ON THE QUANTUM MECHANICS OF A SYSTEM OF PARTICLES

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Abstract

The "*mechanics of a system of particles is developed on the basis of Schrödinger's wave-equation*," without any use of matrices; it is shown that simple cases can be handled as easily by this method as by the matrix method. The passage to classical theory as an approximation stands out very clearly.

The measurement of velocity, momentum, energy and angular momentum as secondary physical qualities is discussed, the usual probability amplitudes being obtained.

The conservation of momentum and of energy is treated in terms of the new theory.

THE foundation of quantum mechanics can fairly be regarded as completed since the publication of Heisenberg's paper of last July,[1] in which an inclusive general principle was at last laid down for the physical interpretation of the theory. In that paper the principle was stated in terms of the generalized matrix theory of Dirac and in this form it was slightly amplified and applied to several simple cases[2] by the present author. To many physicists, however, the matrix theory seems at best pretty abstruse, whereas the wave mechanics of of Schrödinger seems much more tangible. In view of this fact it seemed worth while to show how the entire quantum theory of mechanical systems can be erected with no loss of generality or of simplicity upon the Schrödinger wave-equation for the coordinates as the sole basis.[4] It seems reasonable to hope that the quantum theory of electromagnetism, when finally discovered, will be capable of statement in a similar form.

In the present paper the mechanics of systems of particles is so treated and the simple problems that were solved so easily with the help of the matrix theory are shown to be equally easy to solve without it. In this form of the theory velocity, momentum and energy enter naturally as secondary quantities measurable only indirectly by means of observations of position, just as in classical theory; and the passage to the classical theory as an approximation becomes especially simple. The quantum laws of "Conservation of Momentum and Energy" are easily stated.

[1] W. Heisenberg, Zeits. f. Physik **43**, 172 (1927).

[2] E. H. Kennard, Zeits. f. Physik **44**, 326 (1927).

[4] Cf. also C. G. Darwin, Roy. Soc. Proc. **A117**, 258 (1927). This paper was unaccountably overlooked during the preparation of the present paper. The tenor of both papers is the same, yet there is so little overlapping in detail that it has seemed best not to shorten the present one; it would be beyond the powers of the present author to equal Professor Darwin's fascinating discussion of the general situation, but on the other hand the Heisenberg principle that physical quantities must be physically observable is perhaps carried out here a little more clearly.

1. *The Quantum Mechanics of a System of Particles.* In classical theory one treated a dynamical system of n degrees of freedom by deducing from Newton's Laws of Motion expressions for the values of the coordinates as functions of $2n$ constants of integration and the time. In non-relativistic quantum-mechanics we have, in place of values at a time t, a probability amplitude $\psi(q,t)=\psi(q_1,q_2, \cdots, q_n,t)$ with the property that

$$dP=\psi\psi^* dq_1 dq_2 \cdots dq_n \tag{1}$$

is the probability that an exact experimental determination of the q's at the time t would yield values lying within $dq_1, dq_2, \cdots, dq_n$. (ψ^* is the complex-conjugate value of ψ). In lieu of Newton's Laws we assume with Schrödinger that

$$\epsilon\partial\psi/\partial t=H(-\epsilon\partial/\partial q, q)\psi \tag{2}$$

where $\epsilon=h/2\pi i$, h being Planck's constant, and $H(p,q)$ or $H(p_1,p_2 \cdots p_n; q_1,q_2 \cdots q_n)$ is the Hamiltonian function; the p's can be regarded as merely symbolic, an assumption as to the form of H replacing in the new theory the assumption of a particular dynamical structure for the system. The constants of integration representing the initial conditions are then replaced in the new theory by known values of ψ at some given time.

Equation (1) requires that

$$\int dP=\int \psi\psi^* dq=1, \tag{3}$$

where $dq=dq_1 dq_2 \cdots dq_n$, the integral extending, as do all integrals with unspecified limits in this paper, over the entire physically admissible range of the variables. Either ψ or $P=\psi\psi^*$ may be called with Heisenberg the "probability packet" for the coordinates. The amplitude $\psi'(q',t)$ for another set of n variables q' which are functions of the q's is then

$$\psi'(q',t)=J^{1/2}\psi(q,t)e^{i\zeta(t)} \tag{4}$$

where J is the Jacobian of $q_1 \cdots q_n$ with respect to $q_1' \cdots q_n'$ and $\zeta(t)$ is an arbitrary real function.

We shall employ hereafter Cartesian coordinates $x_1, x_2 \cdots x_n$ for the $n/3$ particles of the system. Then

$$H=\sum_{r=1}^{n}(\xi_r^2/2m_r)+V(x), \tag{5}$$

ξ_r being the "momentum" for x_r and m_r the mass to which x_r refers, while $V(x)$ or $V(x_1, x_2 \cdots x_n)$ is the potential energy.

