THE DIFFUSION PROBLEM FOR A SOLID IN CONTACT WITH A STIRRED LIQUID

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Abstract

A cylindrical solid of length a in a direction x and arbitrary cross section normal to x is in contact on its plane face $x=a$ with a well stirred liquid. The face $x=0$ of the solid and the lateral surface are impervious to heat. The liquid extends from $x=a$ to $x=a+b$, there being no loss of heat across the face $x=a+b$. The initial temperature of the solid and liquid being given, a Volterra integral equation of the second kind with discontinuous kernel is obtained for the temperature of the liquid as a function of the time. The solution of this integral equation is obtained in terms of the roots of a transcendental equation and the roots of an infinite system of linear equations. By means of the theory of singular integral equations, it is shown that the differential equation and boundary conditions possess but one solution of required type, which solution is obtained from the integral equation. The connection of this problem with a case of material diffusion is shown, and a numerical illustration is given. The theory suggests a new method of determining directly and accurately the thermal conductivity of solids.

1. Statement of the Problem.

The problem here considered is that of the one-dimensional flow of heat which occurs when a solid, of length a in a direction x, is in contact, on the plane face $x=a$, with a well stirred liquid, the other face $x=0$ being impervious to heat. In order that the flow of heat be one dimensional, it is necessary, as in all similar problems, either that the solid and the liquid be each infinite in extent in directions normal to x, or that they be cylindrical bodies of the same cross-section whose lateral surfaces are impervious to heat. Let the liquid extend from $x=a$ to $x=a+b$, there being no loss of heat, by radiation or conduction, across the face $x=a+b$. Let $u(x,t)$ represent the temperature in the solid at any point x and time t, and let $v(t)$ represent the temperature of the liquid at any time t. The function $v(t)$ must be continuous for all values of t. Then

$$\alpha^2\partial^2u/\partial x^2=\partial u/\partial t, \quad t>0, \tag{1}$$

where α^2, the diffusivity of the solid, is the quotient of its thermal conductivity k by the product of its density ρ and specific heat c.
Also,

$$\partial u/\partial x=0, \quad x=0, \quad t>0, \tag{2}$$

and

$$\lim_{t \doteq 0} u(x,t)=u_0(x), \quad 0\leqq x\leqq a, \tag{3}$$

where $u_0(x)$ is the initial temperature of the solid. The thermal contact of the solid and the liquid at $x=a$ gives the condition,

$$u(a,t)=v(t), \quad t>0. \tag{4}$$

If the initial temperature of the liquid be v_0, then its temperature at time t is given by

$$v(t) = v_0 - \frac{k}{K} \int_0^t \left(\frac{\partial u}{\partial x}\right)_{x=a} dt,$$

where K is the heat capacity of a prism of the liquid of unit cross section and length b. From this equation one obtains, by differentiation, the condition

$$K\partial v/\partial t = -k(\partial u/\partial x)_{x=a}. \tag{5}$$

The question of the existence and uniqueness of the solution of Eqs. (1) to (5) will be considered in Section 3.

In an experiment in colloidal chemistry, a mass M of a material is uniformly distributed, at time zero, in a jelly of depth a. This jelly is covered with a depth b of water which is kept stirred. The material diffuses through the jelly and into the water, where its concentration is determined as a function of time by analyzing samples. It is desired to determine the coefficient of diffusion of this material in the jelly from the experimental time—concentration curve of the liquid. If one assumes this problem to be entirely analogous to the thermal problem stated above, the solution of the concentration—diffusion problem can be obtained at once from the solution of the heat problem. For by comparison of the fundamental thermal equation

$$\Delta(\text{temperature}) = \Delta(\text{heat})/c\rho V,$$

where ρ is density, V volume, and c specific heat; and the equation

$$\Delta(\text{concentration}) = \Delta(\text{mass})/V$$

it is seen that one need only replace ρc by 1 to pass from one problem to the other. *Thus when u and v are interpreted to mean concentrations, α^2 is to be replaced by k, and K is to be replaced by b.*

It is not proposed to discuss here the legitimacy of the identification of the two problems. It is, however, clear that Eq. (4) is the doubtful one. If it is found that, when a steady state has been finally reached, the concentration of the diffusing material is the same in the jelly and in the liquid, then this experimental fact furnishes reasonably convincing evidence for this assumption. For it indicates that there is indeed equilibrium when the concentrations are equal in the liquid and in the jelly; and sufficiently thin layers of the jelly and liquid, adjacent to the plane of contact, may be assumed to be in continuous equilibrium as the diffusion proceeds. Actual stirring of the liquid cannot, of course, assure at all moments a uniform concentration in the liquid down to the surface of the jelly, but the diffusion proceeds slowly, in an actual case, so that the assumed conditions are closely approximated.

