THE MOTION OF IONS IN CONSTANT FIELDS

By Leigh Page

Abstract

It is shown that the effect of a constant electrical or gravitational force $\boldsymbol{F}$ on ions passing through a constant magnetic field $\boldsymbol{H}$ is to cause the circular or helical ion paths to advance in a direction at right angles to both $\boldsymbol{F}$ and $\boldsymbol{H}$ with the constant velocity

$$\boldsymbol{u}=c[\boldsymbol{F}\times\boldsymbol{H}]/eH^2.$$

Ion paths relative to a rotating earth are discussed on the assumption that the earth's field is purely magnetic relative to the inertial system of the center of the earth. The essential features of the theory are shown to be unaltered if the constant mass of the classical theory is replaced by the variable mass of the relativity theory.

IN DISCUSSING the motion of ions in a magnetic field it is often stated that the ions describe helices about the lines of force without consideration being given to the effect on the ion paths of other forces, such as those due to electrical or gravitational fields. The equations of motion of an ion in constant electric and magnetic fields have been given by many authors,[1] but usually in such a form that the nature of the path is not immediately evident. While the following discussion contains nothing essentially new beyond the method of treatment employed, the very simple description of the ion paths which is obtained may be of use, particularly to those who are interested in the effect of the earth's magnetic field on ions in the upper atmosphere.

We shall utilize the classical dynamics in our main discussion, showing later that the essential features of the theory are unaltered if the constant mass of the classical theory is replaced by the variable mass of the relativity theory.

Classical Dynamics

The equation of motion of an ion of mass m and charge e subject to a constant electrical or gravitational force $\boldsymbol{F}$ and a constant magnetic field $\boldsymbol{H}$ is, in Heaviside-Lorentz units,

$$mdv/dt=\boldsymbol{F}+e[v\times\boldsymbol{H}]/c \tag{1}$$

relative to the inertial system $x\ y\ z$ in which $\boldsymbol{F}$ and $\boldsymbol{H}$ are measured. The magnetic field has no effect on the component of the motion along the lines of force, the ion being subject only to a uniform acceleration equal to the component of $\boldsymbol{F}$ along the lines of force divided by m in so far as the motion in that direction is concerned. Therefore it is only the motion at right angles to $\boldsymbol{H}$ which requires discussion, and the generality of our results will be in

[1] J. J. Thomson, Cond. of Elect. through Gases, p. 112, Eq. (7), (8). (Thomson seems to use left-handed axes); L. Page, Phys. Rev. **24**, 284, Eq. (1), (2), (3).

no way impaired by assuming *ab initio* that both $\boldsymbol{v}$ and $\boldsymbol{F}$ are perpendicular to $\boldsymbol{H}$ and therefore that the motion is confined to a plane. For the sake of definiteness we shall suppose $\boldsymbol{H}$ to be along the z axis, $\boldsymbol{v}$ and $\boldsymbol{F}$ lying in the $x\ y$ plane.

Now consider a set of axes $X\ Y\ Z$ parallel respectively to $x\ y\ z$ and moving relative to the latter with the constant velocity

$$\boldsymbol{u}=c[\boldsymbol{F}\times\boldsymbol{H}]/eH^2. \tag{2}$$

If we denote the velocity of the ion relative to $X\ Y\ Z$ by $\boldsymbol{V}$ then

$$\boldsymbol{V}=\boldsymbol{v}-\boldsymbol{u}, \quad d\boldsymbol{V}/dt=d\boldsymbol{v}/dt, \tag{3}$$

and the equation of motion (1) becomes

$$md\boldsymbol{V}/dt=e[\boldsymbol{V}\times\boldsymbol{H}]/c \tag{4}$$

when referred to the new axes. By referring the motion to these axes we have eliminated the field $\boldsymbol{F}$. The motion described by (4) is motion with constant speed V in a circle of radius

$$\rho=mcV/eH \tag{5}$$

in the clockwise sense for positive e, the angular velocity being

$$\boldsymbol{\omega}=-e\boldsymbol{H}/mc, \tag{6}$$

which, incidentally, is just twice the Larmor precession obtained by neglecting the centrifugal reaction and taking into account only the Coriolis reaction.