(2) now becomes

$$\epsilon\partial\psi/\partial t=\tfrac{1}{2}\epsilon^2\sum_{r=1}^{n}\frac{1}{m_r}(\partial^2\psi/\partial x_r^2)+V\psi, \tag{2'}$$

Writing[3]

$$\psi = Re^{2\pi i h^{-1}\theta}, \quad P = R^2, \tag{6}$$

and separating real and imaginary parts in (2′), we find

$$\frac{\partial R}{\partial t} = \sum_\tau \frac{1}{m_\tau}\left[\frac{\partial R}{\partial x_\tau}\frac{\partial \theta}{\partial x_\tau} + \frac{R}{2}\frac{\partial^2 \theta}{\partial x_\tau^2}\right], \tag{7}$$

$$\frac{\partial \theta}{\partial t} = -\frac{h^2}{8\pi^2}\sum_\tau \frac{1}{m_\tau R}\frac{\partial^2 R}{\partial x_\tau^2} + \frac{1}{2}\sum_\tau \frac{1}{m_\tau}\left(\frac{\partial \theta}{\partial x_\tau}\right)^2 + V. \tag{8}$$

From (6) and (7)

$$\frac{\partial P}{\partial t} = \sum_\tau \frac{1}{m_\tau}\frac{\partial}{\partial x_\tau}\left[P\frac{\partial \theta}{\partial x_\tau}\right], \tag{9}$$

which has exactly the same form as the equation of continuity for a fluid of density P flowing with a velocity whose components u_j are given by:

$$m_j u_j = -\frac{\partial \theta}{\partial x_j} = -\frac{h}{4\pi i}\frac{\partial}{\partial x_j}\log\frac{\psi}{\psi^*}. \tag{10}$$

Thus θ/m_j or $(h/4\pi i m_j)\log(\psi/\psi^*)$ is the "velocity potential" for the probability. From (8) and (10):

$$m_j\frac{\partial u_j}{\partial t} = \frac{h^2}{8\pi^2}\frac{\partial}{\partial x_i}\sum\frac{1}{m_\tau R}\frac{\partial^2 R}{\partial x_\tau^2} - \frac{1}{2}\frac{\partial}{\partial x_j}\sum m_\tau u_\tau^2 - \frac{\partial V}{\partial x_j}. \tag{11}$$

The rate of change of velocity for a particular element of the probability is

$$\frac{du_j}{dt} = \frac{\partial u_j}{\partial t} + \sum u_\tau\frac{\partial u_j}{\partial t} = \frac{\partial u_j}{\partial t} + \sum u_\tau\frac{m_\tau}{m_j}\frac{\partial u_\tau}{\partial x_j};$$

hence by (11)

$$m_j\frac{du_j}{dt} = \frac{h^2}{8\pi^2}\frac{\partial}{\partial x_j}\sum_{\tau=1}^{n}\frac{1}{m_\tau R}\frac{\partial^2 R}{\partial x_\tau^2} - \frac{\partial V}{\partial x_j}. \tag{12}$$

Thus each element of the probability moves in the Cartesian space of each particle as that particle would move according to Newton's laws under the classical force plus a "quantum force" given by the h-term in (12).

The motion here considered occurs in a space of n dimensions. We can also, however, replace the n-dimensional packet by n separate packets, one for each particle, all moving in the same ordinary space. The probability that one particle, say the rth, should be found in an element $dx_{3r}\,dx_{3r-1}dx_{3r-2}$ of its space is the integral of Pdx with respect to all other x's except x_{3r},

[3] The equivalent of Eqs. (6) to (12) was given by Madelung, Zeits. f. Physik **40**, 322 (1927).

x_{3r-1} and x_{3r-2}, so that if we call the probability function for this particle alone P_r, we have

$$P_r = \int P dx_{(r)} = \int \psi\psi^* dx_{(r)} \tag{13}$$

where $dx_{(r)} = dx_1 \cdots dx_{3r-3} dx_{3r+1} \cdots dx_n$. (9) gives ($P$ being assumed to vanish fast enough at infinity):

$$\frac{\partial P_r}{\partial t} = \int \frac{\partial P}{\partial t} dx_{(r)} = \frac{1}{m_{3r}} \sum_{3r-2}^{\tau=3r} \frac{\partial}{\partial x_\tau} \int \frac{\partial \theta}{\partial x_\tau} dx_{(r)}. \tag{14}$$

The mean components of velocity of the probability in the space of the rth particle, or the mean components of velocity of P_r, are

$$u_{rj} = (1/P_r) \int P u_j dx_{(r)} \ (j = 3r,\ 3r-1,\ 3r-2)\ ; \tag{15}$$

and, substituting from (10) for θ in (14) and using (15),

$$\frac{\partial P_r}{\partial t} = \sum_{3r-2}^{\tau=3r} \frac{\partial}{\partial x_\tau} (P_r u_{r\tau}). \tag{16}$$

P_r and u_{rj} are functions of t and of x_{3r}, x_{3r-1}, x_{3r-2}, which are the coordinates of m_r and can be regarded simply as coordinates in ordinary space. P_r represents a probability packet for this particle alone; it can be regarded as moving at each point in ordinary space with components of velocity u_{rj} as given by (15). But it is not possible to proceed farther and obtain dynamical equations for P_r in terms of u_{rj} alone: the elements of the probability for one particle alone move, not like a homogeneous fluid, but like the molecules of a gas, the elements located momentarily at a given point having various velocities and being variously related to the other particles of the system.

In terms of these results the relationships between quantum and classical mechanics stand out very clearly. If we put $h=0$, the probability becomes a distribution of matter moving classically. In general, *if a system is immersed in a uniform force field, its probability packet will simply execute the classical motion in that field in addition to whatever internal motion it may have* by reason of internal classical forces or the quantum force; for by (12) all elements of the probability experience the same classical acceleration in the field in addition to other accelerations of internal origin, and since the accelerations imposed by the field do not alter the relative positions of the particles the internal forces of all sorts are unaffected. The same statement will hold as an approximation if the field is merely approximately uniform over the region occupied by the probability packet.

