THE PHYSICAL PENDULUM IN QUANTUM MECHANICS

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Abstract

It is pointed out that the Mathieu functions of even order are the characteristic functions of the physical pendulum in the sense of Schrödinger's wave mechanics. The relation of various properties of the functions, as known from purely analytical investigations of them, to the pendulum problem is discussed.

THE problem of the physical pendulum, that is, of the motion of a mass-point constrained to move in a circle and acted on by a uniform force field, has played such a great role in the study of analytic mechanics that a discussion of the same problem from the standpoint of Schrödinger's wave mechanics cannot be without interest. It turns out that the characteristic functions are certain of the Mathieu, or elliptic cylinder, functions and that the arguments from mechanics serve to illustrate in an interesting way many of the properties of these functions.

Let the mass of the particle be μ and let its position in a circle of radius a be designated by the angle θ. It will be supposed that the particle carries an electric charge e, and that there is a uniform electric field acting in such a way that the potential energy function is $-eEa\cos\theta$. That is, the force is in the direction of $\theta=0$. Under these circumstances the wave-mechanical equation becomes

$$\frac{d^2\psi}{d\theta^2}+\frac{8\pi^2\mu a^2}{h^2}[W+eEa\cos\theta]\psi=0,$$

in which W is the energy level parameter. The energy levels are the values of W for which this equation possesses solutions which have the period 2π in θ. Introducing the variable, $x=\frac{1}{2}\theta$, and the abbreviations,

$$\alpha=8\pi^2\mu a^2W/h^2,\quad q=\ 2\pi^2\mu a^2eE/h^2,$$

the equation appears in the usual form for Mathieu's equation,

$$d^2\psi/dx^2+(4\alpha+16q\cos 2x)\psi=0,$$

where, now, one seeks solutions with the period π in x.

The solutions of this equation which have the period 2π in x are known as Mathieu functions.[1] For $q=0$, the case of zero field strength, the problem becomes simply that of a free rotator with fixed axis. The characteristic functions are 1, $\sin 2x$, $\cos 2x$, $\sin 4x$, $\cos 4x\cdots$ with the associated values of α equal to 0, 1, 1, 4, 4 $\cdots$. The problem is degenerate since there

[1] An account of them is given in Whittaker and Watson, *A Course of Modern Analysis*, Chap. 19. (1920). This is, however, quite incomplete now because of the many more recent investigations by British and Scottish mathematicians.

are two distinct characteristic functions associated with each energy level, except the first one. The characteristic function for any degenerate state is an indeterminate linear combination of the two functions associated with that state. In particular, the functions which correspond to progressive rotatory motion in the two opposing senses are e^{2imx} and e^{-2imx}, rather than cos $2mx$ and sin $2mx$. These latter functions correspond in some way to equal numbers of rotators turning in opposite senses and these connect in a continuous manner with the non-degenerate characteristic functions for $q \neq 0$, as $q \to 0$. The standard notation for the Mathieu functions is $\mathrm{ce}_n(x,q)$ and $\mathrm{se}_n(x,q)$ where as $q \to 0$ these become equal to cos nx and sin nx respectively. All of these have period 2π, while those in which n is even have the period π in x and so they are the characteristic functions of the pendulum problem.

It will be observed that the zero from which energy is reckoned in the wave equation is from the position at which $\theta = \pi/2$ or $x = \pi/4$. If instead it is reckoned from $\theta = 0$, the minimum of the potential energy curve, one has to add $4q$ to each value of α.

It is of interest now to consider a number of properties of the functions, which have been obtained by purely analytical means, in the light of the pendulum problem. Firstly, it is clear that the functions will bear an invariant relation to the minimum of the potential energy curve, hence yield such relations[2] as

$$\mathrm{ce}_{2n}(\tfrac{1}{2}\pi - x, \; -q) = (-1)^n \cdot \mathrm{ce}_{2n}(x,q).$$

Jeffreys[3] has shown that there are no allowed values of α such that $\alpha < -4q$. Physically this means that there are no states for which the total energy is less than the minimum potential energy permissible for the system, and therefore appears natural enough. The relation, $\alpha = 4q$, is a critical one for classical mechanics in that for $\alpha > 4q$ the motion is rotatory while for $\alpha < 4q$ it is oscillatory. This value has also shown itself as a critical one in the analytical theory of the functions.

For $\alpha > 4q$, Jeffreys finds (approximately) that α must be such that

$$\int_0^{2\pi} (4\alpha + 16q \cos 2x)^{1/2} \cdot dx = 2\pi m,$$

in which m is an integer. Recalling the meaning of α and q, this requirement becomes

$$\int_0^{2\pi} p_\theta d\theta = \tfrac{1}{2} m h,$$

[2] See e.g. Goldstein, Trans. Cambr. Phil. Soc. **23**, 303, (1927) Par. 1.5. This memoir contains a good many of the newer results not given in Whittaker and Watson.

[3] Jeffreys, Proc. London Math. Soc. **23**, 437, (1924–25). This paper and the one preceding it are especially interesting in that the methods of approximate integration which he uses are closely related to those by means of which the connection between classical mechanics and quantum mechanics is established.

so that the condition reduces to the classical quantum condition for even values of m, the ones which correspond to allowed quantum motions.