The integrated equations of motion relative to the moving axes are obviously

$$\begin{aligned} X&=X_0+\frac{mc}{eH}V\cos\frac{eH}{mc}(t+\delta), \\ Y&=Y_0-\frac{mc}{eH}V\sin\frac{eH}{mc}(t+\delta), \end{aligned} \tag{7}$$

X_0, Y_0, being the coordinates of the center of the circular orbit. We note that these equations contain four arbitrary constants X_0, Y_0, V, δ, and therefore they constitute the complete solution of the differential equations of motion. If we refer the motion to the axes $x\ y\ z$ we are led at once to the equations referred to in footnote 1.

The simple analysis developed above shows that the effect of a constant electrical or gravitational force $\boldsymbol{F}$ *in an inertial system* $x\ y\ z$ *is*: (1) *to change the radius of the circular path of an ion about the lines of magnetic force from*

$$\frac{mc}{eH}v_0 \quad \text{to} \quad \frac{mc}{eH}(v_0^2-2\boldsymbol{u}\cdot\boldsymbol{v}_0+u^2)^{1/2} \tag{8}$$

where v_o *represents the initial velocity of the ion relative to x y z; and* (2) *to cause the circular path to advance relative to x y z with the constant velocity* $\boldsymbol{u}$ *given by* (2) *in a direction at right angles to both* $\boldsymbol{F}$ *and* $\boldsymbol{H}$. In this latter respect the effect of the force $\boldsymbol{F}$ is analogous to that of a force applied to a gyrostat in that it gives rise to a motion at right angles to itself.

The path of an ion relative to $x\ y\ z$ is therefore a cycloid. According as V is greater than, equal to, or less than u, the cycloid is curtate, common, or prolate. If we denote the initial velocity of the ion relative to $x\ y\ z$ by $\mathbf{v}_o$ and the angle between $\boldsymbol{u}$ and v_o by θ then we have from (3)

$$V^2 = v_0^2 - 2uv_0 \cos\theta + u^2. \tag{9}$$

For a common cycloid $V = u$ and either $v_o = 0$ or

$$\cos\theta = v_0/2u. \tag{10}$$

We can classify all possible cases as follows:
(1) $V=0$. This case exists only when $v_o = u$ and $\theta = 0$. The path is a straight line at right angles to F and $\boldsymbol{H}$, the force due to the magnetic field being just balanced by $\boldsymbol{F}$.

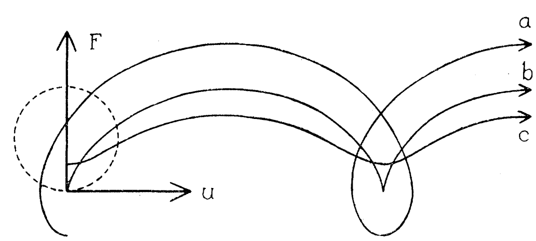

Fig. 1.

(2) $V \neq 0$. We have three subcases, namely:
(a) $v_o = 0$. The path is a common cycloid, the ion starting at the cusp.
(b) $0 < v_o \leqq 2u$. The path is a prolate cycloid, common cycloid or curtate cycloid according as the angle θ is less than, equal to, or greater than that defined by equation (10).
(c) $2u < v_o$. The path is a curtate cycloid for all values of θ.

