Second Series *May, 1929* *Vol. 33, No. 5*

THE

PHYSICAL REVIEW

COMPTON MODIFIED LINE STRUCTURE AND ITS RELATION TO THE ELECTRON THEORY OF SOLID BODIES

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Abstract

A tube especially designed for the study of the Compton effect at large angles of scattering.—The structure of the Compton line obtained with this tube for a scattering angle of nearly 180° with a metallic beryllium scatterer is shown.

Interpretation of the structure of the Compton line.—The diffuse structure of the Compton line is here attributed to a broadening caused by the velocity distribution of the scattering electrons in the solid scatterer analagous to a Doppler broadening and a relation between line structure and velocity distribution is derived. The observed line structure from the beryllium scatterer is compared with theoretical structures computed on several alternative assumptions as to electron velocity distribution. It is assumed:—1. That electrons in the solid scattering substance have the velocity distribution required by a wave-mechanical atom model for a free atom of that substance far removed from neighbors. 2. That electrons may be divided into two classes, one class the metallic or conductive electrons in the state of a degenerate electron gas subject to the Pauli Exclusion Principle and having the velocity distribution derived by Sommerfeld, and the other class as in the 1st assumption unperturbed by the neighboring atoms. 3. That electrons may be divided into two classes as before but that those forming an electron gas have the classical velocity distribution required by the Maxwell-Boltzmann equipartition law. 4. That electrons have the velocity distribution required by the older Bohr-Sommerfeld atom model with point electrons executing Kepler orbits and as in the first assumption unperturbed by neighboring atoms.

The distribution of electron velocities in metals.—The results strongly contradict the classical distribution of electron velocities in solid bodies predicted by the rigid interpretation of the Maxwell-Boltzmann equipartition principle. They are also in contradiction with the older Bohr-Sommerfeld atom model. The results are in accord with the wave-mechanical atom model and constitute favorable evidence for the Sommerfeld distribution of metallic electron velocities and for the degenerate gas state.

A PRELIMINARY report of an experimental study of the structure of Compton line made by the author has recently appeared.[1] The present paper is intended as an interpretation of some of the results of this work. The experimental procedure was not previously described, however, and we therefore here give a brief account of it also.

[1] J. W. Du Mond, Proceedings Natl. Acad., **14**, 875–878 (1928).

The purpose of this investigation has been to study the diffuse structure of the Compton shifted line and to correlate this structure with the internal momenta associated with the atoms of the scattering substance. We have chosen the region near 180° as the most appropriate scattering angle for three reasons. At this angle the inhomogeneity of shift due to the unavoidable inhomogeneity of scattering angle is reduced to a very small minimum. This angle also gives the maximum shift and the maximum breadth of Compton line structure. For this angle double scattering if any be present affects the shift and structure of the line to the smallest extent. As scattering substances aluminum and beryllium have been studied. These were chosen as convenient for use in vacuum inside the x-ray tube described below and as representing two extremes in the atomic number region which could conveniently be studied with Mo K radiation.

Experimental Procedure

Figure 1 shows a cross section of the special tube used.[2] The anticathode is of the copper water cooled type with molybdenum button manufactured by the General Electric Company. The cylindrical box surrounding the anticathode contains the scatterer and the small Seeman type spectrograph.

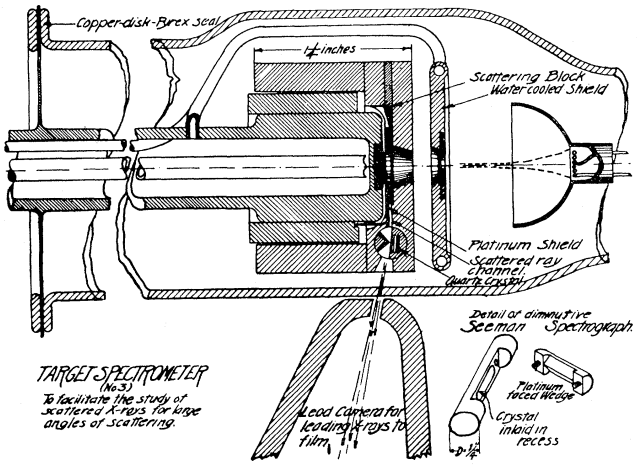

Fig. 1.

Great care was taken in the internal design of this box to insure that the radiation analyzed by the spectrograph came only from the scattering body. Four such tubes were built before satisfactory results were obtained. The water-cooled shield between the box and the cathode was found necessary to prevent stray electron impacts from overheating the box. Before enclosing the anticathode and its box in the x-ray tube the quartz crystal and

[2] J. W. Du Mond, Nature, **116**, 937 (1925).

wedge in the small cylindrical container were carefully oriented to such an angle that the region of the spectrum studied came exactly from the center of the scattering substance. This was done by placing two small sheets of lead across the opening from which the scattering body had previously been removed so as to leave a narrow slit at the center of the opening. Molybdenum radiation from an auxiliary tube passed through this slit diametrically across the cylindrical box and fell on the quartz crystal. This latter was oriented and clamped so that the desired spectral region containing the Mo K spectrum and extending from about 615 X.U. to 800 X.U. was reflected from the crystal. This entire region only occupies a breadth of about 0.4 mm in the scattering block.