The most complete tables of the values of α as a function of q are those of Goldstein (loc. cit.). He has given the values of α for $ce_0(x,q)$, $se_2(x,q)$ and $ce_2(x,q)$ besides several others which are not related to the pendulum problem. He gives an asymptotic expansion good for small m and large q as follows:

$$\alpha \sim -4q+(2m+1)(2q)^{1/2}- \cdots .$$

In this one recognizes that $(\alpha+4q)$ which is the energy counted up from the minimum of potential energy at $\theta=0$ goes linearly with the number m in just the way that the energy levels of the harmonic oscillator go in wave mechanics. Moreover, the interval between levels, $2(2q)^{1/2}$, when expressed as energy is exactly equal to $h\nu$ where ν is the frequency of the small oscillations of a pendulum in the field of strength E, reckoned on classical mechanics, i.e.,

$$\nu=(eE/4\pi^2\mu a)^{1/2}.$$

The rapidity with which the three lowest energy levels approach the values

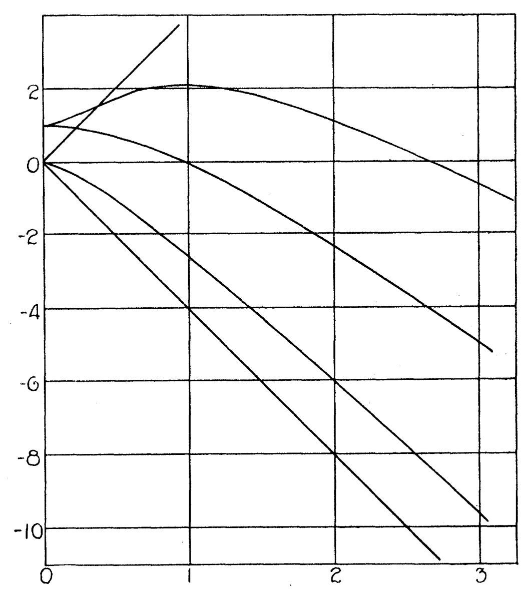

Fig. 1. The three lowest energy levels of the physical pendulum as a function of field strength (abscissas, q. ordinates, α). The straight line of positive slope divides the region of classical rotatory motions from that of classical oscillatory motions. That of negative slope gives the value of the potential energy minimum which forms a more natural origin from which to count the energy of the small oscillation states.

appropriate to the small oscillations theory is indicated in Fig. 1, where α is plotted as a function of q, from Goldstein's tables.

An interesting feature is the way the second and third energy levels cross the line, $\alpha = 4q$, without any discontinuity, this being the critical place at which the associated classical motion changes character from rotatory type to oscillatory by passing through a motion of infinite period. The discontinuous behavior of a pendulum moving with constant energy in a field of slowly decreasing strength caused some trouble in the old mechanics, as noted by Ehrenfest and by Bohr. That this discontinuity is not a feature of the wave mechanical treatment was first recognized by Hund.[4] In a sense this discontinuity still appears in wave mechanics but at $q = 0$ instead of at $q = \alpha/4$, for the characteristic functions, as $q \to 0$, do not join on continuously with those solutions of the equation for the force-free rotator which represent rotatory motion, as already remarked.*

Similarly one expects for small values of the quantum number and large values of q, that the characteristic functions will go over into the characteristic functions of the harmonic oscillator problem. The Mathieu functions do have just such an asymptotic connection with the Hermitian polynomials or parabolic cylinder functions.[5] This behavior is just what one would expect from the relation to the pendulum problem. Thus it makes clear the theorem of Ince (loc. cit.) concerning the clustering of the zeros of the Mathieu functions within a region of the order of magnitude $-Kq^{-\frac{1}{4}} < x < +Kq^{-1/4}$ as $q \to \infty$, where K is a constant. This is because the wave functions in wave mechanics only show a distinctly oscillatory character inside the region of the associated classical motion, where the de Broglie wavelength, h/p, of the system is real. A simple calculation shows that the amplitude of the classical motion for a given quantum state tends to zero as $q^{-1/4}$ for $q \to \infty$.

It thus appears that the principal properties of the Mathieu functions of even order are simply related to the mechanical problem of which they are the characteristic functions in Schrödinger's wave mechanics.

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[4] Hund, Zeits. f. Phys. **40**, 742, (1927) especially footnote, p. 750.

[5] See e.g., Ince, Jl. London Math. Soc. **2**, 46, (1927).

* *Added in proof:* The ψ function that joins on with e^{i2nx} is, of course,

$$\mathrm{ce}_{2n}(x, q)e^{2\pi i E_1 t/h} + i\,\mathrm{se}_{2n}(x, q)e^{2\pi i E_2 t/h}$$

where E_1 and E_2 are the energy levels associated with ce_{2n} and se_{2n} respectively. If one compute the quantum mechanical expression for the current associated with this ϕ, it will be seen that it depends on the time through a factor $\cos 2\pi(E_1 - E_2)t/h$. For small q, $E_1 - E_2 \to 0$ so the current reverses itself with a small frequency which becomes zero for $q = 0$. This behavior reminds one of the "gallows problem" of Ehrenfest and Tolman (Phys. Rev. **24**, 287 (1924)), although here there are no forces due to twist of thread to cause the long period reversal!