substituted in this equation give a relation between the ρ's and then by means of Eqs. (52), (55) and (62) we find the following relation between the currents

$$\begin{aligned} I_{eM}[1-(\pi E_c/4V_{SM})^{1/2}+E_c/2V_{SM}] \qquad\qquad (64)\\ =I_p(m_p/m_e)^{1/2}[1-(E_p/V_{SM})^{1/2}+E_p/2V_{SM}]-I_{eS}(\pi E_e/V_{SM})^{1/2}\end{aligned}$$

We have defined S as a place where dV/dx is zero. At this region we may also put $\rho=0$, so that

$$\rho_{pS}=\rho_{eS}+\rho_{eS}. \tag{65}$$

Expressing the ρ's in terms of the I's by Eqs. (52), (55) and (62) and using this equation to eliminate I_{eS} we obtain

$$I_{eM}/I_p=(m_p/m_e)^{1/2}\frac{1-(E_p/V_{SM})^{1/2}(1+E_e/2E_p)+E_p/2V_{SM}}{1-(\pi E_c/4V_{SM})^{1/2}-(E_e-E_c)/2V_{SM}}. \tag{66}$$

An examination of this result shows that when $(1/2)E_e<E_p<V_{SM}$ an increase in either E_p or E_e causes a decrease in I_{eM}, Thus the effect of the initial velocities of both the ions and the electrons at the sheath edge is to decrease the electron current that can flow from a hot cathode. As a typical example we may take $E_e=2$ volts (corresponding to $T_e=23000°$), $E_p=1$ volt, $E_c=0.2$ volt. The ratio I_{eM}/I_p instead of being 607 for mercury vapor, will be only 31 percent of this for $V_{SM}=5$ volts, 53 percent for 10 volts and 71 percent for 25 volts.

THE SHEATH EDGE

If the initial velocities of the plasma electrons and ions could be neglected we would be justified in regarding the edge of the sheath as sharp. In our study of the double sheath we must now recognize that the velocities of the ions which are measured by E_p are derived from the weak fields that extend from the sheath a considerable distance into the plasma.

To develop clear conceptions of the relation between the plasma and the sheath we will analyze in some detail a typical although somewhat simplified example. Let us imagine two parallel plane electrodes with a plane between them which acts as a source of ions and electrons, but which is not to be regarded as an electrode. Thus the potential of the source will be determined by space charges and cannot be arbitrarily varied as that of an electrode could be.

If the source emits only ions having no initial velocities, the potential will rise until the ions can flow to the two electrodes in accord with the space charge equation for ions (similar to Eq. (1)). Thus at the source the potential gradient is zero. Without loss of generality we may take the source as the origin both for the potential V and for the distance x and for simplicity may consider phenomena only on one side of the source, i.e. for positive values of x. Let I_p be the positive ion current density that flows from the source in this positive direction.

Now let us introduce electrons having a Maxwellian velocity distribution corresponding to a temperature T_e. We assume that the electrode is at a negative potential sufficient to prevent any appreciable number of electrons from reaching it, and thus the Maxwellian distribution is not disturbed. We assume also that the electrons pass freely through the source in both directions, which is equivalent to assuming that the source is a perfect reflector for electrons. Let I_{e0} be the electron current density passing in one direction through the source (random current). At any point having the potential V the positive ion space charge is

$$\rho_p = I_p(m_p/-2Ve)^{1/2}$$

and the electron charge density according to Eqs. (29), (41) and (53) is

$$\rho_e = I_{e0}(2\pi m_e/E_e e)^{1/2} \exp (V/E_e).$$

We now substitute the total space charge $\rho_p - \rho_e$ into the Poisson equation and then reduce the result to a dimensionless form by introducing the variables η and x_p defined by

$$\left.\begin{aligned} \eta &= -V/E_e \\ I_p &= (1/9\pi)(2e/m_p)^{1/2} E_e^{3/2} x_p^{-2} \end{aligned}\right\} \tag{67}$$

This last equation is similar in form to the space charge Eq. (1). Thus x_p is the positive ion sheath thickness calculated for a collector voltage E_e ignoring the effects of initial velocities. The Poisson equation thus gives

$$d^2\eta/d\lambda^2 = (4/9)\left[\eta^{-1/2} - 2\pi^{1/2}\alpha_{e0} \exp (-\eta)\right] \tag{68}$$

where λ is a dimensionless parameter proportional to length defined by $\lambda = x/x_p$ and α_{e0} in accordance with Eq. (46) is

$$\alpha_{e0} = (I_{e0}/I_p)(m_e/m_p)^{1/2}. \tag{69}$$

Multiplication by $d\eta/d\lambda$ and integration gives

$$d\eta/d\lambda = (4/3)\{\eta^{1/2} - \pi^{1/2}\alpha_{e0}[K - \exp(-\eta)]\}^{1/2} \tag{70}$$

where K is a constant of integration. The potential distribution is given by

$$\lambda = (3/4)\int_0^\eta \{\eta^{1/2} - \pi^{1/2}\alpha_{e0}[K - \exp(-\eta)]\}^{-1/2} d\eta. \tag{71}$$

For the case that we are now considering where the ions start from the source without initial velocity, we can determine K by the condition $d\eta/d\lambda = 0$ when $\eta = 0$, and find from Eq. (70) that $K = 1$. Let us now consider the nature of this solution to our problem given by Eqs. (70) and (71).

