EFFECT OF THE EARTH'S MAGNETIC AND ELECTRIC FIELDS ON ION PATHS IN THE UPPER ATMOSPHERE

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Abstract

It is shown that the lines of force of the earth's magnetic field can be treated as if rotating with the earth in so far as the calculation of ion paths are concerned only if there exist positive charges over the poles accompanied by negative charges over the equator. The earth would have a total charge of −72 coulombs although observers on the earth would be aware of no electric field. Assuming that the earth is an uncharged, conducting, uniformly magnetized sphere, rotating about its magnetic axis with angular velocity Ω, it is shown that ion paths progress to the west, the velocity of progression increasing with altitude so as to approach the limiting value $-\boldsymbol{\Omega}\times\boldsymbol{r}$ which measures the progression that would exist if the earth's field were solely magnetic relative to observers who do not partake of the rotation. The earth would have an *apparent* charge of +72 coulombs although actually uncharged. It is shown that a uniformly distributed charge q on the earth merely changes the value of the limiting westward velocity found above, increasing the westward progression if q is positive, and decreasing it if q is negative.

In a recent paper[1] it was shown that the effect of a constant electrical or gravitational force $\boldsymbol{F}$ on ions passing through a constant magnetic field $\boldsymbol{H}$ is (1) to cause the circular or helical path of an ion about the magnetic lines of force to advance with the constant velocity

$$\boldsymbol{u}=\frac{c}{e}\,\frac{\boldsymbol{F}\times\boldsymbol{H}}{H^2} \tag{1}$$

in a direction at right angles to both $\boldsymbol{F}$ and $\boldsymbol{H}$, and (2) to change the radius of the circular or helical path from

$$\frac{mc}{eH}v_0 \quad\text{to}\quad \frac{mc}{eH}(v_0{}^2-2\boldsymbol{u}\cdot\boldsymbol{v}_0+u^2)^{1/2} \tag{2}$$

where $\boldsymbol{v}_0$ represents the component of the initial velocity perpendicular to $\boldsymbol{H}$.

The effect of the earth's gravitational field being small, we shall consider in the present paper only the effect of the earth's electric field in addition to its magnetic field. Then $\boldsymbol{F}=e\boldsymbol{E}$, and Eq. (1) becomes

$$\boldsymbol{u}=c\,\frac{\boldsymbol{E}\times\boldsymbol{H}}{H^2}=\frac{\boldsymbol{s}}{H^2} \tag{3}$$

where $\boldsymbol{s}$ is the Poynting flux, the velocity of progression at right angles to the fields being independent of the charge or mass of the ion as well as of its initial velocity.

[1] L. Page, Phys. Rev. **33**, 553 (1929).

The object of the present paper is to investigate three related problems, as follows:

(1) The intense electric field existing at the surface of the earth probably extends to only a relatively low altitude. Is it theoretically possible then, that, in the upper atmosphere, the lines of magnetic force of the earth's field may be treated as if they rotate with the earth in so far as the calculation of ion paths is concerned? More precisely, under what conditions could the earth's field in the upper atmosphere, as determined by observers stationed on the earth itself, be solely magnetic, unaccompanied by any electric field?

(2) Assuming the earth to be an uncharged, conducting, uniformly magnetized sphere, rotating about its magnetic axis, we shall investigate its electric and magnetic fields and their effects on ion paths in the upper atmosphere. The effects of the collisions of ions with one another or with neutral molecules will be ignored.

(3) The modification of the theory of case (2) necessary to take account of a charge q on the surface of the earth will be considered.

In the course of this investigation we shall have frequent occasion to make use of the transformations of $\boldsymbol{E}$ and $\boldsymbol{H}$ between the inertial system S of the center of the earth and the inertial system S' of an observer stationed on the earth and partaking of its rotation. If we neglect squares of the ratio of the relative velocity to the velocity of light these transformations[2] take the form

$$\boldsymbol{E}'=\boldsymbol{E}+[\boldsymbol{v}\times\boldsymbol{H}]/c, \tag{4}$$

$$\boldsymbol{H}'=\boldsymbol{H}-[\boldsymbol{v}\times\boldsymbol{E}]/c, \tag{5}$$

where $\boldsymbol{v}$ is the velocity of S' relative to S, that is to say, the peripheral velocity of the earth at the point considered due to its rotation.

