LIGHT QUANTA AND WAVE MECHANICS

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Abstract

Light quanta are treated by wave mechanics by analogy with electrons. It is shown that their wave equation is the ordinary optical wave equation. Heisenberg's principle of indeterminateness becomes a description of diffraction. In problems in which localization of quanta is found experimentally, wave packets are to be set up; these are applied to the experiments of Bothe and Geiger, and of Compton. The paths of quanta coincide with the rays of geometrical optics, with a deviation of the error in geometrical optics; by the principle of indeterminateness, more accurate laws for the paths are neither necessary nor possible.

The wave mechanics has brought out the fundamental resemblance between electrons, with their accompanying Schrödinger waves, and corpuscular light quanta, with their electromagnetic waves. The statistical relations between electrons and Schrödinger waves have been discussed mathematically by Dirac and Jordan,[1] and Heisenberg[2] has supplied a physical description of the situation, in his principle of uncertainty. By analogy, the connections between quanta and light waves can be definitely stated, and no doubt are familiar to many persons. Nevertheless, since no treatment of the subject has apparently been published, and since it furnishes the answer to a problem which has been much discussed, and at the same time a very simple illustration of the principle of uncertainty, it seems worth while to give an explicit treatment. As will be seen from the paper below, for many of the most important problems one can use a simple theory in which the waves exist in a three-dimensional space, and the amplitudes are ordinary c-numbers, rather than q-numbers as Dirac has assumed.[3] The result may be stated as follows: that all experiments can be explained by the hypotheses that the intensity in the radiation field measures the probability of the presence of quanta; and that in those cases where a definite localization of quanta seems to be observed, this localization is to be described, to the accuracy with which it can be observed, by setting up wave packets.

The wave mechanics of a quantum can be formulated by the same principles used with electrons. First one selects coordinates (x, y, z, t) and corresponding momenta $(p_x, p_y, p_z, -E)$. Then, in order to find the function $\psi(x,y,z,t)$ which has the property that $\psi\psi^* dxdydz$ is the probability that a quantum with given energy at time t is in the volume $dxdydz$, one finds a functional relation between p_x, p_y, p_z, $-E$, and, if necessary, x, y, z, t: say $f(p_x,p_y,p_z,-E,\ x,y,z,t)=0$. In this function, one replaces each

[1] Dirac, Proc. Roy. Soc. **A113,** 621 (1927). Jordan, Zeits. f. Physik **40,** 809 (1927).
[2] Heisenberg, Zeits. f. Physik, **43,** 172 (1927).
[3] Dirac, Proc. Roy. Soc. **A114,** 243 (1927).

momentum by $h/2\pi i$ times the derivative with respect to the corresponding coordinate, so that f becomes an operator; one forms the differential equation

$$f\left(\frac{h}{2\pi i}\frac{\partial}{\partial x},\ \frac{h}{2\pi i}\frac{\partial}{\partial y},\ \frac{h}{2\pi i}\frac{\partial}{\partial z},\ \frac{h}{2\pi i}\frac{\partial}{\partial t},\quad x,y,z,t\right)\psi(x,y,z,t)=0$$

and the solution is the desired function ψ. To find the relation f, there is no general method given by wave mechanics. But for a quantum, we may take Einstein's equation $E=h\nu$, and de Broglie's[4] relation $(P_x^2+P_y^2+P_z^2)^{1/2}=h/\lambda$, combine with the equation $v/\lambda=\nu$, where v is the velocity, and obtain $v(P_x^2+P_y^2+P_z^2)^{1/2}=E$; in order not to have derivatives under the radical, we may equally well take instead $v^2(P_x^2+P_y^2+P_z^2)=E^2$. Then making our substitution, the equation is

$$v^2(h/2\pi i)^2\nabla^2\psi=(h/2\pi i)^2\partial^2\psi/\partial t^2,\ \text{or}\ \nabla^2\psi-(1/v^2)\partial^2\psi/\partial t^2=0,$$

the familiar wave equation of optics.