In explanation of the method of solution which occurs in the next section, it may be remarked that the characteristic functions of (1) under the conditions (2) and (5) are not orthogonal; so that the usual method meets a difficulty in attempting to satisfy (3). It should also be noted that Eqs. (1), (2), (3), (4), and (5) are satisfied, for the special case $u_0(x)=u_0$, a constant, by the values $u=u_0$, $v=u_0$. If $u_0 \neq v_0$, it is obvious that the values just written do not furnish a solution of the physical problem. The mathematical condition which is not met, is the condition that $v(t)$ be a continuous function for all values of t including $t=0$. The method of solution must proceed in such a way as to safeguard this required continuity of $v(t)$.

2. Solution of the Problem.

In this section a solution of Eqs. (1) to (5) inclusive will be obtained, neglecting for the moment certain questions of rigor. These questions will be raised and answered in the succeeding section.

Let

$$u=u_1+u_2,$$

where u_1 and u_2 satisfy (1) and the conditions

$$\partial u_1/\partial x=0, \quad x=0: u_1(a,t)=0, \quad t>0 \quad ; \quad \lim_{t \doteq 0} u_1(x,t)=u_0(x). \tag{7}$$

$$\partial u_2/\partial x=0, \quad x=0: u_2(a,t)=v(t), \quad t>0: \quad \lim_{t \doteq 0} u_2(x,t)=0. \tag{8}$$

The function u_1 is given by the known formula

$$u_1=\sum_{n=0}^{\infty} A_n e^{-\alpha_n t} \cos \mu_n x \tag{9}$$

where $$\alpha_n=\alpha^2\mu_n^2 \; ; \quad \mu_n=(2n+1)\pi/2a \; ; \quad A_n=2a^{-1}\int_0^a u_0(x) \cos \mu_n x dx. \tag{10}$$

The Fourier expansion here used which involves only odd multiples of $\pi/2a$ can be readily obtained by writing an ordinary cosine expansion in the interval from $x=0$ to $x=2a$ of a function $u(x)$ defined by $u(x)=u_0(x)$, $0 \leqq x \leqq a$; $u(x)=-u_0(2a-x)$, $a \leqq x \leqq 2a$.

If, in particular, $u_0(x)$ be a constant, say u_0, then

$$A_n=\frac{4}{\pi}\,\frac{(-1)^n}{2n+1}\,u_0. \tag{11}$$

Now let $H(x,t)$ be the temperature which the solid would have, at a point x and time t, if it were initially at zero temperature, if its face $x=a$ were maintained at temperature unity, and if its face $x=0$ were impervious to heat. It is easily seen that the function

$$H(x,t)=1-\frac{4}{\pi}\sum \frac{(-1)^n}{2n+1} e^{-\alpha_n t} \cos \mu_n x \tag{12}$$

satisfies these requirements. The function u_2 is now given, in terms of $H(x,t)$ by Duhamel's theorem,[1] as

$$u_2 = \int_0^t v(\tau)\frac{\partial}{\partial t}H(x,t-\tau)d\tau = -\int_0^t v(\tau)\frac{\partial}{\partial \tau}H(x,t-\tau)d\tau,$$
$$= v_0 H(x,t) + \int_0^t \left(\frac{\partial v}{\partial \tau}\right) H(x,t-\tau)d\tau. \tag{13}$$

Let the expressions for u_1 and u_2 be added to form u, and the value of $H(x,t)$ be inserted from (12). It then follows that

$$\left(\frac{\partial u}{\partial x}\right)_{x=a} = -\frac{K}{k}\sum_0^\infty C_n e^{-\alpha_n t} + \frac{2}{a}\int_0^t \frac{\partial v}{\partial \tau}\sum_0^\infty e^{-\alpha_n (t-\tau)} d\tau, \tag{14}$$

where
$$C_n = \kappa\left[\tfrac{1}{4}A_n(-1)^n(2n+1)\pi - v_0\right], \quad \kappa = 2k/aK. \tag{15}$$