We shall designate the vector angular velocity of the earth by $\boldsymbol{\Omega}$ and the parallel magnetic moment by $\boldsymbol{M}$. Assuming H to be 0.5 gauss at the equator

$$M=-4.7(10)^{26}\ \text{H.L.U.}$$

the negative sign indicating that M has the opposite sense to Ω, and the uniform intensity of magnetization I to which we shall attribute the magnetic properties of the earth is

$$I=-0.42\ \text{H.L.U.}$$

Inside the earth the magnetic field has the constant value

$$\boldsymbol{H}=2\boldsymbol{M}/4\pi a^3=2\boldsymbol{I}/3, \tag{6}$$

where a is the radius of the earth and $\boldsymbol{H}$ in this formula represents the mean field strength or magnetic induction. Outside the earth the field is that due to a dipole of moment $\boldsymbol{M}$ placed at its center, that is,

[2] L. Page, Introduction to Electrodynamics, p. 24.

$$\boldsymbol{H}=\frac{1}{4\pi}\left\{3\frac{\boldsymbol{M}\cdot\boldsymbol{r}}{r^5}\boldsymbol{r}-\frac{\boldsymbol{M}}{r^3}\right\}, \tag{7}$$

where $\boldsymbol{r}$ is the position vector of the point under consideration referred to the center of the earth as origin.

1. *Can the Earth's field in the upper atmosphere be solely magnetic relative to an observer stationed on the earth?*

Let us assume that an observer stationed on the earth finds no electric field in the upper atmosphere. Then $\boldsymbol{E}'=0$ and Eq. (4) gives

$$\boldsymbol{E}=-[v\times\boldsymbol{H}]/c.$$

We shall use spherical coordinates, representing the polar angle or co-latitude by θ and the radius vector by r. Using Eq. (7) and the relation

$$v=\boldsymbol{\Omega}\times\boldsymbol{r}$$

we find for the components of $\boldsymbol{E}$ in the directions of increasing r and θ respectively

$$E_r=(M\Omega/4\pi cr^2)\sin^2\theta, \tag{8}$$

$$E_\theta=-(2M\Omega/4\pi cr^2)\sin\theta\cos\theta.$$

Therefore the electric field in S possesses the potential

$$V=(M\Omega/4\pi cr)\sin^2\theta. \tag{9}$$

If due to a surface charge, this field would require a charge per unit area equal to E_r, and in all events a total charge on the earth equal to the outward flux of $\boldsymbol{E}$, namely

$$q=2M\Omega/3c. \tag{10}$$

Hence we reach the surprising conclusion that if observers on the earth had found no evidence of an electric field the correct conclusion to draw from their observations would be that the earth possessed the charge given by (10). If we make use of the values of M and Ω appropriate to the earth

$$q=-0.76(10)^{12}\text{ H.L.U.}=-72\text{ coulombs},$$

which is a small part of the negative charge actually found on the earth.

The potential (9), however, does not satisfy Laplace's equation. Therefore, for the field under consideration to exist, we must have a volume density of charge outside the earth equal to

$$\rho=-\nabla\cdot\nabla V=(2M\Omega/4\pi cr^3)(1-3\cos^2\theta). \tag{11}$$

Since the factor in parentheses is a surface harmonic the total charge between two spheres of radii r and $r+dr$ is zero, the volume charge outside the earth consisting merely of a separation of positive and negative electricity, the positive charges congregating over the poles and the negative over the equator. The charge per unit volume over the pole would amount to

$$\rho=1.4(10)^{-15}\,\text{H.L. U.}=3.9(10)^{-16}\,\text{e.s.u.}$$

which would require only one singly ionized particle per cubic meter. At the equator the magnitude of ρ would be half as great.

2. *Ion paths in the field of an uncharged, conducting, uniformly magnetized sphere rotating about its magnetic axis.*

The field of a conducting, uniformly magnetized sphere rotating about its magnetic axis has been investigated by Swann[3] and Tate.[4] The Ampèrian molecular circuits responsible for the magnetization give rise to an electrical polarization

$$\boldsymbol{P}=[(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{I}]/c$$

in the inertial system S on account of their motion relative to this system. This polarization is equivalent to a uniform volume charge

$$\rho=-\nabla\cdot\boldsymbol{P}=-2\Omega I/c$$

and a surface charge

$$\sigma=P_r=(I\Omega/c)r\sin^2\theta.$$

However, as we are dealing with a conducting sphere these charges will be neutralized by the flow of free electrons and need not be considered further.

