THE PROPAGATION OF SCHROEDINGER WAVES IN A UNIFORM FIELD OF FORCE

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Abstract

The phase-difference between a Schroedinger wave refracted by a uniform field of force and the primary wave is calculated. The results are shown in a table and graphically. As the wave-length increases, the phase-difference decreases, reaches a minimum, and then increases again. It is suggested that the intensity of some crystal reflections should vary anomalously as a result.

The motion of an electron in a uniform field of force has been treated by Kennard[1] from the point of view of the transformation theory and by Darwin[2] from the point of view of Schroedinger waves with particluar attention to Heisenberg's uncertainty principle. In these treatments, however, the simple problem of writing down a solution of Schroedinger's differential equation corresponding to a constant energy value has not been answered. This is done in the present note. Numerical values for the phase of the emerging wave are given. Relativity is neglected.

The electron having a charge e, mass m is moving in a uniform electric field of force directed along OY and having the absolute value F. The potential energy is Fy and the Schroedinger equation is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{8\pi^2 mE}{h^2}\left(1-\frac{Fe}{E}y\right)\psi=0$$

or

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+k^2(1-ay)\psi=0 \qquad (1)$$

where

$$k^2=8\pi^2 mE/h^2, \qquad a=Fe/E \qquad (2)$$

No generality is lost by confining our attention to waves having normals in the yx plane. Write

$$\psi=e^{i\xi x}f(y) \qquad (3)$$

Then

$$d^2f/dy^2+(A-By)f=0 \qquad (4)$$

where

$$A=k^2-\xi^2, \qquad B=k^2a \qquad (5)$$

[1] Kennard, Zeits. f. Physik **44**, 326 (1927).
[2] Darwin, Proc. Roy. Soc. **A117**, 258 (1927).

Let
$$\zeta = B^{-2/3}(A - By) \tag{6}$$

Then
$$d^2f/d\zeta^2 + \zeta f = 0 \tag{7}$$

The general solution of this equation (7) is known to be

$$f = \zeta^{1/2} C_{1/3}(2\zeta^{3/2}/3) \tag{8}$$

where $C_{1/3}$ is the general cylindrical fuction of order 1/3.

Suppose the field of force F exists only for values of $y > 0$ and suppose it is required to find the phase of the reflected Schroedinger waves if they are incident from the side of $y < 0$. We must then satisfy the boundary conditions at the surface $y = 0$. For $y > 0$ we have a solution of the form (8). For $y < 0$ the solution consists of the sum of two waves, one being a plane incident and the other a plane reflected wave. We also must satisfy the condition that for $y = +\infty$ the expression (8) should give vanishing values of f.

Let the incident wave be represented by $\psi = a_i e^{i(\xi x + \eta y)}$ and the reflected by $a_r e^{i(\xi x - \eta y)}$. Then $\xi^2 + \eta^2 = k^2$. The refracted wave is of the form (8), i.e.

$$f = \zeta^{1/2}\left\{\alpha J_{1/3}(2\zeta^{3/2}/3) + \beta J_{-1/3}(2\zeta^{3/2}/3)\right\} = \alpha F(\zeta). \tag{9}$$

If the standard definitions of the J_n are used, namely, if the J_n are defined by their power series and continuations of these, it becomes clear on investigating the asymptotic expansions of the J_n that $\alpha = -\beta$. Thus we have the boundary conditions

$$a_i + a_r = \alpha F(\zeta_0) \quad \text{and} \quad i\eta(a_i - a_r) = \alpha(dF/d\zeta)_{\zeta_0}(d\zeta/dy)_{y=0} \tag{10}$$

Remembering (6) $(d\zeta/dy)_{y=0} = -B^{1/3}$. Using (5) $(d\zeta/dy)_{y=0} = -\eta/\zeta_0^{1/2}$ so that the second equation in (10) is

$$a_i - a_r = (i\alpha/\zeta_0^{1/2})(dF/d\zeta)_{\zeta_0} \tag{10'}$$

Combining this with the first equation in (10)

$$\frac{a_i - a_r}{a_i + a_r} = \frac{i}{\zeta_0^{1/2}}\left(\frac{d \log F}{d\zeta}\right)_{\zeta_0} = iK \tag{11}$$

Hence

$$a_i = \frac{1 + iK}{1 - iK} a_r \tag{12}$$

Since K is real, the amplitude of the reflected wave is therefore equal to that of the incident. Its phase, however, is different.