In spite of the proximity of scatterer and source it was not found possible to work satisfactorily with very short exposures as we at first hoped. Exposures of eight hours were sufficient for beryllium but in the case of aluminum fifty hours or more were found necessary. It is not possible to use more than 10 milliamperes at about 50 k.v. in this tube as the slightest overheating causes the molybdenum anticathode to vaporize. The molybdenum vapor sublimes on the scatterer forming a layer opaque to the radiation. The high resolution and dispersion and the short length of crystal and wedge also account in part for the relatively long exposures which we found necessary.

The spectra were photographed at the end of a long lead camera. Eastman duplitized film was used without any intensifying screen. The lead walls of the camera completely enclosed the film.

The experimental curve shown in Fig. 2 at I was obtained by a microphotometric analysis of the photographic spectra. The optical system of the microphotometer outfit used permitted the exploration of film blackening over a strip only about 6 mm high. The films however had spectral lines about 25 mm high. On account of film grain it was necessary to take several runs with the microphotometer across the spectral lines in different regions of their height. The curves so obtained were then numerically averaged to reduce the accidental fluctuations of the film grain. The beryllium curve is the average of three microphotometer curves taken on one film. In the case of beryllium two other films were obtained both being in agreement[3] with the one here shown. The lines appear superposed on a rather heavy smooth continuous background. For this reason the relation between x-ray intensity and microphotometric deflection is closely linear over the small range represented by the line intensities. To test this, calibration films were made using Mo $K\alpha$ radiation with equal steps of intensity produced by means of an accurately cut exposure disk mounted on a motor shaft. The disk exposed the film in different regions for different intervals of time varying from one

[3] These two other exposures were of longer duration than the one whose intensity curve is reproduced here. They were rather dense and had suffered a slight shifting during the exposure also some slight halation was suspected. They showed a modified line structure slightly broader than the one here reproduced.

tenth to full exposure in tenth steps. It was found that over the same range of microphotometric deflection as that of the films of the experiment, the scale was sensibly linear.

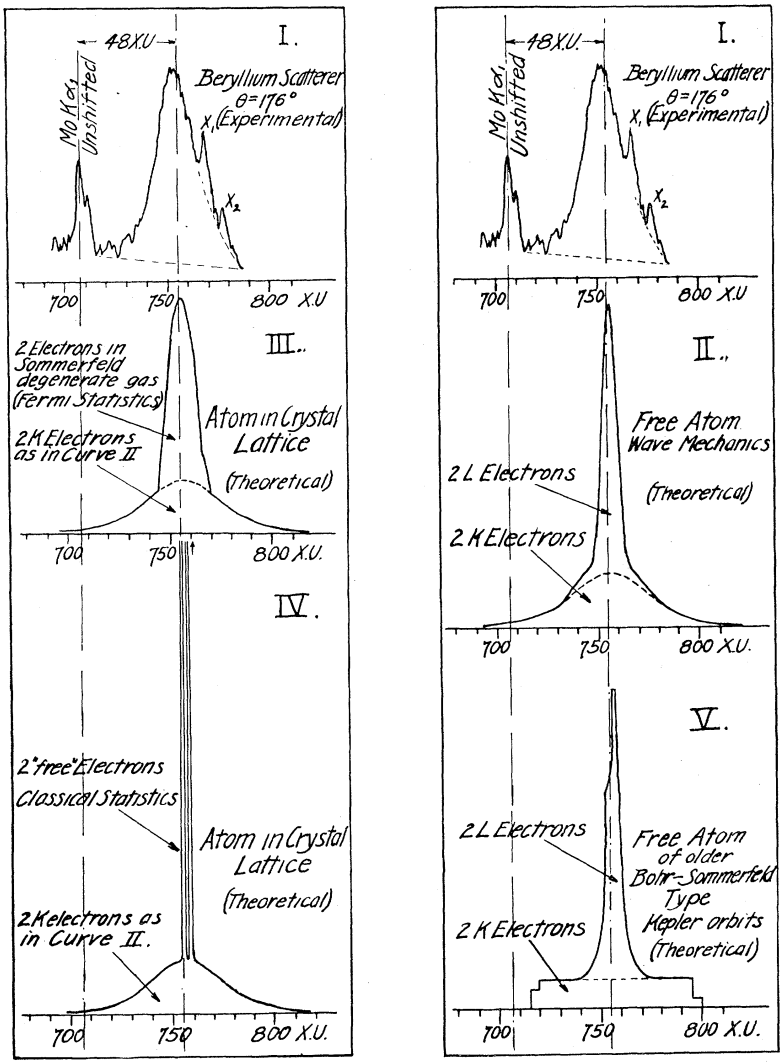

Fig. 2.

The small peaks X_1 and X_2 in Fig. 2 are not yet definitely accounted for. They are perhaps fluorescence lines due to impurities in the beryllium scattering block, perhaps strontium. If they correspond to Smekal transitions (such as those recently observed by Bergen Davis, Phys. Rev., p. 331, Sept. 1928) one would be obliged to suppose energy level differences in

beryllium of much greater magnitude than we can yet explain. The separation of X_1 and X_2 is 8 X.U. about double the separation of the $K\alpha$ doublet of molybdenum. These lines appeared clearly in all the exposures made with the beryllium scatterer.