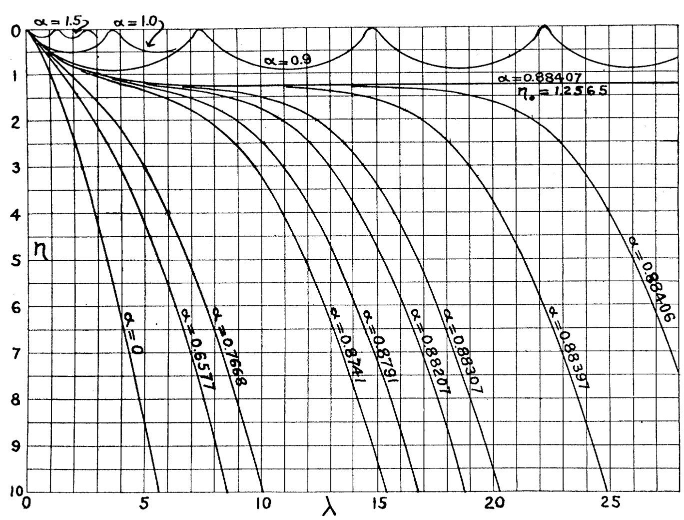

Fig. 2. Effect of electrons of temperature I_e on the potential distribution between planes at one of which ions are generated.

The family of curves shown in Fig. 2 represents the solutions of Eq. (71) for the case $K = 1$ for a series of different values of α_{e0}. By plotting $-\eta$ as ordinate instead of η the curves give directly the potential distribution V as a function of x in terms of E_e and x_p as units. The data for the curves shown in Fig. 2 have been calculated accurately either by numerical quadrature of Eq. (71) or by using some of the equations given below.

If $\alpha_{e0} = 0$, that is, if no electrons are present, Eq. (71) becomes

$$\lambda = \eta^{3/4}$$

which corresponds to the ordinary space charge equation for ions.

For values of α_{e0} small compared to unity, expansion of Eq. (71) gives

$$\lambda = \eta^{3/4}[1 + 0.3\pi^{1/2}\alpha_{e0}\eta^{1/2} + (9/56)\pi\alpha_{e0}^2\eta - \cdots]$$

if η is small, while if $\eta > 4$

$$\lambda = \eta^{3/4} + 2.66\alpha_{e0}\eta^{1/4} - 2.41\alpha_{e0}.$$

Thus the effect of the electrons is to increase λ and decrease the potential gradient.

When α_{e0} is zero there is an infinite positive space charge at $\eta = 0$ and elsewhere the charge decreases continuously as η increases. Electrons, when introduced, are confined almost wholly to regions where η is less than 3 or 4. These electrons tend to produce a minimum in the space charge. The condition for the occurrence of such a minimum is that

$$d^3\eta/d\lambda^3 = (2/9)[-\eta^{-3/2} + 4\pi^{1/2}\alpha_{e0} \exp(-\eta)]d\eta/d\lambda = 0. \tag{72}$$

Before a negative space charge can occur as α_{e0} increases, it is necessary that $d^2\eta/d\lambda^2$ shall pass through zero. Equating both the third and the second derivatives to zero we find $\eta = 0.5$ and $\alpha_{e0} = (\epsilon/2\pi)^{1/2} = 0.6577$. The curve obtained with this value is plotted in Fig. 2.

When α_{e0} increases beyond 0.6577 there are two points of inflection between which there is a negative space charge. At the upper one ($\eta < 0.5$) the potential gradient $d\eta/d\lambda$ is a maximum while at the lower one it is a minimum. At a certain value of α_{e0} which we shall call α_{e1} the slope at this second inflection becomes zero. We find this value by placing the second members of Eqs. (68) and (70) each equal to zero, thus obtaining

$$\alpha_{e1} = 0.88407\ ; \quad \eta_1 = 1.2565.$$

This solution as shown in Fig. 2 is one that gives a region where the space charge and the potential gradient are both zero, conditions which characterize an ideal plasma. We shall return to a consideration of this case later.