Now in order that the free electrons carried around by the rotating earth may be in equilibrium they must be subject to the electric field

$$\boldsymbol{E}=-[(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{H}]/c$$

relative to S, where $\boldsymbol{H}$ is the magnetic induction in the interior of the earth given by (6). Therefore we require an electric field in the interior of the earth whose components in the directions of increasing r and θ are respectively

$$E_r=-\frac{2M\Omega}{4\pi ca^2}\,\frac{r}{a}\sin^2\theta,$$

$$E_\theta=-\frac{2M\Omega}{4\pi ca^2}\,\frac{r}{a}\sin\theta\cos\theta.$$

This field possesses the potential

$$V_i=\frac{M\Omega}{4\pi ca}\left\{\frac{r^2}{a^2}\sin^2\theta+C\right\}$$

and requires the constant volume distribution of charge

$$\rho=-\nabla\cdot\nabla V_i=-M\Omega/\pi ca^3$$

[3] Swann, Phys. Rev. **15**, 365 (1920). The sign appears to be wrong in equation 17.
[4] Tate, Bull. Nat. Res. Council **4**, 75 (1922).

in addition to that necessary to neutralize the polarization referred to above. The total volume charge is therefore

$$q_v = 4\pi a^3 \rho/3 = -4M\Omega/3c.$$

The potential V_0 outside the earth must be a solution of Laplace's equation in inverse powers of r agreeing with V_i for $r=a$. It is evidently

$$V_0 = \frac{M\Omega}{12\pi ca}\left\{\frac{a^3}{r^3}(1-3\cos^2\theta)+\frac{a}{r}(3C+2)\right\}.$$

The surface charge is

$$\sigma = -(\partial V_0/\partial r)_a + (\partial V_i/\partial r)_a,$$

which gives

$$\sigma = (M\Omega/12\pi ca^2)(11-15\cos^2\theta+3C).$$

Therefore the total surface charge is

$$q_s = \int \sigma 2\pi a^2 \sin\theta d\theta = M\Omega(2+C)/c.$$

In order that the earth may be uncharged as a whole the surface charge q_s must be equal and opposite to the volume charge q_v. Hence the constant C has the value $-2/3$ and the potential outside the surface becomes

$$V_0 = \frac{M\Omega}{12\pi ca}\frac{a^3}{r^3}(1-3\cos^2\theta), \tag{12}$$

giving rise to a field whose components in the directions of increasing r and θ are

$$\begin{aligned} E_r &= \frac{M\Omega}{4\pi ca^2}\frac{a^4}{r^4}(1-3\cos^2\theta), \\ E_\theta &= -\frac{2M\Omega}{4\pi ca^2}\frac{a^4}{r^4}\sin\theta\cos\theta. \end{aligned} \tag{13}$$

This is just the field that would be produced by an electric quadruplet at the center of the earth, the moment of its moment being

$$M\Omega a^2/3c.$$

The next step is to find the fields relative to an observer stationed on the earth. Using the transformations (4) and (5) as applied to (7) and (13) and neglecting second order terms in the expressions for the components of $\boldsymbol{H}$, we have

$$\begin{aligned} E_r' &= \frac{M\Omega}{4\pi ca^2}\left\{-\frac{a^2}{r^2}(1-\cos^2\theta)+\frac{a^4}{r^4}(1-3\cos^2\theta)\right\}, \\ E_\theta' &= \frac{M\Omega}{4\pi ca^2}\left\{2\left(\frac{a^2}{r^2}-\frac{a^4}{r^4}\right)\sin\theta\cos\theta\right\}; \end{aligned} \tag{14}$$

$$H_r' = \frac{2M}{4\pi r^3} \cos\theta, \qquad H_\theta' = \frac{M}{4\pi r^3} \sin\theta. \tag{15}$$

We notice that at the surface of the earth E_θ' vanishes although E_r' does not, save at the equator. Therefore only at the equator is the field purely magnetic relative to an observer stationed on the earth, and there only at sea level.

An observer on the earth would conclude that the earth carried a charge q' equal to the flux of E_r' through the surface, that is

$$q' = -2M\Omega/3c = 72 \text{ coulombs}.$$

This *apparent* charge is equal in magnitude but opposite in sign to the charge which would actually exist in the case (1) considered previously.