The Relation Between Compton Line Structure and Electron Velocity Distribution

The structure of the modified line is a key to the distribution of velocity of the electrons which scatter the modified radiation, the line being broadened by the random motion of these electrons in a way similar to that of the Doppler broadening of optical lines emitted by moving atoms. It is shown by means of certain approximative assumptions in an appendix to this paper that an ensemble of electrons all moving at one speed,[4] v, in random directions will modify initially monochromatic radiation by scattering so as to give (as a first approximation) a spectral distribution consisting of a shifted rectangular band with a flat top and vertical discontinuous limits or edges, see Fig. 3*A*. The spectral width of the band $\Delta\lambda$ is proportional to the speed, v, of the electrons. This band is nearly symmetrically distributed about the shifted position corresponding to free stationary electrons. The approximation involves an error of the order of β^2/unity (though β therefore need not be negligible compared to unity). In particular when the radiation is scattered at 180° or thereabouts the width due to such an ensemble is shown

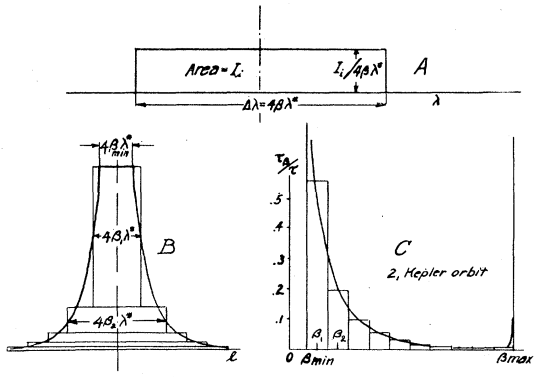

Fig. 3.

$$\Delta\lambda = 4\beta\lambda^* \tag{1}$$

where $\lambda^* = \lambda + h/mc$ and $\beta = v/c$, the speed of the electrons divided by the speed of light. (The theory presented here is completely non-commital as to the *mechanism* of this so called Doppler broadening and employs only the principles of conservation of energy and momentum. It is not therefore an attempt to explain the Compton effect on classical principles.) The area of the rectangular spectral band, Fig. 3*A*, is proportional to the number of electrons in the ensemble and to the time during which they are exposed to the radiation if we assume that all electrons have the same *a-priori*

[4] Throughout this paper the word "velocity" will be used to indicate a vector quantity while the word "speed" will be used to indicate the absolute value of the velocity.

probability of scattering (see footnote 12). We shall therefore choose an area I_i corresponding to the area of the modified band produced by one ensemble of electrons, i having a standard population of say one electron per atom and corresponding to the total time of the exposure. The height of the corresponding rectangle is then given by $I_i/4\beta\lambda^*$. The choice of I_i establishes the scale of ordinates (intensities) and is so chosen as to normalize the final total computed curve with respect to the area under the experimentally observed curve.

In order to compute the shifted distribution produced by all ensembles of electrons an assumption as to the relative *a-priori* probabilities of scattering by electrons in the different ensembles or classes must be made. We here assume these *a-priori* probabilities the same for all electrons.

With these considerations and assumptions we can compute a modified distribution for the scattered radiation from any atom model. For example, for a Bohr hydrogen-like model, having elliptical orbits account can be taken of the variable velocity in the Kepler motion by a study of the proportionate times τ_β/τ spent by the electron in different speed ranges β to $\beta+d\beta$ between the maximum and minimum values $\beta_{\max}$, $\beta_{\min}$ (perihelion and aphelion), where τ is the orbital period. The resulting distribution will be formed of infinitesimal elementary rectangles as shown in Fig. 3*B*. The widths of rectangles are made proportional to the speeds and their areas are given by $(\tau_\beta/\tau)I_i$ so as to make the area of the total curved distribution equal to I_i. The height of each rectangle is given by $(\tau_\beta/\tau)I_i/4\beta\lambda^*$.[5]

The curve representing the shape or structure of the Compton modified line may then be obtained by assuming finite speed ranges and calculating finite rectangles to form a staircase distribution whose discontinuities are finally smoothed out by drawing a smooth curve as shown in Fig. 3*B*. Where practicable, however, it is better to obtain the curve by the following analytical method.

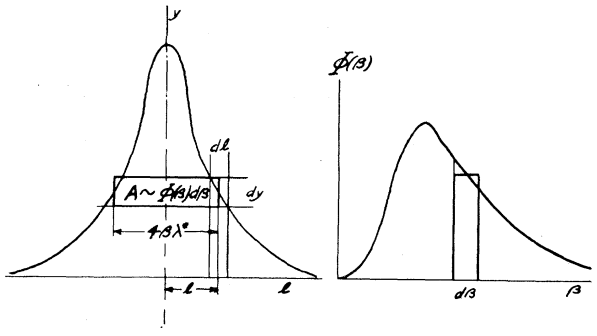

Fig. 4.

Referring to Fig. 4, a function $\Phi(\beta)$ is supposed given which expresses the probability of encountering an electron with speed between β and $\beta+d\beta$ as a function of β. The electron ensemble having this speed β and

[6] This method of treatment is slightly different in form but identical in substance to the theory developed by G. E. M. Jauncey (Phys. Rev. **25**, 314–322 (1925) 723–736 (1925)) for computing line structures due to electrons in Kepler orbits. It is developed in the form here given to permit of extension to wave-mechanical atom models. The author wishes explicitly to acknowledge his indebtedness to Dr. G. E. M. Jauncey.

randomly distributed velocity orientations contributes to the total line structure the elementary rectangle A whose area is proportional to $\Phi(\beta)d\beta$ and whose width is given as mentioned above by $4\beta\lambda^*$. The area of this rectangle is therefore

$$-2ldy = k\Phi(\beta)d\beta$$

where k is a constant determining the scale of y, and $l=\lambda'-\lambda-2h/mc$ is the abscissa of the structure curve measured from its median point. $l=2\beta\lambda^*$ and $\beta=l/2\lambda^*$ we can thus replace $d\beta$ by $dl/2\lambda^*$ and $\Phi(\beta)$ by $\Phi(l/2\lambda^*)$.