To solve the problem of the motion of quanta, then, one solves the optical wave equation, and the intensity of the resulting wave at any point measures the probability of the existence of a quantum at that point. This of course is a connection between waves and quanta frequently proposed before the development of the wave mechanics. Previously, it had to be admitted that this theory was incomplete, in that it did not precisely define the paths of the individual quanta; but now it is seen that this indefiniteness is just what one would expect from Heisenberg's principle of indeterminateness. That principle really involves two statements: first, that it is not mathematically possible in wave mechanics to set up a description of a motion in which the initial conditions (coordinates and momenta) are all precisely defined, but that rather the initial coordinates and momenta must be supposed subject to uncertainties, or probable errors, whose product is at least of the order of magnitude of h; second, that this restriction in the mathematical theory is not of physical importance, for any possible experimental method of fixing the initial conditions would introduce probable errors of at least this magnitude. Applied to optics these statements become, first, that on wave theory one cannot set up a solution in which the initial position and momentum of a quantum are both precisely defined (on account of the diffraction inherent in a wave theory); second, that this is never necessary anyway, since real quanta are deflected to follow the diffraction patterns. From this we can see that definite laws for the motion of quanta are neither possible nor necessary.

The simplest form of solution of the wave equation is a plane wave, $\psi=e^{2\pi i\nu(t-(lx+my+nz)/v)}$. This solution, as we can easily see, corresponds to a quantum whose momentum, with components $lh\nu/v$, $mh\nu/v$, $nh\nu/v$, and energy $h\nu$, are precisely determined, but whose coordinates, x, y, z, t,

[4] This formula may be taken either as a generalization of the equation that momentum of a quantum $=h\nu/c=h/\lambda$, to the case where the velocity is v instead of c; or as an analogy to de Broglie's equation $\lambda=h/$(momentum), or (momentum) $=h/\lambda$, holding for an electron.

are entirely indefinite (thus obeying Heisenberg's rule). The indefiniteness of the determination of the coordinates is seen by observing that $\psi\psi^*=1$, independent of position, so that the quantum has equal probability of being at any point of space. On the other hand, the momenta are $\psi(h/2\pi i)(\partial/\partial x)\psi^*=lh\nu/v$, etc., and the energy is $-\psi(h/2\pi i)(\partial/\partial t)\psi^*=h\nu$ quite independent of the coordinates, and definitely determined. This sort of wave is, of course, the one needed in most optical problems; for generally we can determine with great accuracy the direction and frequency of a beam, but we have no idea as to the location of the quanta in it.

One can easily set up an experiment, however, in which there is definite information about the location of quanta, as well is about their momenta. It is only necessary to take a perfectly plane monochromatic wave, allow this to fall on an opaque screen containing a small hole closed by a shutter, and open the shutter, closing it again very soon. Then on the far side of the screen we have light, of which we know that the quanta were at the position of the hole at the time the shutter was open, and were travelling in the direction of the initial beam. It might seem that, by making the hole smaller and smaller without limit, and the shutter faster and faster, we could make as accurate a determination of position and time as we please, without corresponding sacrifice in the accuracy of determination of the momentum. We note, however, in agreement with the first part of Heisenberg's rule, that this would be impossible in the wave theory. For since the wave passes through a small hole, the problem is one of diffraction, and the smaller the hole, the more bending of direction of the wave there is (that is, diversity of the momentum is introduced by the process of passing through the hole); and since the wave train which passes through is limited in time, there is a broadening of the spectrum (resulting in a diversity of energies). The wave theory does not permit of a solution limited to a certain region of space and time, and yet without diffraction and broadening of the spectrum. Next, agreeing with the second part of Heisenberg's rule, one observes that quanta actually travel so as to fill all parts of the diffraction pattern predicted by wave theory; we infer that quanta, originally all of the same momentum and energy, are actually bent and changed in energy by the mere process of passing through a hole and shutter, so that it is useless to try to get exactly precise initial conditions or laws of travel.

The numerical part of Heisenberg's rule is easily verified in our case. Suppose the beam is travelling along the x axis. Let the diameter of the hole be D, the length of time the shutter is open, T. The wave after diffraction can be at once analyzed into plane waves, for this is just what the telescope does in Fraunhofer diffraction. In the theory of that subject, one sees that there will be waves of appreciable intensity in the diffracted beam making angles up to the order of λ/D with the original direction. The spectrum can be found by Fourier analysis, and it appears that frequencies differing by as much as $1/T$ from the original frequency will be present. Thus one infers that in the beam which has passed through, ν will vary from its original value by as much as $1/T$, m and n will differ from zero by as much as λ/D, and l will equal 1 to the first order of small quantities. The uncertainty of $p_x = lh\nu/v$ will then be of the order of h/vT, of $p_y = mh\nu/v$ of the order of $(h\nu/v)(\lambda/D) = h/D$, of p_z of the order h/D, and of $E = h\nu$ of the order h/T. But the uncertainties in the determination of the coordinates are vT for x (the length of the wave bundle in the direction of its travel), D for y and z, and T for t. Thus in all cases the uncertainty introduced into the momentum, multiplied by the uncertainty in the determination of the coordinate, equals h, as the rule requires.