When α_{e0} increases beyond α_{e1} the second point of inflection becomes imaginary and the curve no longer dips indefinitely downward but after passing through a minimum rises again to $\eta = 0$. Thus for larger values of α_{e0}, η must remain so small that we can replace $1 - \exp(-\eta)$ by η and then Eq. (71) after integration gives

$$\lambda = (3/2)\pi^{-3/4}\alpha_{e0}^{-3/2}[\sin^{-1}\nu - \nu(1-\nu^2)^{1/2}] \tag{73}$$

where

$$\nu = \pi^{1/4}\alpha_{e0}^{1/2}\eta^{1/4}.$$

Thus the potential is a periodic function of the distance λ, the wavelength corresponds to

$$\lambda_1 = (3/2)\pi^{1/4}\alpha_{e0}^{-3/2}$$

and the minimum potential at $\lambda = 1/2\lambda_1$ corresponds to

$$\eta_{max} = 1/\pi\alpha_{e0}^2.$$

Periodic solutions of this kind are illustrated in Fig. 2 for values of α_{e0} of 0.9, 1.0 and 1.5. These data have been obtained directly from Eq. (71) by quadrature since Eq. (73) is only applicable accurately for still larger values of α_{e0}.

Examination of Fig. 2 shows that when α_{e0} is only slightly less than the critical value $\alpha_{e0} = 0.88407$ the curves show an extended region in which the potential gradient is nearly zero. This clearly corresponds to a plasma. Close to the source of ions at $\lambda = 0$ there is a field which draws the ions outward into the plasma. However, for large enough values of λ, the potential begins to drop again and we thus pass from the plasma into a positive ion sheath. As α_{e0} approaches α_{e1} this descending portion rapidly approaches a limiting form in which the successive curves differ from one another merely in a horizontal displacement. Thus for the limit $\alpha_{e0} = \alpha_{e1}$ the lower part of the curve is like that shown in the curves for $\alpha = 0.88207$, 0.88307, 0.88397 etc. except that it lies at an infinite distance to the right.

To investigate this transition from plasma to sheath we may write Eq. (71)

$$\lambda = (3/4)\int_0^n f^{-1/2} d\eta \tag{74}$$

where

$$f = \eta^{1/2} - \pi^{1/2}\alpha_{e0}[1 - \exp(-\eta)]. \tag{75}$$

An expansion in the neighborhood of $\eta = \eta_1$ and $\alpha_{e0} = \alpha_{e1}$ gives to 3 terms

$$f = A\Delta\alpha + B\mu\Delta\alpha + C\mu^2$$

where

$$\Delta\alpha = \alpha_{e1} - \alpha_{e0}$$

$$\mu = \eta - \eta_1$$

and $A = 1.2679$, $B = -0.5046$ and $C = 0.13428$.

Substitution of this value of f in Eq. (74) and integration gives

$$\lambda = 0.75C^{-1/2} \log_\epsilon [f^{1/2} + C^{1/2}\mu + 0.5BC^{-1/2}\Delta\alpha] + \text{const.}$$

When $\Delta\alpha$ is small this reduces to

$$\Delta\lambda = 4.713 \log_{10} [0.424\mu_2(-\mu_3)/\Delta\alpha] \tag{76}$$

where μ_2 and μ_3 correspond to 2 points η_2 and η_3 that lie on opposite sides of η_1, and $\Delta\lambda$ is the distance between the points η_2 and η_3. It is assumed that μ_2 and $-\mu_3$ are large compared to $\Delta\alpha$.

When η is larger than about 4 the exponential term in Eq. (75) can be neglected in comparison with unity. Integration of Eq. (74) for $\alpha_{e0} = \alpha_{e1}$ then gives

$$\lambda = (\eta^{1/2} + 3.134)(\eta^{1/2} - 1.567)^{1/2} - 4.713 \log_{10} \Delta\alpha - 1.93 \tag{77}$$

the integration constant having been determined from the data of Curve VII in Fig. 2 together with Eq. (76).

Our equations enable us to study the transition between plasma and sheath. Let us consider a typical numerical example. A one ampere arc in a 3 cm diameter tube containing mercury vapor saturated at 30°C gives for a collector flush with the tube wall $I_p = 2 \times 10^{-4}$ amperes per cm^2, $T_e = 21000°$ and thus $E_e = 1.81$ volts. Then from Eq. (67) we find $x_p = 0.00686$ cm as our unit of length for the sheath measurements ($\lambda = 1$). If a collector is at -25 volts with respect to the ionized gas then at the collector surface $\eta = 1.256 + 25/1.81 = 15.1$. The effective distance between the collector and the source of ions may be taken to be 1 cm so that in Eq. (77) we put $\lambda = 1/0.0068 = 146$, and obtain $\Delta\alpha = 10^{-29}$. From Eq. (76) we see that the plasma field near the sheath falls to 1/10th for an increase in distance from the sheath equal to 0.032 cm (i.e. 4.71×0.0068).

According to the data of Fig. 2 the potential distribution within the sheath is given by

x	0	0.049	0.073	0.089	0.114	0.149
V	25	5.	1.34	0.44	0.080	0.006

where V is the negative voltage at a point whose distance from the collector surface is x cm. If the initial velocities of the ions and electrons could be neglected the total thickness of the 25 volt sheath, calculated by the ordinary space charge equation, like Eq. (67), would be 0.049 cm.