The differential equation of the curve is given therefore by

$$-2ldy = k\Phi(l/2\lambda^*)dl/2\lambda^*$$

Dividing by $-2l$ and integrating this from $y=0, l=\infty$ to $y=y, l=l$ we obtain the equation of the line structure curve for continuous functions which vanish as $l\rightarrow\infty$

$$y = -k\int_{l=\infty}^{l=l}\Phi(l/2\lambda^*)dl/l. \tag{2}$$

Formula (2) permits us either to start with an observed line structure and to deduce from this the electron velocity distribution function $\Phi(\beta)$ or to start with possible assumed velocity distributions $\Phi(\beta)$ and compute ideal line structures for comparison with the observed structure. The line structures so far determined are not sufficiently well defined to warrant the first mentioned procedure but many interesting conclusions can be drawn by following the second.

General Method of Computing and Normalizing Line Structure Curves

Each electron class (K, L, M, etc.) is assumed to contribute to the total line structure independently. Separate line structure curves are computed for each class of electrons. The ordinate scales are so chosen as to make the area under each of these component curves proportional to the number of electrons responsible for that curve. These curves are then added to obtain the total line structure.

The primary radiation is not truly monochromatic but consists of a doublet ($K\alpha_1$, $K\alpha_2$). Mo $K\alpha_2$ is 4 X.U. longer in wave-length and half as intense as Mo $K\alpha_1$. In order to render the computed structure curves strictly comparable with the observed curves the following procedure was followed in all cases. The total structure curve was computed for a primary wave-length of 708 X.U. (Mo $K\alpha_1$) and the scale of ordinates was so chosen as to make the area under this curve equal to two thirds the area under the *experimentally observed curve*. This gives the contribution due to $K\alpha_1$. To this we add a precisely similar curve shifted however 4 X.U. in the direction of longer wave-lengths and having ordinates half as great as those of the

first curve. This gives the contribution of $K\alpha_2$. The final curve has obviously an area just equal to the experimentally observed curve with which it is to be compared. By following this procedure no arbitrary assumptions need be introduced as to absolute intensities and the ordinate scale is uniquely determined.

The correction for the doublet character of the primary radiation above mentioned introduced a plainly visible asymmetry in all of the computed curves.

Electron Velocity Distributions for Wave Mechanical Atom Models and Derivation of the Corresponding Line Structure Curves

By means of the Dirac transformation theory it is possible to obtain the *momentum* distribution for a wave mechanical atom model given the space distribution (eigenfunction) for the same model. If p_1, p_2, p_3 are the cartesian coordinates in the momentum space and x_1, x_2, x_3 the cartesian coordinates in ordinary space then the probability of encountering a momentum (p_1, p_2, p_3) is given by the square of the absolute value of

$$\phi(p_1,p_2,p_3)=\int\int\int_{-\infty}^{+\infty} e^{-(2\pi i/h)(p_1x_1+p_2x_2+p_3x_3)}\psi(x_1,x_2,x_3)dx_1dx_2dx_3. \qquad (3)$$

The eigenfunctions used are those given by L. Pauling.[6] An effective atomic number Z_i corrected for screening is computed for each class of electrons i and applied in the eigenfunction for that class. The author realizes that this is only a rough approximation which neglects mutual perturbations. An exact solution (if possible) would be very laborious and quite unwarranted for the purposes of this paper.

Equation (3) in polar coordinates becomes, if ψ is independent of θ' and ϕ' and depends on r alone

$$\phi(p)=\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-(2\pi i/h)p r\cos\theta'}\psi(r)r^2\sin\theta' dr d\theta' d\phi' \qquad (4)$$

$\phi(p)$ is generally a complex quantity. The square of its absolute value $|\phi(p)|^2$ gives the density in momentum or the probability of encountering an electron with momentum in the range between p_x, p_y, p_z and p_x+dp_x, p_y+dp_y, p_z+dp_z. The function $\Phi(p)$ which represents the probability of encountering an electron with momentum between $|p|$ and $|p|+|dp|$ is obtained by multiplying $|\phi(p)|^2$ by $4\pi p^2$.

$$\Phi(p)=4\pi p^2\,|\,\phi(p)\,|^2.$$

The resulting functions $\Phi(p)$ can be easily expressed as functions of β or l by the relations

[6] L. Pauling, Proc. Royal Soc. **A114,** 184 (1927).

$$p=mv=m\beta c=mcl/2\lambda^*.$$

The function $\Phi(l)$ thus determined is substituted in the formula (2) for the line structure curve. We tabulate below the electron class, the corresponding eigenfunction, and the resulting line structure curve function. Inessential multiplicative constants have been dropped since the line structure functions are to be subsequently normalized.

