JUNE, 1929 *PHYSICAL REVIEW* *VOLUME 33*

PATHS OF CHARGED PARTICLES IN ELECTRIC AND MAGNETIC FIELDS

By W. Bartky and A. J. Dempster

Abstract

A combination of electric and magnetic fields is described which is suitable for positive-ray analysis experiments, or precise determinations of the ratio of charge to mass. The orbits of charged particles in the radial electric field and transverse magnetic field of the system are computed, and it is shown that a beam of rays of various velocities diverging from a point are brought to a focus provided the variations are not too large. Formulas are given for computing the errors introduced by variations in direction and velocity.

IN DETERMINATIONS of the specific charge of the electron and in methods of positive-ray analysis, the question of bringing a divergent beam of charged particles to a focus is of fundamental importance. Semicircular paths in a constant magnetic field have been used in many experiments.[1] A related problem is the focusing of a beam of charged particles after it has suffered deflections, so that a range of velocities are brought to

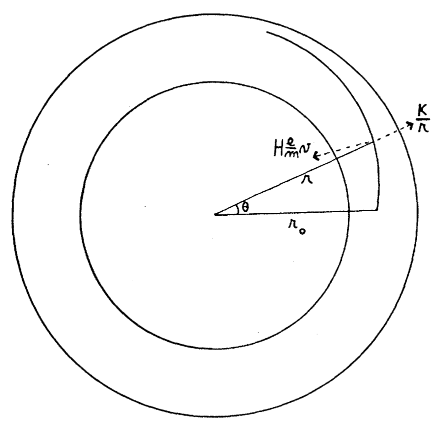

Fig. 1.

the same point. This type of focusing has been developed by Aston in his mass spectrograph.[2] In a previous paper[3] it was suggested by one of the authors that this velocity-focusing might be combined with direction-focusing to obtain greater sharpness than could be obtained by either alone. In this paper one combination of electric and magnetic fields is discussed which seems specially suitable for experimental use.

[1] J. Classen, Phys. Zeits. **9**, 762 (1908); H. Busch, Phys. Zeits. **23**, 438 (1922).
[2] F. W. Aston, Phil. Mag. **38**, 709 (1919).
[3] A. J. Dempster, Phys. Rev. **20**, 638 (1922).

Suppose the charged particles to move in a cylindrical condenser, shown in section in Fig. 1. If a magnetic field H is established perpendicular to the plane of the paper a particle describes an orbit under the influence of a radial acceleration, K/r due to the electric field, and of an acceleration due to the magnetic field which acts at right angles to its direction of motion at any point, and is given by $H(e/m)(ds/dt)$. Let us suppose the electric force to be directed outwards and the magnetic force inwards; the equations of motion, in polar coordinates, are then:

$$\left.\begin{aligned} \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2 &= \frac{K}{r}-H\frac{e}{m}\frac{ds}{dt}\left(r\frac{d\theta}{ds}\right)=\frac{K}{r}-\lambda\frac{rd\theta}{dt} \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= H\frac{e}{m}\frac{ds}{dt}\cdot\frac{dr}{ds}=\lambda\frac{dr}{dt} \end{aligned}\right\} \tag{1}$$

where $\lambda = He/m$. From the second equation

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{\lambda}{2}\frac{dr^2}{dt}, \tag{2}$$

so that $d\theta/dt=\lambda/2+A/r^2$ where $A=r_0^2(\theta_0'-\lambda/2)$
Substituting for $d\theta/dt$ in (1) we obtain

$$\frac{d^2r}{dt^2}=\frac{A^2}{r^3}+\frac{K}{r}-\frac{\lambda^2 r}{4}\cdot \tag{3}$$

We are interested only in orbits which are nearly circular, since only these may be described in the condenser considered.