It follows similarly that the influence of the remainder of the system upon the motion of any one of its particles approximates to the classical influence whenever the classical force upon the particle is approximately the

same in all configurations of the system that stand any appreciable chance of being realized; for then the relative distribution of the probability in the space of that one particle is approximately unaffected by the influence of the remainder of the system. Thus the quantum mechanics predicts no novel interaction between electrons spaced so far apart that their classical interaction is slight.

Finally, let

$$\bar{x}_j = \int x_j dP = \int x_j P dx = \int x_j P_r dx_{3r} dx_{3r-1} dx_{3r-2} \tag{17}$$

denote either the coordinates of the "center of probability" of the n-dimensional packet ($j=1, 2 \cdots n$), or the coordinates of the packet for the rth particle in ordinary space (in which case we limit j to $3r$, $3r-1$ or $3r-2$). Then regarding dP as a moving element:

$$\frac{d\bar{x}_j}{dt} = \int \frac{dx_j}{dt} dP = \int u_j dP, \qquad \frac{d^2\bar{x}_j}{dt^2} = \int \frac{du_j}{dt} dP,$$

$$m_j \int \frac{du_j}{dt} dP = \frac{h^2}{8\pi^2} \int P \frac{\partial}{\partial x_j} \sum_\tau \frac{1}{m_\tau R} \frac{\partial^2 R}{dx_\tau^2} dx - \int P \frac{\partial V}{\partial x_j} dx,$$

by (12). But, putting $P=R^2$ and integrating first by parts with respect to x_j and x_τ in turn, and then with respect to x_j:

$$\int P \frac{\partial}{\partial x_j} \sum \frac{1}{m_\tau R} \frac{\partial^2 R}{\partial x_\tau^2} dx = \int \sum \frac{1}{m_\tau} \frac{\partial R}{\partial x_j} \frac{\partial^2 R}{\partial x_\tau^2} dx$$

$$= \int \sum \frac{1}{m_\tau} \frac{\partial}{\partial x_j} \left(\frac{\partial R}{\partial x_\tau} \right)^2 dx = 0.$$

Hence

$$m_j \frac{d^2\bar{x}_j}{dt^2} = m_j \frac{d}{dt} \int u_j P dx = - \int P \frac{\partial V}{\partial x_j} dx. \tag{18}$$

As applied to the separate particles this equation states that *the center of probability for each particle moves in ordinary space as would the particle itself in classical mechanics under the "mean probable" force as given by the right-hand member of (18)*. This result was obtained for a single particle by Ehrenfest[4] and has been given more generally by Ruark[5]; it makes very clear the reason for the approximate validity of classical mechanics.

The quantum force, represented by the term in (12) containing h, thus affects primarily only the relative motion of different parts of the probability packet. This force is responsible for the characteristic quantum indetermination pointed out by Heisenberg; but it is also responsible for other

[4] P. Ehrenfest, Zeits. f. Physik 45, 455 (1927).
[5] Ruark, Phys. Rev., 31, 533 (1928).

departures from classical behavior such as would not result by Newton's laws merely from an indefiniteness in the initial state (e.g. the phenomenon of the quantization of atomic energy). Unfortunately the separation of these two quite different effects presents grave difficulties. The complexity of the quantum force also seems to make it of little use in thinking out directly the solution of simple cases. Nevertheless equation (12) can be made to suggest the solution in all of the simple cases that were so easily handled by the matrix method, and this we shall proceed to show by solving two of those problems.

2. *The Case of Free Motion.* With $V=0$, Equation (2′) is like an equation for the flow of heat[6] with imaginary "conductivity"; we should expect therefore to obtain a solution that represents the part of the field of P or R^2 that is present initially at each point as moving outward with time. Let us seek a simple form of such polar motion. With $V=0$, equation (12) would be very simple if R were uniform in space; then $u_j=$constant for an element of R^2, and a group of elements leaving the point $x_{10}, x_{20}\cdots x_{n0}$ at time $t=0$ would at time $t=t$ have positions $x_j=(x_j-x_{j0})/t$. Inserting this value of u_j in (10) and (8) and integrating (R being assumed uniform):

$$\partial\theta/\partial t=\tfrac{1}{2}\sum m_\tau(x_\tau-x_{\tau 0})^2/t^2,\quad \theta=-\sum_\tau[m_\tau(x_\tau-x_{\tau 0})^2/2t]+f(x)$$