(K)	1, 0	$\psi_{1,0}(r)=e^{-\sigma_{1,0}r}$	$y_{1,0}=(1+A_{1,0}{}^2l^2)^{-3}$
(L)	2, 0	$\psi_{2,0}(r)=e^{-\sigma_{2,0}r}[-2+2\sigma_{2,0}r]$	$y_{2,0}=2.5(1+A_{2,0}{}^2l^2)^{-3}-7.5(1+A_{2,0}{}^2l^2)^{-4}$ $+6(1+A_{2,0}{}^2l^2)^{-5}$
(M)	3, 0	$\psi_{3,0}(r)=e^{-\sigma_{3,0}r}[6-12\sigma_{3,0}r+4\sigma_{3,0}{}^2r^2]$	$y_{3,0}=4.701(1+A_{3,0}{}^2l^2)^{-3}$ $-37.612(1+A_{3,0}{}^2l^2)^{-4}$ $+110.328(1+A_{3,0}{}^2l^2)^{-5}$ $-133.731(1+A_{3,0}{}^2l^2)^{-6}$ $+57.313(1+A_{3,0}{}^2l^2)^{-7}$

$$\sigma_{n,l}=Z_{n,l}/na_0, \qquad A_{n,l}=(2\pi a_0 n/Z_{n,l})/(2\lambda^* h/mc)$$

$$\lambda^*=\lambda+h/mc=732\cdot 10^{-11}\text{cm}, \qquad a_0=5\cdot 10^{-9}\text{cm},\ h/mc=24\cdot 10^{-11}\text{cm}$$

The effective atomic numbers applied to these functions are as follows: beryllium, $Z_{1,0}=3.81$, $Z_{2,0}=2$.

Line Structure Curve Due to Sommerfeld Conduction Electrons

By the application of the Fermi-Pauli statistics Sommerfeld has recently shown that the conduction electrons in metallic crystal lattices will have the following velocity distribution:

$$\Phi(v)=kv^2 \quad \text{when} \quad |v|<V \tag{5}$$

$$\Phi(v)=0 \quad \text{when} \quad |v|>V \tag{6}$$

where V is given by

$$V=h/m(3n/4\pi G)^{1/3} \tag{7}$$

n being the number of metallic electrons per cc in the crystal lattice and $G=2$.

Expressing this in terms of l and substituting in the formula (2) for the line structure we obtain omitting a constant coefficient and using the indefinite integral

$$y=-\int l^2\frac{dl}{l}+C \quad \text{for} \quad |l|<2\lambda^* V/C \tag{8}$$

since $y=0$ when

$$l=\pm 2\lambda^* V/C, \quad C=\tfrac{1}{2}(2\lambda^* V/c)^2$$

or

$$y=\tfrac{1}{2}(2\lambda^* V/c)^2-\tfrac{1}{2}l^2 \quad \text{for} \quad |l|<2\lambda^* V/c \tag{9}$$

$$y=0 \quad \text{for} \quad |l|>2\lambda^* V/c \tag{10}$$

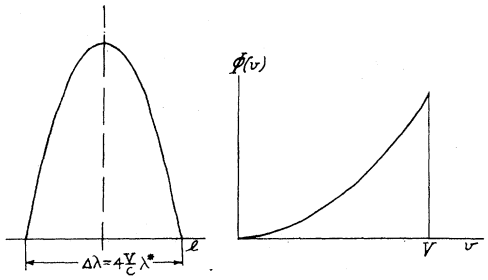

Fig. 5.

This is seen to be a line structure in the form of an inverted parabola of base width $4\lambda^* V/c$. Sommerfeld's electrons have much higher velocities than those required by classical statistics and the result is a much broader line structure. See Fig. 5.

Line Structure Curve Due to Maxwell-Boltzmann Distribution of Conduction Electron Velocities

This is obtained by the same method as the previous curves. In this case

$$\Phi(v) = v^2 e^{-1/2 m v^2/kT} = (cl/2\lambda^*)^2 e^{-m/2(cl/2\lambda^*)^2/kT}.$$

Substituting in the Eq. (2) and integrating

$$y = e^{-m/2kT(c/2\lambda^*)^2 l^2} \tag{11}$$

This is a simple Gaussian error curve which for the temperature of the experiment has a width of only 0.4 X.U. at $1/e$ of maximum value.

Line Structure Curves for Kepler Orbital Velocity Distributions

For the case of circular orbits since β is constant the distribution is evidently a simple rectangle as has been show by G. E. M. Jauncey. Here $\Phi(v)$ degenerates into a simple vertical ordinate at the proper value of v.

For the case of elliptical orbits the function $\Phi(v)$ is of rather complicated form. For this reason the graphical method with finite speed intervals was resorted to. Fig. 3*C* shows a typical curve representing $\Phi(v)$ for a 2_1 orbit. The total speed range between maximum and minimum speed was divided into ten equal steps and the corresponding rectangles were plotted to obtain the component curves whose sum gave the structure curves shown in Fig. 2*V*. Attention is called to the general similarity existing between the structures due to wave mechanics and those due to the older Bohr-Sommerfeld Kepler orbits (see Fig. 2 V and II). The difference as would be expected is that the wave mechanics rounds off the sharp corners of the Kepler curves. *The wave mechanics also permits some slight intensity at very large line breadths.*[7] This is doubtless the explanation for the existence of an unshifted line in cases where Jauncey's theory based on Kepler orbits called for no unshifted line.

Conclusions

It is evident from a comparison of the observed and computed line structures that the assumption of a class of electrons with velocities in agreement with the Maxwell-Boltzmann equipartition of thermal energy is untenable.

[7] The Bohr-Sommerfeld atom would also give a better fit if the azimuthal quantum number k were placed equal to $(l(l+1))^{1/2}$ however.

The two extremely tall and narrow peaks to be expected on this assumption are due to the $K\alpha_1$ and $K\alpha_2$ lines of the primary radiation and should be completely resolved as shown. We have in Curve (IV) Fig. 2 assumed the existence of two electrons per atom with such velocities. Even though this state were but one tenth or one twentieth as populous it would be easily detectable because of the narrowness of the curves called for by electrons with such slow velocities. Such narrow curves or peaks when normalized to give a total area equal to only a small fraction of the area under the observed curve would still have very appreciable ordinates.