In obtaining (14) it has been assumed legitimate to perform the differentiation with respect to x under the sign of integration and the sign of summation. This point will be later considered. In case $u_0(x) = u_0$, a constant, Eq. (15) reduces to the special form

$$C_n = \kappa(u_0 - v_0). \tag{15'}$$

If, now, (14) be substituted in (5) the result is

$$F(t) = f(t) - \int_0^t F(\tau)K(t-\tau)d\tau, \tag{15}$$

where $\quad f(t) = \sum C_n e^{-\alpha_n t}; \qquad K(t) = \kappa\sum e^{-\alpha_n t}; \qquad F(t) = \partial v/\partial t. \qquad (17)$

Unless otherwise stated, sums such as (17) are to be understood throughout the paper to extend from $n=0$ to $n=\infty$.

The Volterra integral Eq. (16) can be solved by an immediate extension of the method developed by Whittaker[2] for the numerical solution of an equation whose kernel is expressed as a finite sum of exponentials. If, in fact, one substitutes in (16) the assumption

$$F(t) = \sum \dot{B}_i e^{-\beta_i t} \tag{18}$$

the resulting equation will be satisfied if the coefficients of $e^{-\beta_i t}$ and of $e^{-\alpha_n t}$ are, for all i and n, the same on the two sides of the equation. Of the two conditions so obtained, the first demands that each β_i satisfy the equation

$$1 + \sum_{n=0}^\infty \frac{\kappa}{\alpha_n - \beta_i} = 0. \tag{19}$$

That is, the numbers β_i are the roots of the transcendental equation

$$1 + \sum \frac{\kappa}{\alpha_n - y} = 0. \tag{20}$$

[1] Carslaw, The Conduction of Heat, 1921, p. 17.

[2] E. T. Whittaker, On the Numerical Solution of Integral Equations. Proc. Royal Soc. London **94A,** 367–383 (1917).

The formula

$$\tan z = \frac{2z}{(\pi/2)^2 - z^2} + \frac{2z}{(3\pi/2)^2 - z^2} + \frac{2z}{(5\pi/2)^2 - z^2} + \cdots$$

permits (20) to be rewritten as

$$\tan z + \lambda z = 0. \tag{21}$$

where

$$y = \alpha^2 z^2/a^2, \quad \lambda = \alpha^2 K/ak = K/a\rho c. \tag{22}$$

It may be noted that λ is the ratio of the heat capacity of the liquid to that of the solid per unit area normal to x.

In the case of the material-diffusion problem, Eqs. (22) are to be replaced by

$$y = kz^2/a^2, \quad \lambda = b/a. \tag{22'}$$

The second condition, referred to above, demands that

$$C_n + \kappa \sum_{i=0}^{\infty} \frac{B_i}{\alpha_n - \beta_i} = 0, \quad n = 0, 1, 2, \cdots. \tag{23}$$

Thus (18) is a solution of (16) if the constants β_i are determined from (21) and (22) and the constants B_i then determined from (23).

Eq. (21) is a generalization of the equation

$$\tan z = z \tag{24}$$

met in many problems of applied mathematics. The method used by Lord Rayleigh[3] to obtain the roots of Eq. (24) can be applied to the more general Eq. (21). There results, as a formula for the n'th positive root (which is numerically equal to the n'th negative root) the equation

$$Z_n = W_n + w_n \tag{25}$$

where

$$W_n = \tfrac{1}{2}(2n+1)\pi \tag{25'}$$

and

$$w_n = \frac{1}{\lambda W_n} - \frac{3\lambda+1}{3\lambda^3 W_n^3} + \frac{30\lambda^2+20\lambda+3}{15\lambda^5 W_n^5} - \frac{1575\lambda^3+1575\lambda^2+483\lambda+45}{315\lambda^7 W_n^7} + \frac{39{,}690\lambda^4+52{,}920\lambda^3+24{,}696\lambda^2+3{,}834\lambda+9}{2835\lambda^9 W_n^9} - \cdots \tag{25''}$$

In Fig. 1 the first four roots, z_0, z_1, z_2, z_3, of Eq. (21) are shown, for values of λ between 1 and 5. The roots beyond the fourth for these values of λ, and any roots for higher values of λ, may be calculated very easily from the equations (25). For smaller values of λ the formula converges very slowly, and the roots can be found more easily by a combination of a graphical and trial and error method. The discussion of this paper will be limited to the case $\lambda \geqq 1$.