Substituting back in (10) (because we differentiated once in obtaining (12)), we find $f(x)=$constant.
Then by (7)

$$\partial R/\partial t=-nR/2t,\quad R=Ct^{-n/2}.$$

This result justifies our assumption as to R. Choosing a special value of C, we thus have

$$S(x_0,x)=(m_1m_2\cdots m_n)^{1/2}(i/ht)^{n/2}\exp\Big[-(i\pi/ht)\sum_\tau m_\tau(x_\tau-x_{\tau 0})^2\Big] \qquad (20)$$

as a solution of (2′) with $V=0$, S is the "transformation function" from x_0 to x of Dirac and Heisenberg; it cannot itself represent a distribution of "probability", for normalization by (3) would require that $C=0$, but it does represent a mathematically possible distribution of R^2. Now let $\psi_0(x)=\psi_0(x_1, x_2\cdots x_n)$ be the amplitude at $t=0$. Then $\psi(x,t)$ given by

$$\psi(x,t)=\int\psi_0(x_{10},x_{20}\cdots x_{n0})S(x_0,x)dx_0, \qquad (21)$$

where $dx_0=dx_{01}\,dx_{02}\cdots dx_{0n}$, is also a solution of (2′) with $V=0$. We shall show that , with the constant factor chosen as in (20), it reduces to ψ_0 at $t=0$. Put

$$x_{\tau 0}=x_\tau+r_\tau(t)^{1/2}\ ;$$

[6] Cf. Ehrenfest, loc. cit.[4]

then (21) and (20) give:

$$\psi(x,t)=(m_1\cdots m_n)^{1/2}(i/h)^{n/2}\int\psi_0(x+rt^{1/2})\exp[-(i\pi/h)\sum m_r r_r^2]dr_1dr_2\cdots dr_n.$$

Since

$$\int_{-\infty}^{\infty}e^{-iay^2}dy=\int_{-\infty}^{\infty}(\cos ay^2-i\sin ay^2)dy=(1-i)(\pi/2a)^{1/2}$$

the right member of the preceding equation reduces to $\psi_0(x)$ for $t=0$.

Thus (21) is the general solution of (2′) with $V=0$ corresponding to the initial condition, $\psi=\psi_0$ at $t=0$; and it agrees of course with Heisenberg's result.

The other simple cases referred to above which can be solved by the same method are the case of uniform force field already treated above by general reasoning, the simple harmonic oscillator in n dimensions, and the effect of a uniform magnetic field. As the last named introduces a new feature, we shall treat concisely a simple case of it.

3. *An Electron in a Homogeneous Magnetic Field.* Taking the z axis parallel to the magnetic intensity, M, we have as Hamiltonian,

$$H=\frac{1}{2m}(p_x^2+p_y^2)+\frac{\omega}{2}(yp_x-xp_y)+\frac{m\omega^2}{8}(x^2+y^2), \tag{22}$$

where $\omega=eM/mc$, $e=$charge and $m=$mass of the electron, and as the wave equation:

$$\epsilon\frac{\partial\psi}{\partial t}=\frac{\epsilon^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\frac{\epsilon\omega}{2}\left(y\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial y}\right)+\frac{m\omega^2}{8}(x^2+y^2)\psi=0. \tag{23}$$

This equation requires some generalization of preceding results because it contains first-order derivatives. Instead of (7) and (9) we find

$$\frac{\partial R}{\partial t}=\frac{1}{m}\left(\frac{\partial R}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial R}{\partial y}\frac{\partial\theta}{\partial y}\right)+\frac{R}{2m}\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)+\frac{\omega}{2}\left(x\frac{\partial R}{\partial y}-y\frac{\partial R}{\partial x}\right), \tag{24}$$

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}(Pu_x)-\frac{\partial}{\partial y}(Pu_y), \tag{25}$$

with

$$mu_x=-(\partial\theta/\partial x)+\tfrac{1}{2}m\omega y,\qquad mu_y=-(\partial\theta/\partial y)-\tfrac{1}{2}m\omega x. \tag{26}$$

In (8) two additional terms appear on the right but upon substituting u from (26) we find simply:

$$\frac{\partial\theta}{\partial t}=-\frac{h^2}{8\pi^2mR}\left(\frac{\partial^2R}{\partial x^2}+\frac{\partial^2R}{\partial y^2}\right)+\frac{m}{2}(u_x^2+u_y^2). \tag{27}$$

Now for a particular element of R^2

$$\frac{du_x}{dt}=\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y};$$

calculating the latter derivatives from (26) and (27), and repeating for u_y, we find:

$$\begin{aligned} m\frac{du_x}{dt}&=\frac{h^2}{8\pi^2 m}\frac{\partial}{\partial x}\left[\frac{1}{R}\frac{\partial^2 R}{\partial x^2}+\frac{1}{R}\frac{\partial^2 R}{\partial y^2}\right]+m\omega u_y,\\ m\frac{du_y}{dt}&=\frac{h^2}{8\pi^2 m}\frac{\partial}{\partial y}\left[\frac{1}{R}\frac{\partial^2 R}{\partial x^2}+\frac{1}{R}\frac{\partial^2 R}{\partial y^2}\right]-m\omega u_x. \end{aligned} \tag{28}$$

With $h=0$ these are the classical equations for the electron (which checks the assumed form for H), and the solution for its position is:

$$\begin{aligned} x&=A\sin\omega t+B\cos\omega t+C,\\ y&=A\cos\omega t-B\sin\omega t+D. \end{aligned} \tag{29}$$

If R is uniform in space, the elements of R^2 will move according to these same equations. Assuming all elements to start out from the point x_0, y_0 at $t=0$, we have $x_0=B+C$, $y_0=A+D$; eliminating the constants by means of these equations and (29) from the values of $u_x=\dot{x}$ and $u_y=\dot{y}$ obtained from (29), we find:

$$\begin{aligned} u_x&=\frac{\omega}{2}\left[\frac{\sin\omega t}{1-\cos\omega t}(x-x_0)+y-y_0\right],\\ u_y&=\frac{\omega}{2}\left[-(x-x_0)+\frac{\sin\omega t}{1-\cos\omega t}(y-y_0)\right]. \end{aligned} \tag{30}$$