For the circular orbit $r=\rho$, the magnetic force is always directed along the radius vector, and from (3) $\lambda^2/4-K/\rho^2=A^2/\rho^4$
Substituting $r=\rho+x$ in (3), we obtain

$$d^2x/dt^2=-x(\lambda^2-2K/\rho^2) \tag{4}$$

If the magnetic and electric fields and initial angular velocity θ_0' are adjusted so that $A=0$ or $\lambda^2=4K/\rho^2$, then (4) reduces to

$$\frac{d^2x}{dt^2}=-\frac{\lambda^2}{2}x \tag{5}$$

and $d\theta/dt=\lambda/2$. The solution of (5) is $x=P\sin\lambda t/2^{1/2}$, showing that a divergent bundle of rays is reunited after a time $t_0=2^{1/2}\pi/\lambda$ when the angle described is $\pi/2^{1/2}$, that is 127° 17′.

If we substitute $r=\rho$ in the first equation of (1) and consider ρ as a function of $d\theta/dt$ we find that ρ passes through a minimum for $d\theta/dt=\lambda/2$, so that for the adjustment of the electric and magnetic fields which makes $A=0$ that is $\lambda^2=4K/\rho^2$, we have particles with slightly different velocities describing approximately the same circular orbit.

To find the sharpness of the focus, it is necessary to compute the orbits described by particles with different directions and velocities to a further approximation.

Since

$$\frac{d^2r}{dt^2}=\frac{d}{d\theta}\left[\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}\right]\frac{d\theta}{dt}=\frac{d^2r}{d\theta^2}\left(\frac{d\theta}{dt}\right)^2+\left(\frac{dr}{d\theta}\right)\cdot\frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)\cdot\frac{d\theta}{dt}$$
$$=\left(\frac{\lambda}{2}+\frac{A}{r^2}\right)^2\frac{d^2r}{d\theta^2}-\frac{2A}{r^3}\left(\frac{\lambda}{2}+\frac{A}{r^2}\right)\left(\frac{dr}{d\theta}\right)^2$$

it follows from (3) that

$$\frac{d^2r}{d\theta^2}-\frac{2A}{r^3}\left(\frac{\lambda}{2}+\frac{A}{r^2}\right)^{-1}\left(\frac{dr}{d\theta}\right)^2=\left(\frac{A^2}{r^3}+\frac{K}{r}-\frac{\lambda^2 r}{4}\right)\left(\frac{\lambda}{2}+\frac{A}{r^2}\right)^{-2} \tag{6}$$

This equation may be solved for the special case in which A is small and the orbits are nearly circular by putting

$$A=\mu a \qquad \text{and} \qquad r=\rho(1+\mu z) \tag{7}$$

where μ is a parameter, and where ρ satisfies

$$\frac{K}{\rho}-\frac{\lambda^2}{4}\rho=0\,; \qquad \rho^2=\frac{4K}{\lambda^2}\,.$$

For on substituting (7) into (6) and neglecting powers of μ greater than the second we find the following equation for z,

$$\frac{d^2z}{d\theta^2}=\left[\frac{\mu a^2}{\rho}-\frac{K}{\rho^2}(2z-\mu z^2)\right]\left[\frac{\lambda}{2}+\frac{\mu a}{\rho^2}\right]^{-2}=-2z+\mu z^2+8\epsilon\mu z+4\epsilon^2\mu \tag{8}$$

when $\epsilon=a/\lambda\rho^2$. Assume the solution of (8) to be of the form

$$z=z^{(0)}+\mu z^{(1)}$$

then

$$\begin{aligned} d^2z^{(0)}/d\theta^2&=-2z^{(0)}\\ d^2z^{(1)}/d\theta^2&=-2z^{(1)}+z^{(0)^2}+8\epsilon z^{(0)}+4\epsilon^2. \end{aligned} \tag{9}$$