[3] Rayleigh, Theory of Sound. 1896. Vol. I, p. 334.

Eqs. (23), from which the numbers B_i are to be determined, form an infinite system of linear equations. In Section 3 it will be shown that this system possesses a unique bounded solution, and in Section 4 a convenient method is given for the actual computation of the numbers B_i.

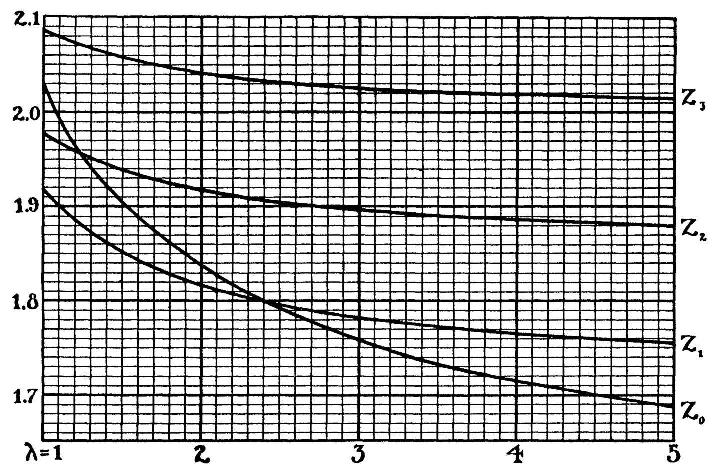

Fig. 1. This figure gives the first four roots, z_0, z_1, z_2, z_3, of the equation $\tan z+\lambda z=0$, for values of λ ranging from 1 to 5. For the curve z_0 the vertical scale reads as shown. For z_1, the values read from the curve should be increased by 3: for z_2, add 6: for z_3, add 9.

From (18) and (17)

$$\partial v/\partial t=\sum B_i e^{-\beta_i t}, \tag{26}$$

so that

$$v=D_0-\sum (B_i/\beta_i)e^{-\beta_i t}.$$

Now the total heat which a cylinder of the solid of unit cross section will have lost in cooling from its initial temperature $u_0(x)$ to a final uniform steady temperature u_∞ is

$$\rho ca(\bar{u}-u_\infty), \text{ where } \bar{u}=a^{-1}\int_0^a u_0(x)dx.$$

The total heat which a similar cylinder of the liquid will have gained is $K(v_\infty-v_0)$, where, moreover, $v_\infty=u_\infty$. Thus $v_\infty=D_0=(\bar{u}+\lambda v_0)/(1+\lambda)$.

Consequently

$$v=\frac{\bar{u}+\lambda v_0}{1+\lambda}-\sum \frac{B_i}{\beta_i}e^{-\beta_i t}. \tag{27}$$

For $t=0$, this equation reads, after reduction,

$$\sum (B_i/\beta_i)=(u-v_0)/(1+\lambda). \tag{28}$$

It must be possible to establish this surprisingly simple value for the sum of the quotients of B_i by β_i directly from Eqs. (19) and (23). Apart from its intrinsic interest, Eq. (28) furnishes a very convenient and convincing check on computed values of B_i and β_i.

If (26) be now substituted in (13), the resulting value of $u=u_1+u_2$ is given by

$$u=u_1+v_0H+\int_0^t \sum_{i=0}^{\infty} B_i e^{-\beta_i\tau}\left[1-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}e^{-\alpha_n(t-\tau)}\cos\mu_n x\right]d\tau.$$

The term obtained, from the integral, when the factor 1 of the square bracket is used, is $v-v_0$. The remaining portion of the integral may be written

$$-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}e^{-\alpha_n t}\cos\mu_n x\sum_{i=0}^{\infty}B_i\int_0^t e^{(\alpha_n-\beta_i)\tau}d\tau$$

$$=-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\mu_n x\sum_{i=0}^{\infty}\frac{B_i e^{-\beta_i t}}{\alpha_n-\beta_i}+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\mu_n x e^{-\alpha_n t}\sum_{i=0}^{\infty}\frac{B_i}{\alpha_n-\beta_i}$$

The second of these last written two terms may be simplified. In fact, using (23) and (17), this second term may be rewritten as

$$-\sum_{n=0}^{\infty}A_n\cos\mu_n x e^{-\alpha_n t}+\frac{4v_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\mu_n x e^{-\alpha_n t}=-u_1-v_0[H(x,t)-1].$$