The line structure to be expected from an atom with point electrons executing Kepler orbital motions with its resulting angularities and discontinuities is also seen to be discordant with the experimentally observed structure (compare Fig. 2 V and I).

Of the two remaining theoretical Curve III Fig. 2 corresponds to the assumption that the electron momentum is distributed as it would be in a free atom of beryllium and hence neglects the perturbing effect of the close proximity of the atoms in the crystal lattice. Curve II corresponds to the assumption that on account of the proximity of atoms two electrons per atom are not closely associated to any particular atom but constitute a degenerate electron gas while the remainder of the electrons are distributed in momentum as they would be in a free atom of beryllium neglecting the perturbing effect of neighboring atoms. This arbitrary division of electrons into two distinct classes is doubtless only a very rough approximation to the truth. It is highly probable that no sharp boundary divides the electron gas from the bound electrons there being an intermediate state in which electrons execute motions very different from those to be expected in a single free unperturbed atom but motions which nevertheless are to a large extent conditioned by the fields of one or more atoms. Such a class of electrons is neither completely free nor completely bound. It is difficult to take account of this intermediate class of electrons quantitatively[8] but it is easy to see qualitatively the effect on the computed form of the line structure. Attention is called in Curve III Fig. 2 to the sharp breaks at the two points where the parabolic structure due to the electrons of the degenerate gas state meets the broad bell shaped structure due to the bound electrons. These breaks would certainly be absent if it were possible to take account of the continuous nature of the transition between the bound electrons and those in the degenerate gas state.

We now call attention to those parts of Curves II and III near the maximum. It is at once evident that Curve III is blunter and broader than Curve II and that in this respect Curve III is in better accord with the observed line structure. It is precisely in this portion of the curve that we should expect the degenerate gas approximation to give a good representation of the facts.

We may therefore conclude that the experimental curve supports the Sommerfeld theory of metallic electrons as a degenerate gas in just those regions for which this theory is designed to apply.

[8] Bloch. Zeits. f. Physik, Jan. 1929.

It is notable also that Curve III has a maximum ordinate slightly *higher* than that experimentally observed. Were Curve III to be corrected in order to take account of the continuous nature of the transition between the bound electrons and those in the degenerate gas state it would be necessary to *lower* the maximum ordinate in order to maintain a constant area without changing the shape of the peak. This would doubtless improve the agreement of the observed and computed maximum ordinates.

To sum up our conclusions then we may say:

The Maxwell-Boltzmann statistics applied to conduction electrons give results discordant with the observed structure of the Compton line for scattering from beryllium.

The electron velocity distribution of the older Bohr-Sommerfeld atom model gives results discordant with the observed structure of the Compton line.

Velocity distributions based on the wave mechanics of a beryllium atom on the one hand and on the Sommerfeld theory of degenerate electron gas on the other give line structures in accord with the experimentally observed curves in the regions to which these theories are in each case designed to be applicable.

It is interesting to note that the conclusions drawn from this work constitute confirmatory evidence for the Sommerfeld theory of gas degeneration as applied to conduction electrons *in a field of phenomena quite remote from that for which the theory was developed.*

A simple computation shows that the recoil momentum taken by any electron in the case here discussed is sufficient to throw that electron completely out of the range of velocities forbidden by the presence of other electrons according to the Pauli "Verbot."

Work is now under way at this laboratory in an attempt to obtain experimental distribution curves with greater precision. A study of line structure scattered by nonconductors should prove interesting and is now being started. Work is being continued by the double crystal method of Bergen Davis and also by means of a multiple crystal spectrograph recently constructed here which permits the use of converging x-ray beams.

My sincere thanks are due to Professor W. V. Houston for his kindness in acquainting me with the results of the electron theory of metals in the Fermi statistics, and to Professor L. C. Pauling and Mr. Edwin McMillan for the help they have given me in the development and computation of the functions $\Phi(\beta)$ for wave mechanical atom models. I am much indebted to Professor Gregor Wentzel for his discussions and criticism in conversations in Paris and in subsequent letters. I am most grateful, also, to Professor R. A. Millikan for his faith and encouragement in this study which for a long time gave but meager promise of interesting results.

APPENDIX

The explanation of Compton line breadth as a Doppler effect of the motion of bound and conduction electrons is an approximation. It has the advantage of offering a familiar pictorial explanation of the facts but the disadvantage of limited applicability. To treat the problem

rigorously one must set up a wave function for the crystal lattice and then compute the modified radiation by a method similar to that of G. Wentzel.[9] This no one yet has succeeded in doing satisfactorily.

The present approximate theory is applicable to low electron velocities, long primary wave lengths and relatively free electrons or expressed precisely

$$v \ll c, \quad h/mc \ll \lambda, \quad E_B/E_R \ll 1 \tag{12}$$

E_B is the binding energy of the electron, E_R the energy received by the electron in the scattering process:—

$$E_R = h\nu - h\nu' = hc(\lambda' - \lambda)/\lambda\lambda'$$

For any particular level the ratio E_B/E_R is evidently a function of l the abscissa we have used in describing the line structure. It increases in the direction of shorter wave lengths but over the region $\lambda > 730$ X.U. does not exceed 0.2 for any level. For most levels it is very small indeed. The region $\lambda > 730$ X.U. includes all of the interesting portion of the modified line structure.