From the foregoing analysis it appears that the motions of the positive ions in the plasma are not to be chosen arbitrarily but are fixed by the electron temperatures and by the geometry of the ion source. With the plane source we have assumed above, we found $\eta_1 = 1.2565$ which means according to the nomenclature of Eq. (66) that $E_p = 1.2565 E_e$.

This particular assumption, however, is of course not directly applicable to experimental conditions. Dr. Tonks has been able to solve the integral equations involved in the problem of potential distribution resulting from a uniform generation of ions throughout the plasma (plane, cylindrical or spherical) and also for the more usual case where the rate of generation at each point is proportional to n_e the concentration of electrons at that point. It is intended to publish these results soon. The way in which the distribution of ion generation in the plasma affects the conditions in the sheath is by altering the average velocity of the ions which enter the sheath. The equations that we derived for the double sheath, such as Eq. (66), are applicable if we choose the proper value for the ratio E_p/E_e. For the case of ion generation proportional to n_e and an infinite plane collector the effective value of this ratio is 0.751 for very high intensity of ionization, decreasing somewhat as the intensity of ionization decreases but probably never falling as low as 0.5 which according to our discussion of Eq. (72) is the limit at which only periodic plasma solutions are possible.

For most experimental conditions we are thus justified in placing $E_p = 0.7E_e$ in Eq. (66).

Experimental Data on Double Sheaths

In determining the space potential by the hot probe method,[15] the volt-ampere characteristic of a small filament is determined first when the filament is cold and then again when heated to a temperature at which electrons are emitted. Only when the filament is below the space potential do the electrons escape. Thus the voltage at which the two curves separate is the space potential. At lower potentials the difference between corresponding ordinates measures the electron current from the probe.

In using this method in ionized gases it was soon found, when the probe potential was lowered below space potential, that the electron current did not rise abruptly to the saturation emission corresponding to the probe temperature, but often a transition region of several volts was observed. Clearly in this range the current is limited by space charge in spite of the fact that an abundant supply of positive ions is present in the ionized gas. The double sheath theory should apply in such cases.

Experiment 559. Similar limitation of current by space charges in double sheaths is observed in hot cathode tubes containing gas if the current to the anode is lowered below its saturation value by using a large resistance in series with the anode. For example, a spherical bulb 12.5 cm in diameter contained a tungsten filament 1 cm long and 0.18 mm diameter, a disk shaped anode 2.2 cm diameter, and a mica-backed collector 1.1 cm in diameter. The filament was heated to a temperature at which it gave a saturation current of 22.5 ma. The volt-ampere characteristics of the anode were measured when the bulb contained mercury vapor saturated at 20°C, while simultaneous readings were made of the positive ion current flowing to the disk collector, this being kept at −80 volts with respect to the anode. Until the anode potential was raised to about 12 volts above that of the cathode, negative charges on the walls of the tube prevented any appreciable current from flowing. At 15 volts, however, the current rose to about 1 ma and as the voltage was increased to 25 the current increased only slowly to 1.9 ma. Electrons of 25 volts energy have an ionizing power 3.3 times as great as those of 15 volts. This small increase in current in spite of the considerable rise of voltage is probably due to the fact that the sheaths on the walls are relatively thick. As the ion production increases the sheaths become thinner so that the area of the sheath boundary, (which collects the ions that pass to the walls) increases and prevents any large rise of ion concentration. The small ion currents to the collector, ranging from 5 to 8 microamperes, indicate that the sheaths must have been several cm thick.

At voltages above 27 volts the behavior is quite different; as the current is raised from 2 to 22.5 ma the voltage rises slowly from 27 to a sharp maximum 33 at 15 ma and falls to 28 at 22 ma. As saturation is approached the voltage can be raised to 60 or more without increasing the current above 22.5.

In the range of relatively constant voltage (27–33) the collector current rose from 8 to 80 microamperes while the electron current changed from 2

[15] I. Langmuir, J. Franklin Inst. **196**, 754 (1923).

to 22. With such low intensity of ionization the collector currents do not measure accurately the ion current density since the sheath on the collector is many mm thick.

Similarly the diameter of the double sheath on the cathode must have been 10 to 30 times that of the cathode itself. Rough calculations based on the space charge theory of sheaths indicate that the electron current from the cathode, at the higher currents (20 ma) was about 2000 times as great as the positive ion current that flowed to the cathode and at lower currents the ratio may have increased to 4000:1.