In the problem we have just discussed, there is a determination of both the initial coordinates and momenta of the quanta; the determination is not perfectly definite, it is true; yet, when one considers the relatively small magnitude of diffraction effects, one is tempted to say that ordinarily the definiteness of the determination is more striking physically than the indefiniteness, as the rectilinear propagation of light is more striking than diffraction. The light transmitted through the hole travels as a fairly compact packet of waves, following the path of the "ray" of geometrical optics, and gradually spreading out. Since the intensity of this wave determines the probability of the presence of quanta, all quanta transmitted will travel somewhere within the packet. Thus we can say that the paths of the quanta are the rays of geometrical optics, with an error of the order of magnitude of the error in geometrical optics. To a greater accuracy than this, we neither have nor require any law of the paths of quanta. In particular, one notes that individual quanta must be deflected, and change their frequency, as they pass through the hole; yet we need not infer from this the existence of forces producing the deflections. For the laws of conservation of energy and momentum, and the concept of force, need not apply precisely to the individual quanta; any more than any other law. They hold, as does the whole of ordinary dynamics, in the approximation to which we may use geometrical optics; beyond that, they need no longer be assumed to apply.

One error, in connection with the wave packet which we have set up by our experiment of the hole and shutter, is particularly to be avoided. This is the supposition that our wave packet is a dynamical "model" of a light quantum, or that it can give any information about the "dimensions" of a quantum. Such questions are entirely foreign to the theory; the dimensions of the wave packet, as we have seen, are derived entirely from the initial conditions of the problem, as regulated by experimental circumstances.

In the actual experiments that have been performed, there are only two in which the localization of light quanta has played a conspicuous part. These are the experiments on the Compton effect, performed by Bothe and Geiger, and by Compton.[5] In them, a plane wave of x-rays fell on atoms which scattered them. Both the recoil electrons, and the photo-electrons ejected by the scattered light, were observed. Bothe and Geiger observed that photo-electrons were ejected simultaneously with recoil electrons; Compton found that, correlating recoil electrons with photo-electrons, the

[5] Bothe and Geiger, Zeits. f. Physik **32**, 639 (1925); A. H. Compton, Proc. Nat. Acad. Sci., **11**, 303 (1925).

scattered quanta must have travelled in such a direction as to have the relation to the direction of the recoil electron demanded by the laws of conservation of energy and momentum. These experiments are somewhat more complicated to discuss than the cases we have taken, for we do not know the exact nature of the interaction processes between light and atoms. Nevertheless, a qualitative explanation can be given, which seems undoubtedly true in its broad outline.

If we knew merely that the plane wave was being scattered by the atoms, we should use a solution of the wave equation in which a plane wave struck the atoms, and spherical scattered waves came off from all of them. From this sort of solution, we can compute the probability of ejection of a recoil electron from an atom; we can also, knowing the intensity of the scattered light, compute the probability of the emission of a photo-electron; and our theory gives us no reason to expect a correlation between the two. But actually in the experiment our information is more precise than this; for in investigating the photo-electrons, it must be assumed that we know from which atom the recoil electron was ejected, and we know when and with what direction and velocity it was sent off. We must set up our wave representing the scattered light, making full use of these observations. The atomic wave function representing the scattering atom is to be so constructed that a wave packet representing the ejected electron is sent off within the limits of time and limits of angle observed in the experiment. Although we do not know in detail how to set up the scattered light wave in such a case, it seems certain that it would be in the form of a wave packet, emitted roughly during the time the recoil electron's packet was being ejected, and in roughly the direction demanded by the corpuscular theory of the Compton effect. The diffuseness of the wave packets, demanded by wave theory, is again to be correlated with the fact that the laws of the quanta are not known in detail, so that they do not precisely obey the law of conservation of momentum and energy in their encounters. We now have a probability of finding scattered quanta only within this wave packet; and the fact that photo-electrons were ejected only in the proper direction and time follows directly from the probability interpretation of the wave. This is a good example of Heisenberg's remark that the mere process of making an observation (in this case, observing the recoil electron), may greatly affect the probability of subsequent observations (in this case, restricting the possible places for the ejection of photo-electrons to a certain small beam); more generally, it illustrates the universal property of statistics, that the probability of an occurrence depends conspicuously on what is assumed known to start with.

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