We now substitute these expressions for u_x and u_y in (27), omit the term in h and integrate, obtaining:

$$\theta=-\frac{m\omega}{4}\frac{\sin\omega t}{1-\cos\omega t}[(x-x_0)^2+(y-y_0)^2]+f(x,y).$$

Substitution of this value for θ in (26) and comparison with (30) shows that $f(x,y)=\frac{1}{2}\omega(y_0x-x_0y)+\text{const.}$ Then, putting the value of θ into (24) and assuming R independent of x and y, we find after integration

$$R=C(1-\cos\omega t)^{-1/2}.$$

Thus

$$\begin{aligned} S(x_0,y_0;x,y)=\frac{im\omega}{2h}\left(\frac{2}{1-\cos\omega t}\right)^{1/2}\exp\left\{-\frac{i\pi m\omega}{2h}\left[\frac{\sin\omega t}{1-\cos\omega t}((x-x_0)^2\right.\right.\\ \left.\left.+(y-y_0)^2)+2(x_0y-y_0x)\right]\right\} \end{aligned} \tag{31}$$

and

$$\psi(x,y,t)=\int\psi_0(x_0,y_0)S(x_0,y_0\,;x,y)dx_0dy_0 \tag{32}$$

are solutions of (23). Analysis of ψ by the method used above shows that $\psi\rightarrow\psi_0(x,y)$ as $t\rightarrow 0$. Hence ψ as given by (32) and (31) is the general solution of (23) corresponding to the initial amplitude ψ_0. This result agrees with that obtained by the matrix method.[7]

4. *The measurement of Velocity and Momentum.* The ordinary fundamental definition of velocity as space divided by time requires two successive specifications of position. But two *exact* experimental determinations of position made upon the same particle will always yield the velocity of light, because of the indefinitely great Compton-effect or equivalent disturbance produced by the first one; and two inexact determinations leave an indefiniteness in the measured value. There seems to be only one case in which an observation can be made which deserves to be called an exact measurement of velocity, namely, the case of free motion.

Suppose that at time t the amplitude $\psi(x,t)$ differs appreciably from zero only within a distance D from a certain point $(\alpha_1,\ \alpha_2\cdots\alpha_n)$ (i.e. for $\sum(x_\tau-\alpha_\tau)^2<D$), and suppose that at a very much later time t' the position of the system is accurately determined, the values thus found for the coordinates being x'. Then we can regard the quantities

$$v_j=(x_j'-\alpha_j)/(t'-t)\qquad(j=1,2\cdots n) \tag{33}$$

as experimental values of the components of velocity of the particles; they become exact as $t'\rightarrow\infty$, since the indefiniteness in the initial positions of the particles is less than $D/(t'-t)$ and vanishes in the limit. We can reasonably regard these values as referring specifically to the time t because the result of such an observation for given $(t'-t)$ would vary statistically in a manner independent of t and it is therefore reasonable to postulate that the lapse of time until t' causes no error.

The amplitude for x' will be, by (20) and (21):

$$(m_1\cdots m_n)^{1/2}[i/h(t'-t)]^{n/2}\int\psi(x,t)\exp\{-[i\pi/h(t'-t)]\sum m_\tau(x_\tau'-x_\tau)^2\}dx \tag{34}$$

and for v by (4) and (33):

$$(m_1\cdots m_n)^{1/2}(i/h)^{n/2}\int\psi(x,t)\exp\{2\pi ih^{-1}\textstyle\sum_\tau m_\tau[v_\tau x_\tau-$$
$$\tfrac{1}{2}(x_\tau-\alpha_\tau)^2(t'-t)^{-1}-\alpha_\tau v_\tau-\tfrac{1}{2}v_\tau^2(t'-t)]\}dx\cdot \tag{35}$$

Here the last two terms in the bracket are independent of the variables of integration and so result merely in multiplying ψ by a factor of absolute value unity, which has no effect on the probability: this factor, together,

[7] E. H. Kennard, loc. cit. p. 348.

with i before the integral sign, we shall simply omit. The second term in the bracket vanishes as $t' \to \infty$. Hence in the limit we can take as the amplitude for v at time t

$$M(v) = (m_1 \cdot \cdot \cdot m_n)^{1/2} h^{-n/2} \int \psi(x,t) \exp[2\pi i h^{-1} \sum_\tau m_\tau v_\tau x_\tau] dx \quad (35)$$

and for the momenta $\xi_j = m_j v_j$, by (4):

$$M(\xi) = h^{-n/2} \int \psi(x,t) \exp[2\pi i h^{-1} \sum_\tau \xi_\tau x_\tau] dx. \quad (36)$$

The last equation is the familiar result yielded by matrix theory; $M(\xi)$ $M^*(\xi)$ $d\xi$ is the probability that an exact measurement of the momenta made as described above would yield values lying within $d\xi$. Equation (36) leads at once to the Heisenberg relation between the degrees of indetermination of a coordinate and its momentum.

