HIGH FREQUENCY SOUND RADIATION FROM A DIAPHRAGM

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Abstract

Calculation of the intensity of the high frequency sound radiation from a circular oscillator.—By a hydrodynamical-acoustical method a calculation is made of the intensity of the high frequency sound radiation from a circular piston-like oscillator at a distance from the oscillator greater than $2a$, where a is the radius. It is shown that there is no parallel "beam" of sound of cross-sectional area equal to the area of the oscillator, but that nevertheless most of the sound energy is contained in a cone of solid angle $\pi(0.45\lambda/a)^2$ ster-radians where λ is the wave-length of the radiation. Solution of the problem for points at *great* relative distance from the source then yields a result analogous to that obtained for the Fraunhofer diffraction of light through a circular aperture. The corresponding formula is $\pi(0.61\lambda/a)^2$. Comparison is made between the two methods and they are shown to be essentially the same, the difference in the formulae being due to difference in interpretation solely.

Introduction

The increasing use of high frequency sound radiation for subaqueous signalling has rendered important the calculation of the intensity of such radiation at a distance from the source, usually a piezo-electric quartz oscillator. Crandall[1] has given an approximate treatment, and in the French engineering literature[2] there is reference to the fact that the high frequency radiation from a piston-like source is confined to a beam of solid angle $\pi(0.61\lambda)^2/a^2$, where λ is the wave-length of radiation and a is the effective radius of the circular source, and $\lambda < a$. This would mean that the lateral spreading of the sound is confined to a plane angle $2 \text{ arc} \tan(0.61\lambda/a)$. The calculation of these latter quantities is presumably based on the analogous optical problem of the Fraunhofer diffraction of light by a circular aperture in an infinite screen. The present writer believes that it is worth while to give a more complete treatment of the problem from the standpoint of hydrodynamics and sound.

Calculation of Intensity of Sound Radiation from a Circular Oscillator

We shall assume that the source or oscillator can be considered as replaced by an equivalent circular piston of radius a, of such a character that the normal maximum displacement velocity at *each* point of the piston surface is the *same* and equal to $\dot{\xi}_0 e^{i\omega t}$ where $\dot{\xi}_0$ is thus constant and ω is $2\pi\nu$, where ν is the frequency of the vibration. The problem is to calculate the intensity due to this source at a point in a plane parallel to the piston surface and

[1] Crandall, Theory of Vibrating Systems and Sound, 1926, p. 137, ff.
[2] See, for example, F. Collin, Le Génie Civil, Vol. 86, 1925, pp. 38–40.

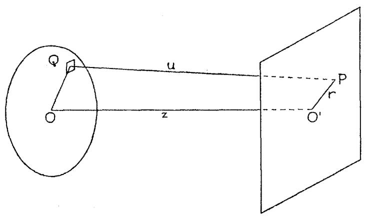

Fig. 1.

distant z from O, the center of the piston. Let such a point be P (see Fig. 1) and let its coordinates with respect to O' as origin be x_0, y_0, and its distance from O' then is $r=(x_0^2+y_0^2)^{1/2}$. The first step is to find the velocity potential at P. Lord Rayleigh[3] proved the general theorem that if a bounded surface radiates sound into the region on one side of it the velocity potential at a distant point P is given by

$$\phi_P = -(1/2\pi)\iint(\partial\phi/\partial n)(e^{-iku}/u)dS \tag{1}$$

wherein $\partial\phi/\partial n$ is the normal component of the displacement velocity at the surface, u is the distance from the surface element dS to the point P and the integration is taken over the whole radiating surface. In the present case if we take the surface element at point Q with coordinates x, y, z referred to O' the integral becomes

$$\phi_P = -\frac{\xi_0 e^{i\omega t}}{2\pi}\int_{-a}^{+a}\int_{-(a^2-x^2)^{1/2}}^{+(a^2-x^2)^{1/2}}\frac{e^{-iku}}{u}dx\,dy \tag{2}$$

Now from the figure, we have

$$u=\left[z^2+(x-x_0)^2+(y-y_0)^2\right]^{1/2} \tag{3}$$

Transforming from rectangular to polar coordinates $x-x_0=\rho\cos\theta$; $y-y_0=\rho\sin\theta$ yields

$$\phi_P = -\frac{\xi_0 e^{i\omega t}}{2\pi}\int_{\theta_1}^{\theta_2}\int_{\rho_1(\theta)}^{\rho_2(\theta)}\frac{e^{-ki(\rho^2+z^2)^{1/2}}}{(\rho^2+z^2)^{1/2}}\rho d\rho d\theta \tag{4}$$

where the limits for ρ and θ will depend on the relative magnitudes of r and a. For $r<a$ and $r>a$ they are such that the resulting integration becomes extremely complicated. We shall therefore confine ourselves to the case where $r=a$. We then have $\rho_1(\theta)=0$, $\rho_2(\theta)=2a\cos\theta$; while $\theta_1=-\pi/2$ and $\theta_2=+\pi/2$. Transforming back to u and θ, we finally have for the velocity potential

$$\phi_P = -\frac{\xi_0 e^{i\omega t}}{\pi}\int_0^{\pi/2}\int_z^{(z^2+4a^2\cos^2\theta)^{1/2}} e^{-iku}du d\theta \tag{5}$$