In this range the electron current from the cathode is evidently limited by the rate of arrival of positive ions at the cathode sheath. If the total current is fixed by an external resistance, the voltage drop in the sheath adjusts itself to a value which will give just the required number of ions. Since the number of ions formed in the plasma is proportional to the current and increases rapidly with voltage, we have an explanation of the relatively constant voltage.[16]

Tube with Oxide Coated Cylindrical Cathode

Conditions more nearly approaching those of the theory we have developed are to be found in discharges using relatively large oxide coated cathodes. The following data were obtained by F. B. Vogdes. The cathode was a nickel cylinder of 0.63 cm diameter, 2.5 cm length closed at one end and heated by radiation from an inner tungsten helix. The outer cylindrical surface was coated with barium oxide. This cathode was mounted in a well exhausted spherical bulb 10 cm diameter provided with an appendix containing mercury. A collector disk 0.6 cm diameter was placed 2 cm opposite the central part of the cathode, the axis of the cathode lying in the plane of the disk. Shortly after starting to operate the tube (appendix at 12°C) with an anode current of 4 amperes, the arc drop was 50 volts and the current depended greatly on the cathode heating current. Thus the current was limited by the cathode emission. The positive ion current density given by the collector was 14 ma per cm^2 and the electron temperature T_e was 140000°. The ratio I_e/I_p was thus 57.

After operation for some time, the activity of the cathode suddenly increased, the arc drop fell to 17 volts, and T_e became 38,000. On the following day the characteristics were steady and reproducible and the data given in the accompanying table were taken. The arc current was now independent of the cathode heating current and adjusted to 4 amperes by means of a resistance in series with the anode.

[16] Approximate calculations of the ratio of electron current to ion current for cylindrical sheaths indicate that this ratio should increase from $(m_p/m_e)^{1/2}$ for very thin sheaths up to 2.25 $(m_p/m_e)^{1/2}$ for sheaths of large diameter. However, this theory neglects the fact that for sheaths of large diameter many of the ions may escape from the sheath after describing orbits about the cathode. Thus the ratio i_e/i_p may increase to values much greater than that just given. In the case of mercury vapor, values of this ratio as high as 2000 or 4000 are therefore consistent with the theory.

Appendix Temp.	Pressure baryes	Arc drop	T_e	I_e ma·cm^{-2}	I_p ma·cm^{-2}	I_e/I_p obs.	I_e/I_p cal.	Ratio $\frac{\text{obs.}}{\text{cal.}}$
19.7°	1.5	16.0	24000	800	5.6	143.	372	0.385
8.8	0.55	20.2	35000	800	6.0	133.	350	0.380

The effect of lowering the vapor pressure of the mercury in a ratio of about 3:1 was to increase the electron temperatures and the arc drop but the ratio of the observed electron to ion currents remained nearly constant at about 140:1. In the last column of the table are given the ratios calculated by the double sheath theory according to Eq. (66) placing $E_p/E_e = 0.7$; $E_e = T_e/11600$, $E_c = 0.1$ and V_{SM} = arc drop + 0.5.

We see that the small variation of the observed value of I_e/I_p from 143 to 133 is in accord with the effect to be expected from the increased electron temperature. However, the observed ratio is only 38 percent of that given by our theory. Such a difference may probably be explained by a lack of uniformity in the emission from the oxide coated surface.

Experiment 560. In this experiment the intensity of ionization and the voltage drop in the sheath could be varied independently of one another. Two cathodes were used, one to produce the ionization while the volt-ampere characteristics of the second were determined. The spherical bulb having a mercury appendix was of 12.5 cm diameter and contained a cathode K of tungsten wire 0.25 mm diameter wound as an open helix, and an oxide coated cathode C like that used before: a nickel cylinder 2.5 cm long and 0.63 cm diameter heated by an internal tungsten coil. There was also a collector disk B 1.1 cm diameter, backed by mica and a disk shaped anode A 2.2 cm diameter. The collector was so placed in the bulb that primary electrons reflected from the sheath on the bulb were not focussed upon it, for if this precaution is not taken serious errors may sometimes result, especially at low pressures of mercury vapor.

In exhausting the tube all metal electrodes were heated to bright red by high frequency induction and the bulb was baked at about 400°C. During the whole experiment the tube remained connected to a glass condensation pump through a liquid air trap, the readings of a McLeod gauge showing pressures of non-condensible gases less than 10^{-5} barye.

A typical set of data obtained with this tube containing saturated mercury vapor at 20°C is illustrated in Fig. 3. In this case the tungsten cathode K was maintained at -50 volts with respect to the anode (taken as zero potential) and heated to a temperature at which it emitted 60 ma. of electrons. The ionization produced by these 50 volt electrons was such as to give 0.127 ma of ion current to the collector B (kept at -75 volts). This corresponds to $I_p = 65 \times 10^{-6}$ amps. cm^{-2} after correction for the edge effect.[17]

[17] Reference 3, p. 540.

Curve I represents the observed volt-ampere characteristic of the oxide-cathode C when the heating current in the tungsten coil was 5 amperes,—the cathode then being at a temperature too low to emit an appreciable electron current. The rise in current as the voltage approaches zero is due to electrons collected from the plasma. At positive voltages C becomes anode and the current is then limited by the emission from K. By plotting the ratio[18] of the electron currents to C and to the anode A (the sum being approximately 60 ma.) on semi-logarithmic paper against the voltage of C, a straight line was obtained extending over a range of 6 volts, in which the ratio increased 10^4 fold. From the slope of this line, the temperature of the plasma electrons was found to be 8500° which gives $E_e = 0.733$ volt.