These three types of path are illustrated in Fig. 1 for positive e, the magnetic field being perpendicular to the paper and directed toward the reader. The radius of the rolling circle

$$r = mcu/eH \tag{11}$$

is independent of the initial velocity v_o. It is determined solely by the fields $\boldsymbol{F}$ and $\boldsymbol{H}$ and by the charge and mass of the ion. The distance of the generating point from the center of the rolling circle, given by (5), depends in addition upon the magnitude and direction of the initial velocity v_o, as is shown by (9). Curve a represents a curtate cycloid, b a common cycloid and c a prolate cycloid, all for the same value of u. The common rolling circle is indicated by a broken line. As is clear from the fact that u is not a function of v_0 the drift velocity of progression perpendicular to $\boldsymbol{F}$ and $\boldsymbol{H}$ is independent of the initial velocities of the ions or the particular type of cycloid described.

If the non-magnetic force is due to an electric field, $\boldsymbol{F}=e\boldsymbol{E}$, and the charge disappears from the equation (2) for the velocity of progression. Therefore positive and negative ions drift in the same direction and with the same speed in an electric field whatever their masses or the magnitudes of their charges may be. The paths of the negative ions are obtained from those of the positive ions depicted in the figure by rotating the diagram through 180° about a horizontal axis.

On the other hand, if the non-magnetic force is due to a gravitational field, $\boldsymbol{F}=m\boldsymbol{g}$ and (2) becomes

$$u=cmg/eH$$

In this case ions of opposite sign drift in opposite directions with speeds inversely proportional to the ratio of e to m. Thus a current is produced, which, in the case of the earth, is from west to east and therefore in such a direction as to tend to demagnetize the earth.

In order to obtain an estimate of the magnitude of the drift which would exist if no collisions took place, let us consider an ion of atomic mass A projected across the lines of force of the earth's magnetic field at the equator and subject to the earth's electric and gravitational fields. For the purpose of calculation assume

$$H=0.5 \text{ gauss}=0.14 \text{ H. L. U.} \quad \text{(Heaviside-Lorentz Units)}$$
$$E=100 \text{ volts/meter}=0.94\ (10)^{-3} \text{ H. L. U.}$$
$$e=4.77\ (10)^{-10} \text{ E. S. U.}=1.69\ (10)^{-9} \text{ H. L. U.}$$
$$m=1.65\ (10)^{-24}\ A \text{ gm}$$

The drift velocity due to the earth's electric field is then

$$u_e=cE/H=2(10)^8 \text{ cm/sec}$$

and that due to the earth's gravitation field is

$$u_g=c\frac{mg}{eH}=0.2A \text{ cm/sec}.$$

For ions projected into the earth's magnetic field near the surface of the earth with velocities of the order of magnitude of $(10)^8$ cm/sec the drift due to gravity is negligible, although that due to the earth's electric field is of the same order of magnitude as the velocity of projection.

Finally we shall consider the relation between the ion paths relative to an inertial system S with origin at the center of the earth and those relative to the inertial system S' of an observer stationed on the earth at the equator. On account of the rotation of the earth S' has an eastward velocity relative to S of $4.7\ (10)^4$ cm/sec.

We shall neglect the small drift due to the earth's gravitational field and furthermore we shall suppose that relative to S the earth's field is purely magnetic, no accompanying electric field being present. Then the ions, if projected with velocities of $(10)^8$ cm/sec relative to S, will describe stationary circles relative to this inertial system of a radius of 200 A meters according to equation (5). Therefore the ion paths will progress to the west with a drift velocity of 4.7 $(10)^4$ cm/sec relative to the observer stationed on the earth at the equator.

How is the observer in S' to account for this progression? The transformations for the electromagnetic field give the answer at once. If $\boldsymbol{v}$ designates the eastward velocity of 4.7 $(10)^4$ cm/sec of S' relative to S, the observer in S' is aware[2] of the electric field (induced electromotive force)

$$\boldsymbol{E}' = k[\boldsymbol{v}\times\boldsymbol{H}]/c$$

in addition to the magnetic field $\boldsymbol{H}' = k\boldsymbol{H}$, where

$$k \equiv (1 - v^2/c^2)^{-1/2}$$

This electric field gives rise to a drift

$$\boldsymbol{u}' = c[e\boldsymbol{E}'\times\boldsymbol{H}']/eH'^2 = -\boldsymbol{v}$$

according to (2). Therefore the analysis of the problem from the point of view of the observer stationed on the earth agrees with that of the observer in the inertial system S.