Let us make the assumption that $z>2a$. Then to a first approximation the evaluation of (5) depends on the calculation of the integrals

$$\int_0^{\pi/2}\cos k\left[z+(2a^2/z)\cos^2\theta\right]d\theta \text{ and } \int_0^{\pi/2}\sin k\left[z+(2a^2/z)\cos^2\theta\right]d\theta \tag{6}$$

[3] Rayleigh, Theory of Sound, Vol. II, §278, §302.

which in turn depend on the calculation of

$$\int_0^{\pi/2} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\left(\frac{2a^2k}{z}\cos^2\theta\right)d\theta$$

Making use of the identity $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$, we have

$$\int_0^{\pi/2} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\left[\frac{a^2k}{z}+\frac{a^2k}{z}\cos 2\theta\right]d\theta$$

$$=\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{a^2k}{z}\cdot\int_0^{\pi/2}\cos\left(\frac{a^2k}{z}\cos 2\theta\right)d\theta \;\left.\begin{matrix}-\sin\\ +\cos\end{matrix}\right\}\frac{a^2k}{z}\cdot\int_0^{\pi/2}\sin\left(\frac{a^2k}{z}\cos 2\theta\right)d\theta \quad (7)$$

The substitution $\psi=\pi/2-2\theta$ yields

$$\frac{1}{2}\cdot\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{a^2k}{z}\cdot\int_{-\pi/2}^{+\pi/2}\cos\left(\frac{a^2k}{z}\sin\psi\right)d\psi-\frac{1}{2}\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}\frac{a^2k}{z}\cdot\int_{-\pi/2}^{+\pi/2}\sin\left(\frac{a^2k}{z}\sin\psi\right)d\psi$$

of which the second term is zero since the sine is an odd function, while the first is simply

$$\frac{\pi}{2}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{a^2k}{z}\cdot J_0\left(\frac{a^2k}{z}\right) \quad (7a)$$

where J_0 denotes the Bessel's function of zero order.

Now for the *excess pressure* at P, we have from acoustical theory[4]

$$p=-\rho_0\dot{\phi}_P=\frac{i\omega\rho_0\dot{\xi}_0e^{i\omega t}}{2\pi}\iint\frac{e^{-iku}}{u}du\,d\theta \quad (8)$$

where ρ_0 is the density of the medium (in this case water). Using the value given in (7) and (7a) we have for the *real* part of p

$$p_{\text{Real}}=-\tfrac{1}{2}\rho_0c\dot{\xi}_0\{\cos\omega t\cdot[J_0(a^2k/z)\cos k(z+a^2/z)-\cos kz] \\ +\sin\omega t\cdot[J_0(a^2k/z)\sin k(z+a^2/z)-\sin kz]\} \quad (9)$$

In nearly all acoustical problems the most useful expression for the intensity, that is the rate of flow of sound energy per sec. per unit area, is[5]

$$I=\overline{p^2_{\text{Real}}}/\rho_0c \quad (10)$$

where c is the velocity of sound in the medium under consideration and the bar indicates the time average. Introducing (9) there results for the intensity of the radiation at the point P where $r=a$

$$I_P=(\rho_0c\dot{\xi}_0^2/8)[1-2J_0(a^2k/z)\cos(a^2k/z)+(J_0(a^2k/z))^2] \quad (11)$$

Inspection of (11) shows at once that the only value of a^2k/z for which $I=0$, is itself equal to zero, corresponding to an infinite z. This means that there

[4] See, for example, Crandall, loc. cit., p. 115.
[5] See, for example, Crandall, loc. cit., p. 92.

is no parallel beam of sound of cross sectional area equal to the area of the oscillator. Nevertheless *most* of the sound is confined to a relatively narrow conical region extending outward with its apex at the oscillator. We can see this as follows. The velocity potential at the point O' on the axis (i.e. for $r=0$) may be obtained by carrying out the integration indicated in equation (5) using as u limits z and $(z^2+a^2)^{1/2}$ and as θ limits 0 and 2π. The result is

$$\phi_{0'}=(2/ik)\xi_0 e^{i\omega t}\left[e^{-ik(z^2+a^2)^{1/2}}-e^{-ikz}\right] \tag{12}$$

whence using the same reasoning as above we arrive at the expression for the intensity at O'

$$I_{0'}=2\rho_0 c\xi_0^2 \sin^2 (ka^2/4z) \tag{13}$$

We now seek the value of ka^2/z such that

$$I_p/I_{O'}=1/10 \qquad (13)$$

and find by the use of tables that this is true for $ka^2/z=2.82$ to two decimal places. Hence in the diagram (Fig. 2) we have

$$\delta=2 \text{ arc tan } (a/z)=2 \text{ arc tan } (0.45\lambda/a) \tag{15}$$

recalling that $k=\omega/c=2\pi/\lambda$ where λ is the wave-length of the sound. This gives the lateral "spread" of the outgoing radiation. The corresponding solid angle at O is thus $\pi(0.45\lambda)^2/a^2$ steradians. It is clear that for practical purposes most of the radiation is confined to this cone, which has a somewhat smaller vertical solid angle than that given by the expression $\pi(0.61\lambda^2/a^2$.