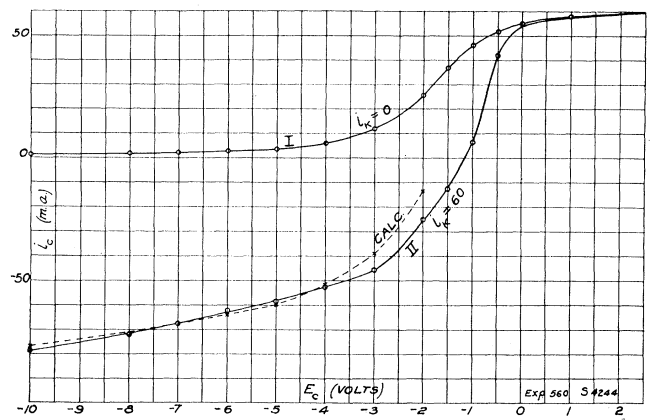

Fig. 3. Volt-ampere characteristics of oxide coated cylindrical cathode; E_c is voltage of the cathode C with respect to the anode A; $E_k = -50$; $i_k = 60$ ma (disregard "$i_k = 0$" on curve I).

When the current through the tungsten coil was raised to 9.5 amperes, the saturation emission from C, calculated by Richardson's equation from measurements of saturation currents at lower temperatures, increased to 2.2 amperes. We see, however, from the observed volt-ampere Curve II, Fig. 3, that the electron currents were less than 80 ma. These currents remained unchanged if the heating current was lowered to 8.0 amperes (calculated emission 180 ma.) but the electron current became saturated at -14 ma. with a heating current of 7.0 amps. Thus the currents of Curve II are limited by space charge.

For positive voltages the two curves coincide since no electrons can escape. The difference between corresponding ordinates for negative voltages thus represents the current of thermionic electrons that leaves the cathode, which we shall call i_e. As the voltage is lowered below 0 volts, a few electrons escape because of their initial velocities which correspond to about $E_c' = 0.1$ volt. The point of inflection at about $E_c = -0.5$ volt thus lies close to the space potential. This would seem to indicate a positive anode drop

[18] Langmuir and Jones, Phys. Rev. **31**, 396 (1928).

of 0.5 volt. However, because of the contact potential of the oxide on the cathode the effective cathode potential is probably a volt or so more positive than indicated by the value of E_c.

The values of i_e obtained by subtracting the ordinates of Curve II from those of Curve I are tabulated in Column 6 of Table III. A corresponding set of data obtained with an ionizing current $i_k=20$ ma and a heating current of 9.5 amperes for the oxide cathode C is given in Column 3. We see that although these currents are independent of the temperature of C they are nearly proportional to the current from K showing that the electron current is determined by the supply of ions to the cathode.

Let us first examine these data from the point of view of Eq. (66), that is, let us see whether the variation of i_e with E_c can be explained as the effect of the velocities of the plasma electrons and ions. Placing $E_e=0.733$ volt

TABLE III. *Variation of electron current with cathode potential.*
Mercury vapor sat. at 20° $E_k=-60$ Emission of C 2.2 amps.

1	2	3	4	5	6	7	8
		$i_k=20$ ma; $I_p=0.025$ ma cm^{-2}			$i_k=60$ ma; $I_p=0.061$ ma cm^{-2}		
E_c	I_e/I_p	i_e obs.	i_e cal.	x cm	i_e obs.	i_e cal.	x cm
−3	310	22.4	20.7	0.040	57.7	50.9	0.026
−4	347	23.9	23.7	0.048	58.9	57.8	0.031
−5	375	25.5	26.2	0.056	62.2	63.4	0.036
−6	393	27.0	27.9	0.063	65.2	67.1	0.040
−8	420	31.5	30.9	0.076	73.8	73.5	0.048
−10	437	34.0	33.0	0.087	79.9	78.1	0.056

from the data of Curve I, $E_p=0.7E_e=0.513$ volt and $E_c'=0.1$ volt, we calculate from Eq. (66) the values of the ratio I_e/I_p given in Column 2. Dividing the observed values of i_e by this factor we should obtain the positive ion current to the cathode. However, although I_p in the plasma may be assumed to be independent of E_c (since $-E_c$ is less than the ionizing potential) the positive ion current will vary slightly with E_c because of the changes in the sheath thickness. The fifth and eighth columns give the sheath thickness in cm calculated by the space charge equation making allowance for the initial velocities of the ions.[19] The values of I_p used in these calculations, according to a method which will be described later, were taken to be 0.025 and 0.061 ma cm^{-2} for $i_k=20$ and 60 ma respectively.