Using recurrence relations for Bessel fuctions, we find from (11) and (9)

$$K = \frac{J_{2/3} + J_{-2/3}}{J_{1/3} - J_{-1/3}} = \tan \epsilon \tag{13}$$

Here $\zeta_0 = AB^{-2/3}$ and so $2\zeta_0^{3/2}/3 = 2(k^2m^2)^{3/2}/3k^2a = 2km^3/3a = 4\pi m^3/3\lambda a$ where λ represents the wave-length for $y<0$. In the table below the values of K and ϵ are given as functions of the parameter

$$\kappa = 2\zeta_0^{3/2}/3 = 4\pi m^3/3\lambda a \tag{14}$$

Making use of Dinnik's tables[3] for $J_{1/3}$, $J_{-1/3}$, $J_{2/3}$, $J_{-2/3}$

TABLE I. *Values of K and ϵ for different values of κ.*

κ	K	$\tan^{-1}K$	2ϵ	κ	K	$\tan^{-1}K$	2ϵ
0.2	−1.91	−62.3°	−124.6	2.0	.337	18.7°	−322.6
0.4	−2.50	−68.1	−136.2	2.4	− .082	− 4.7	−369.4
0.6	−4.20	−76.4	−152.8	2.8	− .496	−26.3	−412.6
0.8	−21.8	−87.3	−174.6	3.2	−1.13	−48.5	−457.0
1.0	6.34	81	−198	3.6	−2.84	−70.5	−501.0
1.2	2.56	68.7	−223.6	4.0	18.1	86.9	−546.2
1.4	1.49	56.1	−247.8	4.4	2.04	63.9	−592.2
1.6	.939	43.1	−273.8	4.8	.820	39.3	−641.4
1.8	.589	30.5	−299.0	5.2	.288	16.1	−687.8

By (12) we have $a_i = e^{2i\epsilon}a_r$, the values of 2ϵ belonging to different κ being tabulated above. The phase of the reflected wave at $y=0$ is therefore retarded behind the phase of the incident wave by the amount $|2\epsilon|$. If $\kappa=0$, i.e., if the wave-length is large or if the incidence is glancing, the phase difference is 180°, as it should be because for long wave-lengths we deal

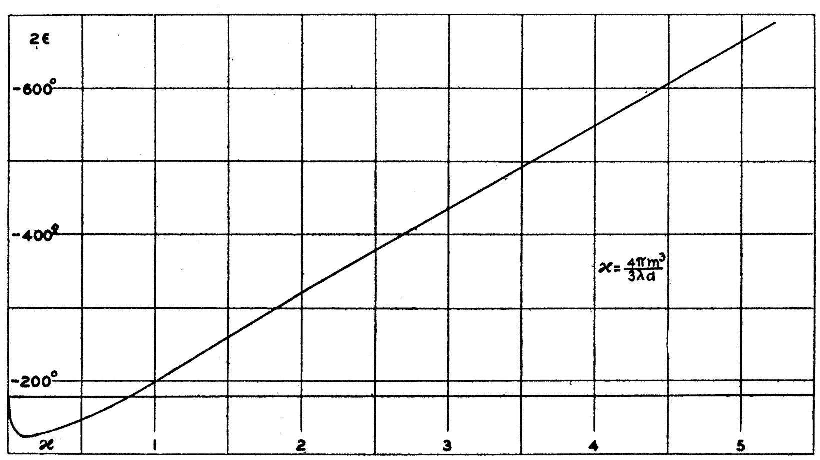

Fig. 1.

with pure reflection. If $\kappa > 2$ we have approximately a linear dependence of 2ϵ on κ. This is the region where phase relations can be accurately described by refraction and using geometrical optics. It must be noted, however, that even for $\kappa > 2$ an effect of the reflection is still felt in the form of

[3] Dinnik, Archiv d. Math. u. Phys. **18,** 337 (1911).

an additive constant of about 90° in the phase. This is particularly clearly seen on the attached graph, the asymptote to the phase curve cutting the axis of $\kappa=0$ at about 90°, while simple ray tracing leads to the expression

$$(-2\epsilon)=2\kappa \tag{15}$$

so that a change of 2π in 2ϵ should give a change of π in κ as is seen to be the case on the figure.

It seems possible that in the case of reflections of electrons by crystals the peculiar shape of the curve for $\kappa<2$ may be of importance. It is conceivable that α varies periodically over the surface of the crystal. The result of this would be a change in κ and therefore in 2ϵ. Hence the phase difference between the surface reflections from portions of the crystal having different values of a would be expected to vary with the wave-length and the angle of incidence. It is therefore to be expected that for electron reflections of the *plane grating type* a crystal cannot be described by a fixed distribution of reflecting matter as has been usually done in the case for x-rays.

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