We shall verify the normalization of $M(\xi)$; one of the formulas obtained in doing so will be required later. Putting $I = \int M(\xi)\, M^*(\xi) d\xi$ we have:

$$I = h^{-n} \int d\xi \int\int \psi(x') \psi^*(x'') \exp[2\pi i h^{-1} \sum_\tau \xi_\tau (x_\tau' - x_\tau'')] dx' dx'', \quad (37)$$

$$I = h^{-n} \int \psi(x')\, dx' \int d\xi \int \psi^*(x'') \exp\, [2\pi i h^{-1} \sum_\tau \xi_\tau (x_\tau' - x''_\tau{}')]\, dx''.$$

This change in the order of integration could be made without question if both $\psi(x')$ and $M(\xi)$ vanished outside of certain finite limits (besides satisfying the usual requirements). Now it happens that (as is not hard to show) ψ and M cannot have this particular property simultaneously. Accordingly we shall simply assume that ψ and M vanish at infinity fast enough so that the change of order is possible (caution in regard to this is really necessary; for instance, the integration with respect to ξ cannot be carried out first of all). We now recognize in the double integral following dx' simply the Fourier integral expansion of $\psi^*(x')$, multiplied by h^n. Hence

$$I = \int \psi(x') \psi^*(x') dx', \quad (38)$$

and $I = 1$ by (3).

The physical definition of velocity contained in equations (33) is not immediately applicable to cases other than free motion. According to Heisenberg we are at liberty to imagine the forces abolished at any instant, so that the motion becomes free and exact measurements of velocity and momentum can then be made. If we do not wish to adopt such an assumption then we might still call $M(\xi)$ as defined by (36) the "amplitude for the momenta," without asking whether this quantity has an immediate physical meaning or not; this will be done during the remainder of this paper.

5. *The "Conservation of Momentum."* Equ. (36) leads to an important connection between momentum and force. The *total* component of momentum

of all particles in one direction, say that of ξ_α where $\alpha = 1$, 2 or 3, is $\sum_{k=0}^{n-1}\xi_{\alpha+3k}$: the "mean probable" momentum of the whole system is therefore

$$\Gamma_\alpha = \int \sum_{k=0}^{n-1}\xi_{\alpha+3k}M(\xi)M^*(\xi)d\xi \tag{39}$$

$$= h^{-n}\int\sum\xi_{\alpha+3k}d\xi\int\int\psi(x')\psi^*(x'')\exp\left[2\pi i h^{-1}\sum\xi_\tau(x_\tau'-x_\tau'')\right]dx'dx''$$

$$= -\frac{1}{2\pi i h^{n-1}}\sum\int d\xi\int\int\frac{\partial\psi(x')}{\partial x'_{\alpha+3k}}\psi^*(x'')\exp\left[2\pi i h^{-1}\sum\xi_\tau(x_\tau'-x_\tau'')\right]dx'dx''$$

by (36), after an integration by parts in each term of the sum (ψ being assumed to vanish at the limits). The integrals in the last equation are the same as that in (37) except that a derivative replaces $\psi(x')$; hence, making the corresponding change in (38):

$$\Gamma_\alpha = -\frac{h}{2\pi i}\sum\int\frac{\partial\psi(x)}{\partial x_{\alpha+3k}}\psi^*(x)dx.$$

We can also shift the differentiation onto ψ^* by an integration by parts; hence we can write

$$\Gamma_\alpha = \frac{h}{4\pi i}\sum_{k=0}^{n-1}\int\left(\psi\frac{\partial\psi^*}{\partial x_{\alpha+3k}}-\psi^*\frac{\partial\psi}{\partial x_{\alpha+3k}}\right)dx. \tag{40}$$

Now from (10)

$$\sum_{k=0}^{n-1}m_{j+3k}\int u_{\alpha+3k}Pdx = \frac{h}{4\pi i}\sum\int\left(\psi\frac{\partial\psi^*}{\partial x_{\alpha+3k}}-\psi^*\frac{\partial\psi}{\partial x_{\alpha+3k}}\right)dx. \tag{41}$$

Γ_α therefore equals the first member of the last equation and from (18)

$$\frac{d\Gamma_\alpha}{dt} = -\sum_{k=0}^{n-1}\int P\frac{\partial V}{\partial x_{\alpha+3k}}dx. \tag{42}$$

Eqs. (40) and (41) might be interpreted as identifying the total "momentum of the probability" with that of the particles. *Eq.* (42) *shows that the time rate of change of the total mean probable momentum of the system is equal to the total mean probable classical force upon it, internal forces cancelling out in the usual manner. This is the theorem of the conservation of momentum in the new mechanics.*

The same theorem can be deduced for one particle alone, or for any other separate portion of the whole system, but we shall not elaborate the details.