For any particular level the modified spectral distribution becomes discontinuous to the left of the point where the ratio $E_B/E_R = 1$. In this region the Smekal lines recently observed by Bergen Davis[10] appear. The unshifted line may be considered as a special case of these. In the cases here discussed this region is too close to the unshifted line to be resolved.

The following assumptions which are all that are necessary for the derivations of formulas used in this paper seem justified for the region of approximation defined by the three inequalities (12).

(1). All electrons are assumed to have the same *a-priori* probability of scattering independent of their velocity or quantum state.

For the region defined above the classical scattering formula giving the total scattered intensity proportional to the total number of electrons for a wide variety of atoms is known to hold.[11]

(2). Conservation of momentum and energy is assumed in the interaction of light quanta and electrons taking into account only the momentum and energy of the electron in the atom or in the crystal lattice just before scattering and similar quantities for the scattered quantum and the recoil electron. The modifying effect on the shifted radiation of any momentum or energy transferred or otherwise given to the rest of the atom in the scattering process is neglected.

We first investigate the general case of an initially moving electron which scatters a quantum at an angle of scattering θ. Referring to Fig. 6 let ν_1 = initial frequency of quantum propagated in the direction of positive x-axis the interaction occurring at the origin. Let $\beta_1 c$ be the speed of the electron before interaction a_1, b_1, c_1, the direction cosines of its velocity and $a_1 = \cos\theta_1$ so that θ_1 is the angle between the electron's initial velocity and x-axis. Let the scattered quantum have a frequency ν_2 and a direction of propagation defined by the direction cosines p, q, r making angle ϕ with the initial velocity of the electron and an angle θ with OX. Then

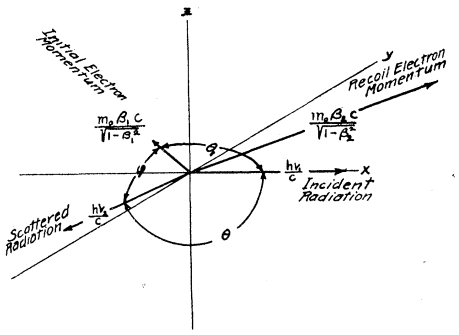

Fig. 6.

$$\cos\phi = (a_1 p + b_1 q + c_1 r)$$

$$p = \cos\theta$$

[9] G. Wentzel, Zeits. f. Physik **43**, 1–8 (1927); **43**, 779–787 (1927).
[10] Bergen Davis, Phys. Rev. **32**, 33 (1928).
[11] Bergen Davis, Phys. Rev. **25**, 737–739 (1925).

Let the recoil electron have a final speed $\beta_2 c$ in the direction defined by the cosines a_2, b_2, c_2. (Cf., de Broglie "Ondes et Mouvements" Fasicule 1 pp. 94–95.) Assumption (1) gives us the four equations.

$$h\nu_1+m_0c^2/(1-\beta_1^2)^{1/2}=h\nu_2+m_0c^2/(1-\beta_2^2)^{1/2} \tag{13}$$

$$h\nu_1/c+(m_0\beta_1c/(1-\beta_1^2)^{1/2}a_1=(h\nu_2/c)p+(m_0\beta_2c/(1-\beta_2^2)^{1/2})a_2 \tag{14}$$

$$(m_0\beta_1c/(1-\beta_1^2)^{1/2})b_1=(h\nu_2/c)q+(m_0\beta_2c/(1-\beta_2^2)^{1/2})b_2 \tag{15}$$

$$(m_0\beta_1c/(1-\beta_1^2)^{1/2})c_1=(h\nu_2/c)r+(m_0\beta_2c/(1-\beta_2^2)^{1/2})c_2 \tag{16}$$

Eliminating a_2, b_2, c_2 and β_2 letting $\alpha=h\nu_1/m_0c^2$ we have

$$\nu_2=\nu_1\frac{1-\beta_1\cos\theta_1}{1-\beta_1\cos\phi+2\alpha(1-\beta_1^2)^{1/2}\sin^2\theta/2}. \tag{17}$$

Substituting $\nu=c/\lambda$ and neglecting β^2 in comparison to unity we have for the shift

$$\lambda_2-\lambda_1=\frac{\beta_1(\cos\theta_1-\cos\phi)}{1-\beta_1\cos\theta_1}\lambda_1+\frac{2\alpha\lambda_1\sin^2\theta/2}{1-\beta_1\cos\theta_1} \tag{18}$$

in which the second term accounts for the simple "Compton Shift" and the first term represents the deviation from this shifted position caused by the electron's initial velocity. If now we substitute

$$l=\lambda_2-\lambda_1-2\alpha\lambda_1\sin^2\theta/2 \tag{19}$$

so that the new wave length coordinate l has for its origin the "center" of the shifted line (position for scattering by free *stationary* electrons) we obtain for the shift away from that new reference point

$$l=\frac{\beta_1\left\{\cos\theta_1\left(\lambda_1+2\alpha\lambda_1\sin^2\frac{\theta}{2}\right)-\cos\phi\lambda_1\right\}}{1-\beta_1\cos\theta_1}.$$

This can be considerably simplified by introducing a notation based on the special case of scattering by an initially stationary electron (Compton case).