Relativity Dynamics

If we make use of the variable mass of the relativity theory in place of the constant mass of the classical theory the essential features of the preceding theory remain unchanged. Suppose that we have mutually perpendicular constant fields $\boldsymbol{E}$ and $\boldsymbol{H}$ relative to an inertial system S. We shall orient our axes so that the Y axis is parallel to $\boldsymbol{E}$ and the Z axis to $\boldsymbol{H}$.

Now consider a second initial system S' which has a velocity $\boldsymbol{u}$ along the X axis equal to

$$\boldsymbol{u} = c[\boldsymbol{E}\times\boldsymbol{H}]/H^2 \tag{12}$$

The transformations[2] for $\boldsymbol{E}$ and $\boldsymbol{H}$ give us for the electric field $\boldsymbol{E}'$ and magnetic field $\boldsymbol{H}'$ in S'

$$\begin{aligned} \boldsymbol{E}' &= k\left\{\boldsymbol{E} + \frac{1}{c}\,\boldsymbol{u}\times\boldsymbol{H}\right\} = 0, \\ \boldsymbol{H}' &= k\left\{\boldsymbol{H} - \frac{1}{c}\,\boldsymbol{u}\times\boldsymbol{E}\right\} = k\boldsymbol{H}\left(1 - \frac{E^2}{H^2}\right), \end{aligned} \tag{13}$$

where $k \equiv (1 - u^2/c^2)^{-1/2}$.

[2] L. Page, Introduction to Electrodynamics p. 24.

As in the previous treatment we have eliminated the electric field by passing over to the inertial system S'. The equation of motion relative to S' is

$$\frac{d}{dt'}\left\{\frac{m\mathbf{V}'}{(1-V'^2/c^2)^{1/2}}\right\}=\frac{e}{c}\mathbf{V}'\times\mathbf{H}', \tag{14}$$

where $\mathbf{V}'$ is the velocity of the ion relative to this system. This is the equation of motion in a circle with constant speed V', the radius of the circle and the angular velocity being given by (5) and (6) respectively provided we replace $\mathbf{V}$ by our present $\mathbf{V}'$ and put the transverse mass

$$m(1-V'^2/c^2)^{-1/2}$$

of the relativity theory in place of the constant mass m of the classical theory.

Therefore the relativistic treatment of the problem leads to essentially the same results as that based on the classical dynamics, although it emphasizes the fact that we must confine our conclusions to cases where E is less than H so that u may be less than c.

Note. Dr. Hulburt has kindly drawn the author's attention to the fact that some of the conclusions of this paper have already been recognized by others. In the paper of Ross Gunn[3] the average drift velocity of an ion in the direction at right angles to applied electric and magnetic fields has been calculated and shown to be independent of the initial velocity. The nature of the ion paths, however, has not been investigated.

In Nature, Chapman[4] refers to the "drift acquired by charges under the joint action of the magnetic field, gravity and the vertical electrostatic field" of the earth, and makes the statement that "the drift is westward for the electrons and eastward for positive ions." This statement, while true for the gravitational drift, is not correct for the electrical drift.

Finally Maris and Hulburt[5] state that "ions, no matter what their velocities are, under the combined action of gravity and the earth's magnetic field, move at right angles to these two vectors with a velocity approximately mg/He, the positive and negative ions moving in opposite directions."

SLOANE PHYSICS LABORATORY,
YALE UNIVERSITY,
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[3] Gunn, Phys. Rev. **32**, 135 (1928).
[4] Chapman, Nature **122**, 572 (1928).
[5] Maris and Hulburt, Nature **122**, 807 (1928).