From the diameter of the sheath $(0.635+2x)$ the collecting area of the cylindrical sheath was found (length 2.54 cm) and thus a positive ion current density I_p' was calculated from each value of i_e. The averages of these, excluding only those at $E_c=-3$, gave 0.0117 and 0.0299 ma cm^{-2} for $i_k=$ 20 and 60 ma. These multiplied by the sheath areas and the factor I_e/I_p of Column 2, gave the values of i_e (calc.) in Columns 4 and 7. The dotted

[19] Calculated by Eqs. (10) and (11), Reference 3, p. 452. The positive ion "temperature" which enters only as a small correction, was taken to be 5000° and the voltage in the sheath $-E_c$.

curve in Fig. 3 represents the values in Column 7. A comparison of these observed and calculated values (which involve only one empirically determined parameter I_p') shows that the observed variation of i_e with E_c is in satisfactory agreement with Eq. (66) and is thus explained by the velocities of the plasma electrons and ions. Only about 1/4 of the observed variation is due to changes in the sheath thickness.

The slight differences in the shape of the curves obtained from the calculated and the observed values of i_e are probably to be explained;—1. By uncertainties in the contact potentials which would displace the curves horizontally a volt or so;—2. By changes in the distribution of ion currents in in the plasma, resulting from the space charges of the escaping electrons. If the cathode is made more negative than -10 volts ionization is produced by the emitted electrons which may cause considerable departures from the theoretical curve; for this reason we have confined ourselves in this discussion to the region above -10 volts.

Since in deriving Eq. (66) we have assumed that plasma electrons in appreciable numbers are not able to reach the cathode we must not expect this equation to apply when, according to Curve I, any large electron current passes to this electrode. Thus we should not expect agreement between the observed and calculated values of i_e at voltages much higher than -4. We see in fact that considerable differences are observed at -3 volts.

TABLE IV. *Relation between electron and ion currents.*
Hg at 20°C $E_k = -50$ $E_B = -85$ $E_c = -8$
Currents in milliamps.

Hg Temp.	1 i_k	2 i_e	3 i_B	4 I_p	5 i_p	6 x	7 I_p
20°	10	16.7	0.0376	0.0063	0.0923	0.1316	0.0129
	20	28.6	0.0591	0.0168	0.1580	0.0994	0.0237
	30	38.0	0.0765	0.0269	0.2100	0.0854	0.0327
	40	47.9	0.0940	0.0379	0.2646	0.0755	0.0422
	50	57.7	0.1098	0.0484	0.3188	0.0685	0.0517
	60	66.5	0.1270	0.0604	[0.3674]	0.0636	[0.0604]
Exponent		*0.77*	*0.67*	*1.17*	*0.77*		*0.86*
10°	60	63.9	0.117	0.0534	0.328	0.0676	[0.0534]
Exponent		*0.80*	*0.67*		*0.80*		*0.88*

Studies were also made of the variation of i_e with ionizing current i_k, of the relation of i_e to the ion current density I_p and the effect of varying the mercury vapor pressure. For these purposes E_c was kept constant at -8 volts, and, while i_k was varied from 10 to 60 ma measurements were made of i_e and of i_B, the ion current to the collector B, this being at voltages ranging from -70 to -120. The cathode C was heated to a temperature at which it emitted 2.2 amps. Table IV gives data obtained in this way.

Plotting i_e against i_k on double logarithmic paper gives a straight line of slope 0.77 showing that the electron current increases in proportion to the 0.77 power of the ionizing current i_k. The ion current to the collector i_B, plotted similarly gives approximately a straight line of slope 0.67.

Before we can test the proportionality between i_e and i_p, the ion current to the cathode, which is to be expected by our double sheath theory, corrections must be made for the effects of the thickness of the sheaths on C and B. The theory of these edge corrections[17] indicates that $i_B^{1/2}$ should be a linear function of $\nu^{1/2}$, the slope S of this line being

$$S=0.00306g(m_e/m)^{1/4}\mathrm{amp}^{1/2}\mathrm{volt}^{-3/4} \tag{78}$$

where g is a numerical factor approximately equal to unity and which is dependent solely on the geometry of the collector and where

$$\nu=V^{3/2}\{1+0.0247(T/V)^{1/2}\} \tag{79}$$

is the $V^{3/2}$ of the space charge equation multiplied by a correction factor which allows for the effects of the initial velocities of the ions as they enter the sheath. Numerous experiments in which collectors like B have been used for ions in mercury vapor have given $g=0.98$ which corresponds by Eq. (78) to $S=1.21\times10^{-4}$. By this theory, to correct i_B for the edge effects, we subtract $S\nu^{1/2}$ from $i_B^{1/2}$ and thus obtain $i_0^{1/2}$ where i_0 is the corrected current; —that is, the current that would flow if no edge correction were needed. Dividing i_0 by the actual area of the collector gives I_p the ion current density. The figures in Column 4 were obtained in this way. Putting $V=85$ volts, $T=5000$ for the ion temperature, gives $\nu=932$ and $S\nu^{1/2}=3.694\times10^{-3}$ amps$^{1/2}$, so that if i_B is expressed in microamps we subtract 3.694 from $i_B^{1/2}$ to obtain $i_0^{1/2}$, i_0 also being in microamps. Dividing i_0 by 0.95 cm^2, the collector surface, gives I_p.