Let the shifted wave-length for the Compton case be

$$\lambda_c=\lambda_1+2\alpha\lambda_1\sin^2\theta/2$$

then

$$l=\frac{\beta_1\cos\theta_1\lambda_c-\beta_1\cos\phi\lambda_1}{1-\beta_1\cos\theta_1}. \tag{20a}$$

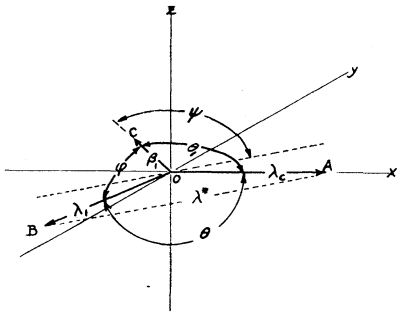

Fig. 7.

In Fig. 7 let OA be the direction of the incident quantum, OB the direction of the scattered quantum, OC the direction of the electron's initial velocity. We make the vector OA equal in length to λ_c, the vector OB equal in length to λ_1 and the vector OC equal in length to β_1. We define a new wave-length

$$2\lambda^*=(\lambda_c^2+\lambda_1^2-2\lambda_c\lambda_1\cos\theta)^{1/2}$$

represented in length by the vector AB.

We now note that the numerator in Eq. (20a) can be represented in terms of the vectors of Fig. 7 (designated by their terminii) as the difference of two scalar products

$$\boldsymbol{C}\cdot\boldsymbol{A}-\boldsymbol{C}\cdot\boldsymbol{B}$$

or

$$\boldsymbol{C}\cdot(\boldsymbol{A}-\boldsymbol{B}).$$

Now the vector whose length is $2\lambda^*$ is precisely

$$(\boldsymbol{A}-\boldsymbol{B}).$$

Hence if ψ is the angle between OC and AB we have the identity

$$\beta_1 \cos\theta_1\lambda_c-\beta_1\cos\phi\lambda_1=2\beta_1\cos\psi\lambda^*$$

so that we can write

$$l=\frac{2\beta_1\cos\psi}{1-\beta_1\cos\theta_1}\lambda^* \tag{20}$$

The direction AB is that of the vector difference between the momentum of the incident quantum and the momentum of the quantum scattered in the special Compton case of an initially stationary electron. It is evident that AB is the appropriate reference axis for the general case of an initially moving electron. The shift l is seen to be nearly proportional to the projection of the electron's initial velocity on this axis AB. It is important to note that this axis is fixed in space and that λ^* is a constant for any given angle of scattering θ and initial wave-length λ_1.

In particular when the angle of scattering $\theta=180°$ or nearly so we have (the case of this experiment)

$$\cos\psi=\cos\theta_1=-\cos\phi$$

$$l=\frac{2\beta_1\cos\theta_1}{1-\beta_1\cos\theta_1}\lambda^* \tag{21}$$

where $\lambda^*=(\lambda+h/mc)$

Holding β_1 the initial speed of the electron constant and varying the directions of the electron's initial velocity defined by $\cos\theta_1$ it is evident from Eq. (21) that

$$-2\beta_1\lambda^*\leqq l\leqq 2\beta_1\lambda^* \text{ when } \beta_1\ll 1 \tag{22}$$

or $-2\beta_1\lambda^*+2\beta_1^2\lambda^*\leqq l\leqq 2\beta_1\lambda^*+2\beta_1^2\lambda^*$ when β_1 is not small compared to 1 but $\beta_1^2\ll 1$. l can therefore vary over the wave-length range given by:—

$$\Delta\lambda=4\beta_1\lambda^* \tag{23}$$

for electrons of constant initial speed β_1 and varying directions θ_1.

LINE STRUCTURE ELEMENT FOR AN ENSEMBLE OF ELECTRONS ALL HAVING SPEED β AND RANDOM VELOCITY ORIENTATIONS

An ensemble of electrons of the type mentioned is represented in velocity space by vectors radiating from the origin and having their terminii distributed with uniform density over the surface of a sphere of radius β with the origin as center.

The probability of an encounter between a quantum and an electron such that the angle between the two trajectories is θ_1 (per unit small angular range $d\theta_1$) is

$$P\theta_1=\tfrac{1}{2}\sin\theta_1. \tag{24}$$

The probability P_l of a given shift, l is obtained from equations (21), (24) and the derivative of (21) by eliminating θ_1 and $dl/d\theta_1$ as

$$P_l=(4\beta\lambda^*)^{-1}(1+l/2\lambda^*)^{-2}.$$

The error introduced by nglecting $l/2\lambda^*$ in comparison to unity is 3 per cent for $l=25$ XU, $\lambda^*=738$ XU and much less for the more important parts of the line structure. Hence

$$P_l=1/4\beta\lambda^*$$

and P_l is thus seen to be independent of l. Hence all shifts in the range permitted by inequalities (22) have the same probability, i.e. the distribution is rectangular.[12] (See Fig. 3 A).

[12] This statement is strictly true if the ordinates of the distribution function or line structure curve are understood to represent the *number of quanta*. If the distribution curve repre-

As indicated by inequalities (22) this rectangle is not quite symmetrically centered about the origin of wave-length abscissa l for very large values of β but this slight correction has been neglected in the present paper.

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February 10, 1929.

Note added in proof, April 8. The author has just received a manuscript from S. Chandrasekhar of India giving an independent theoretical derivation of the parabolic line structure due to the Fermi-Sommerfield electron gas. His results are in accord with those here given.

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sents the *energy* however a slight correction for the variation in the energy per quantum over the breadth of the line is necessary. The computed curves of this paper represent the distribution of number of quanta as a function of wave-length and not the energy. The microphotometer curves on the other hand represent energy rather than number of quanta. No correction was made for this, however, as the discrepancy thereby introduced is much smaller than the experimental uncertainties and in no way affects the conclusions.