With the initial conditions

$$z\Big|_{\theta=0}=c_0 \qquad \frac{dz}{d\theta}\Big|_{\theta=0}=c_1$$

the solution of the first equation is

$$z^{(0)}=c_0\cos 2^{1/2}\theta+(c_1/2^{1/2})\sin 2^{1/2}\theta.$$

Substituting this in the second equation of (9) we obtain

$$\frac{d^2z^{(1)}}{d\theta^2}=-2z^{(1)}+\left(\frac{c_0^2}{2}+\frac{c^2}{4}+4\epsilon^2\right)+\left(\frac{c_0^2}{2}-\frac{c_1^2}{4}\right)\cos 2^{3/2}\theta+\frac{c_0c_1}{2^{1/2}}\sin 2^{3/2}\theta$$
$$+8\epsilon\left(c_0\cos 2^{1/2}\theta+\frac{c_1}{2^{1/2}}\sin 2^{1/2}\theta\right)$$

The solution of the last equation subject to the conditions

$$z^{(1)}\Big|_{\theta=0}=0 \qquad \frac{dz^{(1)}}{d\theta}\Big|_{\theta=0}=0 \text{ is;}$$

$$z^{(1)}=\left(\frac{c_0^2}{4}+\frac{c_1^2}{8}+2\epsilon^2\right)-\frac{1}{12}\left(c_0^2-\frac{c_1^2}{2}\right)\cos 2^{3/2}\theta-\frac{c_0c_1}{6\cdot 2^{1/2}}\sin 2^{3/2}\theta$$

$$+\frac{4\epsilon\theta}{2^{1/2}}\left(c_0\sin 2^{1/2}\theta-\frac{c_1}{2^{1/2}}\cos 2^{1/2}\theta\right)-\left(\frac{c_0^2}{6}+\frac{c_1^2}{6}+2\epsilon^2\right)\cos 2^{1/2}\theta$$

$$+\frac{1}{2^{1/2}}\left(\frac{c_0c_1}{3}+\frac{4\epsilon c_1}{2^{1/2}}\right)\sin 2^{1/2}\theta.$$

For $\theta=\pi/2^{1/2}$, $z^{(0)}$ and $z^{(1)}$ reduce to

$$z^{(0)}=-c_0,\quad z^{(1)}=\frac{c_0^2}{3}+\frac{c_1^2}{3}+4\epsilon^2+2^{1/2}\pi\epsilon c_1.$$

For $A=0$ we have by (2) that the angular velocity is $\lambda/2$. Let the initial angular velocity θ_0' for $A=\mu a$ be $(\lambda/2)(1+\delta)$. Then as a first approximation from (2) $\mu a=\rho^2\lambda\delta/2$. Suppose the parameter $\mu=1$ then the initial conditions for $\theta=0$ become

$$r_0=\rho(1+c_0),\quad dr/d\theta=\rho c_1,\quad \theta_0'=\lambda(1+\delta)/2$$

Since $\epsilon=a/\lambda\rho^2=\delta/2$, the value of r at $\theta=\pi/2^{1/2}$ is;

$$r=\rho\left(1-c_0+\frac{c_0^2}{3}+\frac{c_1^2}{3}+\delta^2+\frac{\pi}{2^{1/2}}c_1\delta\right) \tag{10}$$

If the extreme variation in velocity is 3 percent and if the angular aperture of the beam is 1 in 25 ($c_1=0.02$), the value of r is $\rho(1-c_0)$ with an error of less than 1 in 1000. With the old simple magnetic focusing, the values of r for the different velocities would have varied fifteen times as much.

Equation (10) shows that in general there are two values of the velocity for which r is equal to r_0. These may be made to coincide by altering c_0, that is by altering the electric or magnetic field, since $c_0=(r_0/\rho)-1=(r_0\lambda/2k^{1/2})-1$. If c_1 is sufficiently small, this occurs for $c_0=0$ or $He/m=\lambda=2k^{1/2}/r_0$ thus giving e/m in terms of the electric and magnetic fields and the radius of curvature.

Ryerson Physical Laboratory,
University of Chicago,
March, 1929.