Thus,

$$u=v-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\mu_n x\sum_{i=0}^{\infty}\frac{B_i e^{-\beta_i t}}{\alpha_n-\beta_i}. \tag{29}$$

In many problems of applied mathematics, the solution is expressed as an infinite sum of terms each of which satisfies the differential equation but not the boundary conditions. In the Ritz[4] metnod, the solution appears as a sum of terms each of which satisfies the boundary conditions but not the differential equation. It is interesting to note that u is here expressed as a sum of functions none of which separately satisfies either the differential equation or the boundary conditions. It may be checked, without difficulty, that the function u and v given by (29) and (27) do, by virtue of (19) and (23), satisfy all the conditions (1) to (5).

3. Analytical Details.

(A) *Existence and uniqueness.* It will now be shown that there exists one and only one solution $u(x,t)$ and $v(t)$ of conditons (1) to (5) for which v is a continuous function of t and for which u and $\partial u/\partial x$ are continuous functions of x and t in a region $0\leqq x\leqq a$, $0<\eta\leqq t\leqq T$, where η is an arbitrarily small quantity, and T is as large as one pleases.

We will refer to Eqs. (1) to (5) as Conditions I, and to Eqs. (1), (2), (3), (16) and equation

$$k(\partial u/\partial x)_{x=a}=-KF(t), \quad t>0 \tag{30}$$

as Conditions II. It will be shown that Conditions II permit one and only one solution of the required type, and that any solution of Conditions I is also a solution of Conditions II.

[4] W. Ritz, Gesammelte Werke, p. 251.

Since $\partial u/\partial x$ is to be a continuous function of t, so, also, by (30), is $F(t)$. The kernel $K(t-\tau)$ of the integral Eq. (16) is not continuous for $t=\tau$. It may be readily seen, however, that this equation comes under the theory developed by Evans;[5] thus (16) permits one and only one continuous solution $F(t)$, $0<\eta\leqq t\leqq T$. The function $F(t)$ being uniquely determined by (16), conditions (1), (2), (3), (30) are then a usual set of conditions for a thermal problem, and are known to permit one and only one solution $u(x,t)$.

It remains to show that a solution of I is necessarily a solution of II. That is, it is necessary to show that (16) follows from conditions I. Now if there is any function $F(t)=\partial v/\partial t$, then u is expressible uniquely in terms of $\partial v/\partial \tau$ and v_0 by means of Eqs. (1) to (4) as $u=u_1+u_2$, where u_1 is given by (9) and u_2 by (13). Eq. (16) follows from this formula and from (5) if it is permissible, in forming $\partial u/\partial x$, to differentiate u_1 under the sign of summation, and u_2 under the sign of integration and summation. Since the discussion is restricted to the range $0<\eta\leqq t\leqq T$, this question can be answered affirmatively at once for u_1, and also for the first term of u_2. It is thus only necessary to justify the equation

$$\frac{\partial}{\partial x}\int_0^t \frac{\partial v}{\partial \tau}H(x,t-\tau)d\tau=\frac{2}{a}\int_0^t \frac{\partial v}{\partial \tau}\sum_0^\infty(-1)^{n+1}e^{-\alpha_n(t-\tau)}\sin\mu_n x d\tau. \quad (31)$$

That this equation holds can be shown without difficulty.[6]

(B) *Existence and uniqueness of the roots of the infinite system of equations* (23). To investigate the convergence of the infinite system of Eqs. (23), from which the coefficients B_i are to be calculated, it is convenient to rewrite this system in the form

$$\sum b_{ni}\,D_i\equiv\sum_{i=0}^{\infty}\frac{m_n-(2n+1)^2}{m_i-(2n+1)^2}D_i=[m_n-(2n+1)^2]C_n\equiv C_n', \quad (32)$$

where $$D_i=4a^2\kappa B_i/\pi^2\alpha^2,\quad m_i=4a^2\beta_i/\pi^2\alpha^2=4z_i^2/\pi^2. \quad (33)$$

The system (32) possesses, according to a theorem by von Koch[7] a unique bounded solution for the D_i provided that, for all n,

$$|C_n'|<C, \quad (34)$$

$$\sum_{i=0}^{\infty}{}'|b_{ni}|<1, \quad (35)$$

where C is a positive constant, and where the prime on the summation sign indicates that the term $i=n$ is to be omitted.