The velocity of progression of ion paths relative to an observer on the earth is

$$\begin{aligned} \boldsymbol{u}_1' &= c\boldsymbol{E}'\times\boldsymbol{H}'/H'^2 \\ &= -\boldsymbol{\Omega}\times\boldsymbol{r}\left\{1-\frac{a^2}{r^2}\frac{1+\cos^2\theta}{1+3\cos^2\theta}\right\}. \end{aligned} \tag{16}$$

This westward velocity is less for all latitudes and all altitudes than the velocity $\boldsymbol{v} = -\boldsymbol{\Omega}\times\boldsymbol{r}$ which would exist if the earth's field were solely magnetic relative to S as was assumed in the hypothetical case considered in the previous paper,[1] but approaches $-\boldsymbol{\Omega}\times\boldsymbol{r}$ as a limit as the altitude increases. The drift is zero at the poles at all altitudes and zero at the equator at sea level. Everywhere else it is to the west, increasing in magnitude at any specified latitude as the altitude increases. Table I gives the drift u_1', the limiting velocity v and the ratio of u_1' to u for various latitudes both at sea level ($r=a$) and at an altitude equal to the radius of the earth ($r=2a$).

TABLE I.

Latitude	$r=a$: $-u_1'$	$r=a$: $-v$	$r=a$: u_1'/v	$r=2a$: $-u_1'$	$r=2a$: $-v$	$r=2a$: u_1'/v
±90°	0	0		0	0	
±75°	0.59(10)⁴cm/sec	1.20(10)⁴cm/sec	0.49	2.09(10)⁴cm/sec	2.40(10)⁴cm/sec	0.87
±60°	1.07	2.32	0.46	4.01	4.64	0.87
±45°	1.31	3.38	0.39	5.74	6.76	0.85
±30°	1.15	4.02	0.29	6.60	8.04	0.82
±15°	0.50	4.48	0.11	6.97	8.96	0.78
Equator	0	4.64	0	6.97	9.28	0.75

At sea level the maximum westward drift occurs at latitude 42.6° and at the altitude a at 11.5°.

The significance of Eq. (16) can be illustrated graphically to the best advantage if we write the equation in the form

$$u_1' = \left(-4.64(10)^4 \text{cm/sec} \right) \sin\theta \left\{ \frac{r}{a} - \frac{a}{r} \frac{1+\cos^2\theta}{1+3\cos^2\theta} \right\}$$

and plot u_1' against r/a for a number of different latitudes. For any specified latitude this equation represents a hyperbola, the asymptotes being the u_1' axis and the straight line

$$u_1' = (-4.64(10)^4 \text{ cm/sec}) \sin\theta(r/a)$$

which represents the limiting velocity v for the latitude $\pi/2-\theta$ and the altitude $r-a$. In Fig. 1 are drawn curves for latitudes 0°, 30°, 60°, the ordinates representing the positive quantity $-u'_1$ and the abscissas values of r/a. The asymptotes are indicated by broken lines.

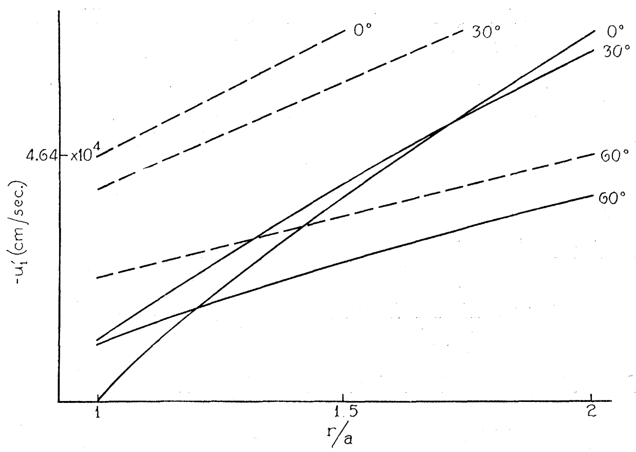

FIG. 1.

We might expect to obtain a closer approximation to the actual fields existing above the surface of the earth by assuming the sphere of conductivity to have a radius b somewhat greater than the radius a of the sphere of uniform magnetization, that is, the radius of the solid portion of the earth. For ions in the lower atmosphere, on account of their short mean free paths, will drift mainly in such directions as to annul any electrical field which may exist relative to S'. Therefore the lower atmosphere will exhibit ordinary conductivity such as has been ascribed to the interior of the earth. The large electric field existing close to the surface can probably be ignored since it can be roughly likened to the field between the plates of a spherical condenser and has no effect on the regions outside.