6. *The Measurement of Energy and other Secondary Magnitudes.* In order to arrive at the Schrödinger theory of atomic energy levels from the present standpoint it is necessary to exhibit each energy value as the result of a mechanical experiment. Now in any measurement the quantity actually observed seems to be always a *position* (on a scale, plate, etc.); from the standpoint of physical observation the coordinates seem to play a funda

mental role, while such things as momentum and energy are arrived at only as subsidiary quantities. The measurement of a magnitude of the latter type can be illustrated as follows:

Let $H\ (p,\ q)=H\ (p_1, \cdots, p_n;\ q_1, \cdots, q_n)$ be the Hamiltonian for a system when the motion of the center of mass is omitted, and let $F\ (p,\ q)$ $=F\ (p_1 \cdots p_n,\ q_1 \cdots q_n)$ be any function of coordinates and momenta which "remains constant in time." In order to "measure" F let us suppose that we are able to immerse the system in a field that exerts a uniform force upon it parallel to the coordinate x of its center of mass and "proportional to F" (an actual example being a non-homogeneous magnetic field exerting a force proportional to magnetic moment, as in the Stern-Gerlach experiment). Then the total Hamiltonian is

$$H_1=H(p,q)+(\xi^2/2M)-\beta F(p,q)x \tag{43}$$

where $\xi=$ momentum corresponding to x, M = mass of system and β is the field constant; and

$$\epsilon\frac{\partial\psi}{\partial t}=H\left(-\epsilon\frac{\partial}{\partial q},\ q\right)\psi+\frac{\epsilon^2}{2M}\frac{\partial^2\psi}{\partial x^2}-\beta xF\left(-\epsilon\frac{\partial}{\partial q},\ q\right)\psi. \tag{44}$$

Putting

$$\psi=\psi_1(q,t)\psi_2(x,t) \tag{45}$$

we find that the variables will separate provided $F(\cdots)\psi=\kappa\psi$ for all t, that is, provided

$$\psi_1(q,t)=\gamma(t,\kappa)u(q,\kappa) \tag{46}$$

and

$$F(-\epsilon\partial/\partial q,q)u(q,\kappa)=\kappa u(q,\kappa), \tag{47}$$

κ being a constant. The usual argument then leads to the equations:

$$H(-\epsilon\partial/\partial q,q)u(q,\kappa)=Wu(q,\kappa), \tag{48}$$

$$\epsilon d\gamma/dt=W\gamma, \qquad \gamma=e^{2\pi ih^{-1}Wt}, \tag{49a,b}$$

$$\epsilon\frac{\partial\psi_2}{\partial t}=\frac{\epsilon^2}{2M}\frac{\partial^2\psi_2}{\partial x^2}-\beta\kappa x\psi_2, \tag{50}$$

where W is a new constant. (We could also add an arbitrary term $\eta(t)\psi_2$ in (50) provided we subtracted $\eta(t)\gamma$ in (49), but this would not alter ψ). We shall suppose that u and ψ_2 are separately normalized to satisfy (3)[9]; to make this possible, κ and W must be "characteristic values" for their respective equations.

If u is to satisfy both of Eq. (47) and (48) it is necessary that the operators F and H should "commute," i.e., if we write $F(\cdots)$, $H(\cdots)$ for

[9] For normalization in the continuous spectrum by the Weyl method cf. E. Fues, Ann. d. Physik 81, 281 (1926)

$F(-\epsilon\partial/\partial q,\ q)$, etc., we must have $F(\cdots)\ H(\cdots)=H(\cdots)\ F(\cdots)$. For, applying $H(\cdots)$ to both sides of (47) and $F(\cdots)$ similarly to (48) and comparing: $H(\cdots)\ F(\cdots)u=\kappa\ H(\cdots)u=\kappa Wu=WF(\cdots)u$ $=F(\cdots)\ H(\cdots)u$. This condition upon F severely restricts the number of mathematical quantities that can be measured physically in this manner; angular momentum is, however, among them.

Assuming that F and H do commute, application of $H(\cdots)$ to both sides of (47) yields the result that

$$F(\cdots)[H(\cdots)u]=\kappa[H(\cdots)u].$$

Thus, if u is a characteristic function of (47), $H(\cdots)u$ is likewise a solution of this equation, and, since we may safely assume that it will satisfy the same boundary condition of finiteness as u, it must also be a characteristic function. If only one such function exists for the value of κ in question, then we have $H(\cdots)u=Au$ where A is a constant, and u satisfies (48) with $W=A$. If κ is a multiple characteristic value, it is easily shown that a corresponding number of linear combinations of the functions belonging to κ can be constructed which are characteristic functions of (48) as well for certain values of W. In short, when F and H commute (47) and (48) have in general a common set of characteristic functions.

Turning now to Eq. (50) this has the same form as the wave-equation for a particle in one dimension subjected to a uniform force $\beta\kappa$(cf. (5) with $n=1$, $v=-\beta\kappa x$). We have seen above (below Eq. (16)) that such a force merely gives to the entire packet the classical displacement $\delta=\frac{1}{2}\beta\kappa t^2/M$. To be sure, the packet also spreads out in the x-direction, just as it would in the absence of the field, but this spreading is easily seen from equation (20) to be ultimately proportional to t, so that the uncertainty in δ due to it can be diminished at will by increasing t. We can then take

$$\kappa=2\delta M/\beta t^2, \tag{51}$$

calculated from observed values of δ and t and the known values of M and β, as a "physically observed" value of F.