For the lower currents at least, these values are certainly over-corrected. We realize this if we consider that the sheath thickness for $i_k=10$ ma calculated from the value of I_p in Column 4 is 0.7 cm as compared to a collector diameter of 1.1 cm. We also find by a double logarithmic plot that I_p is proportional to the 1.17 power of i_k, whereas numerous experiments (unpublished) with guard-ring collectors, that need no corrections, have shown that I_p, as observed by a collector mounted not far from the center of a spherical bulb, varies with a power of the ionizing current that ranges from 0.85 to 0.90.[20]

[20] In mercury arcs, (Reference 3, p. 764) I_p varies in proportion to the 1.25 power of the arc current indicating appreciable ionization of excited atoms. Experiments (Langmuir and Jones, Phys. Rev. **31**, 403 (1928)) with ionization by 30–100 volt electrons in which the *total* ion currents were measured using the walls of the cylinder enclosing the discharge as collector have shown that I_p varies with the 1.0 power of the ionizing current even when this varies in a ratio of more than 1:100. The reason for exponents less than unity is undoubtedly to be sought in a variation of the plasma fields which causes the distribution of ions between a central collector and the glass walls to vary with the intensity of the discharge. The fact that the best

Let us see whether the experimental data of Columns 1, 2 and 3 are consistent with the assumption that the electron current from the cathode is proportional to i_p, the ion current that flows to the cathode. The thickness of the double sheath x according to the last line of Table I, is 1.364 times that calculated by the ordinary space charge equation. The current density used for this calculation should be that at a surface half way between the cathode and the outer edge of the sheath, so that the effective area of the cathode sheath is $\pi(0.635+x)$ 2.54 cm². Thus from Eqs. (1) and (79) taking $T=5000$, $V=8$ and expressing i_p in ma we get

$$x^2=0.00212(0.635+x)/i_p. \tag{80}$$

The collecting area for ions, however, is $\pi(0.635+2x)$ 2.54 cm² and therefore

$$i_p=7.98(0.635+2x)I_p. \tag{81}$$

For the highest current, $i_k=60$, we may assume that the value of I_p in Column 4 is approximately correct. With this value of I_p, Eqs. (80) and (81) solved as simultaneous equations, give $x=0.0636$ cm and $i_p=0.3674$ ma. We obtain the other values of i_p in Column 5 by taking them proportional to i_e in Column 2, and from these the sheath thickness x (Column 6) is found by Eq. (80). Equation (81) then enables us to calculate I_p as given in Column 7. These values of I_p give a straight logarithmic plot against i_k of slope 0.86 which agrees with the exponents 0.85 to 0.90 found in the experiments with guard-ring collectors, and therefore our experimental data are consistent with the double sheath theory that requires a proportionality between i_e and i_p.

The ratio i_e/i_p given by these data is 181:1. By Table III at this cathode voltage (-8) we should have the ratio 420:1 so the observed electron current is 0.431 of that calculated, only a little greater than the ratio 0.38 we found from the data supplied by Mr. Vogdes. Undoubtedly part at least of this discrepancy is accounted for by lack of uniformity of the oxide coating and perhaps the lower temperature of the ends of the cathode cylinder.

The last two lines of Table IV give data obtained with mercury vapor saturated at 10° at which the pressure is only 0.40 that at 20°. The exponents were found from logarithmic plots of data with values of i_k ranging from 10 to 60. The ratio i_e/i_p is here 195, somewhat closer to the theoretical value 420.

Filamentary Cathodes

A large number of experiments (Experiment 562–a) have been made with two tungsten cathodes 0.25 mm in diameter, one (K) to produce the ionization of the mercury vapor and the other (C) to give the volt-ampere characteristics. The values of I_p were determined simultaneously using a guard-

data (with guard rings) give a straight double logarithmic plot, is entirely inconsistent with recombination of ions and electrons as a cause of the low value of the exponent. Recombination would introduce a term varying with I_p^2 which cannot be reconciled with the observed data.

ring collector. With C cold the positive ion currents i_p to C could be measured as a function of voltage. Then with C hot enough to emit a surplus of electrons the electron current was measured. The ratio i_e/i_p depended on I_p and on the voltage, E_c, but was independent of the pressure of mercury vapor (10 to 20°C) and of the voltage E_k of the ionizing electrons. For $E_c = -10$ volts the following values were obtained.

I_p	i_e/i_p
0.140 ma cm^{-2}	940
0.38	530
0.57	450

A more detailed analysis of the curves obtained in these experiments will be reserved for a subsequent paper.

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