The fields in the lower atmosphere, then, will be of the type discussed in section 1 of this paper. The upper atmosphere, however, on account of the long mean free paths of the ions, does not exhibit ordinary conductivity since the drift is mainly at right angles to the electric field. In this region the fields would be expected to be of the type which have been previously considered in the present section.

We have, then, three regions to consider: (1) the interior of the earth ($r<a$), (2) the lower conducting atmosphere ($a<r<b$), and (3) the upper atmosphere which is not conducting in the usual sense ($b<r$).

In the first region we have

$$V_1=\frac{M\Omega}{4\pi ca}\left\{\frac{r^2}{a^2}\sin^2\theta-\frac{2}{3}\frac{a}{b}\right\},\qquad \rho_1=-\frac{M\Omega}{\pi ca^3},$$

which differs from the potential V_i previously found only in that the value of the constant C necessary to make the total charge vanish is slightly changed.

In the second region

$$V_2=\frac{M\Omega}{4\pi ca}\left\{\frac{a}{r}\sin^2\theta-\frac{2}{3}\frac{a}{b}\right\},\qquad \rho_2=\frac{2M\Omega}{4\pi cr^3}(1-3\cos^2\theta),$$

which differs from (9) only in the addition of a constant. The surface charge on the spherical surface separating the first and second regions is easily found to be

$$\sigma_{12}=(3M\Omega/4\pi ca^2)\sin^2\theta.$$

Finally in the region of the upper atmosphere

$$V_3=\frac{M\Omega}{12\pi cb}\frac{b^3}{r^3}(1-3\cos^2\theta),\qquad \rho_3=0,$$

and the surface charge on the sphere of radius b is

$$\sigma_{23}=-(M\Omega/2\pi cb^2)\cos^2\theta\ .$$

In so far as the fields in the upper atmosphere are concerned, the only effect of the present refinement is to change the effective radius of the earth from a to b. Therefore all the conclusions reached in the earlier part of this section are valid provided a is interpreted as the radius of the sphere of ordinary conductivity, a radius only slightly greater than the radius of the solid portion of the earth.

A further refinement would take account of the effect on the earth's magnetic field of the diamagnetic action of the ions in the upper atmosphere, such as has been considered by Gunn.[5] This effect is, however, probably small and will not be taken into account here.

3. *Effect on ion paths of a charge on the surface of the earth.*

If a charge q is distributed uniformly over the surface of the earth the field in the interior investigated in the preceding section will remain unaltered, the radial component of the exterior electric field relative to both S and S' being increased by the amount

$$q/4\pi r^2.$$

[5] Gunn, Phys. Rev. **32,** 133 (1928).

Neglecting squares of the ratio of Ωr to c, this added electric field gives rise to an added drift

$$\boldsymbol{u}_2' = \boldsymbol{\Omega} \times \boldsymbol{r} \frac{(qc/M\Omega)}{1+3\cos^2\theta}, \tag{17}$$

the total drift being the sum of the drift $\boldsymbol{u}_1'$ of Eq. (16) and the drift $\boldsymbol{u}_2'$ of (17). The direction of the drift $\boldsymbol{u}_2'$ depends upon the sign of q, being westward for a positive charge (since M is negative) and eastward for a negative charge. As the actual charge on the earth is negative the added drift due to its charge is in the opposite sense to that calculated in section 2 of this paper. For any specified latitude the ratio of u'_2 to $|\boldsymbol{\Omega} \times \boldsymbol{r}.|$ is independent of the altitude; we may therefore consider that the effect of the charge is to change the limiting velocity of the last section from $-\boldsymbol{\Omega} \times \boldsymbol{r}$ to

$$-\boldsymbol{\Omega} \times \boldsymbol{r} \left\{ 1 - \frac{(qc/M\Omega)}{1+3\cos^2\theta} \right\},$$

the added term being greatest in absolute value at the equator and zero at the poles. It may be noted that q cannot be given such a value as to eliminate the resultant electric field at all points and the drift relative to the earth produced by it.

While the limiting velocity of drift is changed by the presence of a charge on the surface of the earth the difference between the limiting velocity and the actual velocity remains unaltered. These differences, therefore, may be taken directly from the curves in the figure which have been drawn for the case where $q = 0$.

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