Now, in general, we cannot assume that ψ_1 satisfies (45) to (47) initially. Probably we can, however, safely write the initial amplitude for $t=0$ in the form, $\psi_1{}^0(q)\psi_2{}^0(x)$, and we can then expand $\psi_1{}^0(q)$ in terms of the normalized characteristic functions $u(q,\ \kappa)$ of (47), thus:

$$\psi_1{}^0(q)=\sum\nolimits_n c_n u(q,\kappa_n)+\int c(\kappa)u(q,\kappa)d\kappa,$$

the series extending over the point spectrum and the integral over the continuous spectrum for κ. The complete packet at time t is then

$$\psi=\sum\nolimits_n c_n e^{2\pi i h^{-1}W_n t}u(q,\kappa_n)\psi_2(x,t,\kappa_n)+\int c(\kappa)e^{2\pi i h^{-1}Wt}u(q,\kappa)\psi_2(x,t,\kappa)d\kappa, \tag{52}$$

where $\psi_2(x,\ \kappa,\ t)$ is the solution of (50) for the value κ, chosen so as to reduce to $\psi_2{}^0(x)$ at $t=0$, and W_n or W has the value for which $u(q,\ \kappa)$ is a characteristic function of (48) as well.

The factor $\psi_2(x, \kappa, t)$ will then produce in each term of ψ or partial packet the displacement δ described above. The components of the initial packet corresponding to different values of κ will thus be spread out into a physical spectrum; those corresponding to the discrete values, κ_n, will finally become clearly separated from each other, while those belonging to the continuous spectrum will overlap with components belonging to a continually decreasing range of κ as time goes on. It is easily shown that the probability of finding the system in any particular component packet is equal to $c_n c_n^*$, or to $c(\kappa)c^*(\kappa)d\kappa$; also that the probability for particular values of q is $u(q,\kappa_n)u^*(q,\kappa_n)$ or $u(q, \kappa)u^*(q, \kappa)$, respectively.

A process of this general nature seems to be the justification, from the mechanical point of view, for the common assumption that the coefficients c_n and $c(\kappa)$ represent the probability amplitude for the quantity F.

If we put $F=H$, we have the case of a measurement of the "energy." In this case (47) and (48) both become the ordinary Schrödinger equation without the time factor.

7. *The Convervation of Energy and Angular Momentum.* This topic has been discussed by others but perhaps a few additional remarks from the present standpoint may not be out of place.

Let ψ be expanded in the series (52) with W in place of κ, u denoting characteristic functions of Schrödinger's equation, (48). Then if the system is isolated and conservative the quantities $c_n c_n^*$ or $c(W)c^*(W)$, representing probabilities for the energy, are independent of the time: this is the well-known equivalent of the Conservation of Energy for an isolated system.

But we can sometimes measure the energies of separate parts of a system, as of an electron and atom before and after a collision. To illustrate such a measurement let $H=H_1+H_2$ where H_1 and H_2 commute with each other and therefore also with H; then we might subject the system simultaneously to a force in the x-direction proportional to H_1, another in the y-direction proportional to H_2 and a third in the z-direction proportional to H and thereby measure by the above method H_1, H_2 and H all at the same time. In that case we should have three equations in place of (47), viz.

$$H_1(-\epsilon\partial/\partial q, q)u = W_1 u, \quad H_2(-\epsilon\partial/\partial q, q)u = W_2 u,$$

and

$$H_1(-\epsilon\partial/\partial q, q)u + H_2(-\epsilon\partial/\partial q, q)u = Wu,$$

which also represents (48). The partial packet that is brought to any given point in xyz space then represents energies W_1 and W_2 for the parts and W for the whole and it is clear from the equations that

$$W_1 + W_2 = W.$$

Thus, as is usually assumed, the conservation of energy holds exactly between a system and its parts.

Finally, if the potential energy V contains the time, we have classically

$$dH/dt = \partial V/\partial t.$$

On the new theory it can be shown that

$$d\overline{W}/dt = \int P(\partial V/\partial t)dq \tag{53}$$

where

$$\overline{W} = \sum_n c_n c_n^* W_n + \int c(W)c^*(W)W dW \tag{54}$$

and represents the "mean probable" value of the energy; the c's are the coefficients in (52) when $F=H$ and $\kappa = W$ and are functions of the time along with V. The proof of this theorem involves only a straightforward use of eqs. (2) and (48) and of the formulas for the coefficients in an expansion in terms of the u's and will not be given here.

Similar results can be obtained for any quantity F which commutes with the Hamiltonian H. The fact that the c's in (52), which are the probability amplitude for F, are independent of the time whenever H is, constitutes the quantum equivalent of the classical theorem that F is conserved during the motion of a conservative system.

An important case is that of angular momentum and magnetic moment, which has been discussed briefly by Oppenheimer[10] and Weyl[11] and more completely by Darwin[4] with special reference to the Stern-Gerlach experiment.

The theory of quantum mechanics has been presented in the present paper from a strictly mechanical point of view. This approach seems to be the most natural one, but it leads unfortunately to very few predictions that are within the reach of experimental verification. The field of atomic energy levels, in which the theory has so far achieved its principal successes, can be explored only with the aid of radiation, employed not as a geometrical clue to position but with reference to the process of its emission and absorption by matter. This process has been treated by Dirac,[12] but in a very abstract manner. The most urgent need at present seems to be a satisfactory theory of the electromagnetic field; when this has been found, we may hope that it will be capable of statement in a readily apprehended form similar to the theory which Schrödinger and Heisenberg have given us for mechanical phenomena.

CORNELL UNIVERSITY,
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[10] J. R. Oppenheimer, ZSP 43, **27** (1920).
[11] H. Weyl, ZSP 46, **1** (1927).
[12] P. A. M. Diroc, Roy. Soc. Proc. 114, **243** (1927).