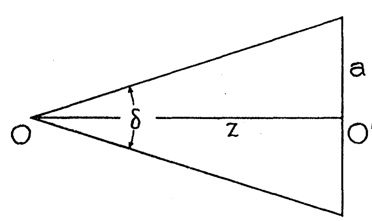

Fig. 2.

As an illustration, consider an oscillator with $a=10$ cm emitting waves of frequency 50,000 cycles, apparently the practical upper limit for signalling purposes because of the increase of the viscosity damping with frequency. This corresponds to a wave-length of 2.92 cm. We then have $\delta=15°$ approximately. According to the formula of the introduction we get $\delta=20°$ nearly. The difference, of course, is due to difference in interpretation of the problem. It may be worth while to carry through the calculation leading to that formula to see wherein the difference lies.

Returning to Eq. (3), we can write it if z is large and if $r>a$ so that x^2+y^2 can be neglected compared with $x_0^2+y_0^2$ and the product terms,

$$u=z\left(1+\frac{x_0^2+y_0^2-2x_0x-2y_0y}{2z^2}\right) \tag{16}$$

The velocity potential then becomes (see (2))

$$\phi_P=-\frac{\xi_0 e^{i\omega t}}{2\pi z}e^{-ik(z+r^2/2z)}\int_{-a}^{+a}\int_{-(a2-x2)^{1/2}}^{+(a^2-x^2)^{1/2}} e^{ik(x_0x+y_0y)/z}dxdy \tag{17}$$

involving evaluation of the integrals

$$\int_{-a}^{+a}\int_{-(a^2-x^2)^{1/2}}^{+(a^2-x^2)^{1/2}} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \frac{k}{z}(x_0x+y_0y)dydx \tag{18}$$

Let $kx_0/z=l$ and $ky_0/z=m$. Now introducing the rotation of axes

$$x=\frac{lx'}{(l^2+m^2)^{1/2}}-\frac{my'}{(l^2+m^2)^{1/2}}\ ; \quad y=\frac{mx'}{(l^2+m^2)^{1/2}}+\frac{ly'}{(l^2+m^2)^{1/2}}$$

we have for an arbitrary function F

$$\int_{-a}^{+a}\int_{-(a^2-x^2)^{1/2}}^{+(a^2-x^2)^{1/2}} F(lx+my)dydx=\int_{-a}^{+a}\int_{-(a^2-x'^2)^{1/2}}^{(a^2-x'^2)^{1/2}} F((l^2+m^2)^{1/2}x')dy'dx'$$

If F is an odd function the result is zero. If F is an even function, we have or the integral

$$2\int_{-a}^{+a} F((l^2+m^2)^{1/2}x')(a^2-x'^2)^{1/2}dx'$$

Now letting $x=a\cos\psi$, we have[6]

$$\int_{-a}^{+a}(a^2-x^2)^{1/2}\cos((l^2+m^2)^{1/2}x)dx$$
$$=a^2\int_0^{\pi}\sin^2\psi\cos(kar\cos\psi/z)d\psi \quad =\pi a^2\frac{J_1(kar/z)}{kar/z} \tag{19}$$

Making use of (19) in the evaluation of (17) and computing I_P as before we find

$$I_p=\frac{\rho_0c\xi_0{}^2k^2a^4}{2z^2}\left[\frac{J_1(kar/z)}{kar/z}\right]^2 \tag{20}$$

Now $I_P=0$ for that value of r for which $J_1(kar/z)=0$. According to Stokes the first root of $J_1(x)=0$ is $x=1.22\pi$. To this corresponds, then $r/z=0.61\lambda/a$, whence the statement made in the introduction follows. The larger factor 0.61 results then from the fact that the intensity is made to go to zero for the value of r sought. This then corresponds to the radius of the inner bright spot produced on a distant screen by the diffraction of light through a circular aperture. Here as before, however, by far the most of the sound is concentrated in a cone the base of which at distance z is a circle of radius r' such that $I_{r'}/I_{r=0}=1/10$. Indeed if we calculate r' from (20) to satisfy this condition we obtain

$$r'/z=0.435\lambda/a \text{ or } \delta=2\arctan 0.435\lambda/a \tag{21}$$

in good agreement with the value in (15) obtained by our somewhat simpler previous method.

Sloane Laboratory,
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May 11, 1928.

[6] See, for example, Weber, Partiellen Dif. Gleichungen der Math. Phys., Vol. 1, §68 or Gray, Mathews, and MacRobert, Bessell Functions, 1922, p. 46.