It may be shown without difficulty that these conditions are satisfied by the quantities b_{ni} and C_n' defined by (32).

(C) *Convergence of double series*. In obtaining (29) and in showing that the differential Eq. (1) and the boundary conditions (3) and (5) are satisfied, certain double series were rearranged. Due to the theorem referred to in

[5] G. C. Evans, The Integral Equation of Volterra of the Second Kind with Discontinuous Kernel. Trans. Amer. Math. Soc. **11** (1910), 393–413.

[6] Carslaw, Fourier Series and Integrals (1921), p. 156.

[7] V. Koch, Jahresbericht d. Deut. Math. Ver. **22**, 289 (1913).

section 3 (A), it is only necessary to investigate the absolute convergence of the series for $0<\eta\leqq t$ and $\tau<t$. Under these simplifying restrictions, the absolute convergence may be readily established.

4. Numerical Example.

To illustrate the computations involved, the solution will now be obtained for the case of diffusion of material, initially uniformily distributed with a concentration which will be called unity, from a solid of length a into a liquid region, initially free of the diffusing material, of the same length a. Then

$$u_0(x)=1, \quad v_0=0, \quad K=b=a, \quad C_n=2k/a^2.$$

For $\lambda=1$ the first eight roots of Eq. (21) are,

$$z_0=\ 2.0288, \quad z_1=\ 4.9132, \quad z_2=\ 7.9787, \quad z_3=11.0855$$
$$z_4=14.1775, \quad z_5=17.3364, \quad z_6=20.4692, \quad z_7=23.6034.$$

These values are, by (22′) to be squared and multiplied by k/a^2 to produce the values β_i. Now Eqs. (23) for the determination of B_i may be written, since the constants C_n do not depend on n, in the form

$$\sum_{i=0}^{\infty}\frac{m_n-(2n+1)^2}{m_i-(2n+1)^2}\,B_i'=\left[m_n-(2n+1)^2\right], \tag{36}$$

where

$$B_i'=4a^2B_i/\pi^2k \tag{37}$$

The actual computation of the numbers B_i' is simplified by the observation that as i increases the constants B_i' all approach the value which the right-hand member of (36) approaches as n increases, namely, $8/\lambda\pi^2$; or, in this example, $8/\pi^2$. In fact, the coefficient of B_n' in the n'th equation is unity, while the coefficients off the principal diagonal become more nearly skew symmetric as n increases. Moreover, the coefficients rapidly decrease in absolute value as one goes from the principal diagonal. The value of the left member therefore arises almost entirely from the principal diagonal term, the other terms almost cancelling, pair by pair, and being, moreover, very much smaller even if they did not sensibly cancel. The evalution of the numbers B_i' was thus accomplished by tentatively assuming that all B_i' $i>3$, were sensibly equal to $8/\pi^2$. The first four equations (36) were then written

$$\sum_{i=0}^{3}\frac{m_n-(2n+1)^2}{m_i-(2n+1)^2}B_i'=\left[m_n-(2n+1)^2\right]-\frac{8}{\pi^2}\sum_{i=4}^{\infty}\frac{m_n-(2n+1)^2}{m_i-(2n+1)^2}\cdot \tag{38}$$

The value of the sum on the right can be readily computed to any desired degree of accuracy by summing, term by term, until the ratio of

$$1/\left[m_i-(2n+1)^2\right] \text{ and } 1/m_i$$

is as nearly unity as is desired, and then noting that, for large i, we have $m_i\doteq(2i+1)^2$.

If the B_i', $i>3$, all differ from $8/\pi^2$ by less than an amount δ, the difference between the value of the last member of (38) when computed as just suggested, and its true value is less, in the case of the first equation, than

$$\frac{8\delta}{\pi^2}[m_n-(2n+1)^2]\sum_{i=4}^{\infty}\frac{1}{(2i+1)^2-(2n+1)^2}=0.036\delta.$$

The values of the first four roots B' as actually calculated from (38) are 0.545, 0.748, 0.789, 0.799. The last of these differs from $8/\pi^2$ by $\delta=.012$. The assumed right member of the first Eq. (38) thus differs from its correct value by less than (0.036) (.012) = 0.0004. Since the right member is approximately 0.668 a correction of the order just computed would have no effect on the significant figures retained.

The first four roots of Eqs. (36) were computed, in this way, for values of λ ranging from $\lambda=1$ to $\lambda=5$. These values are shown in Fig. 2. In order

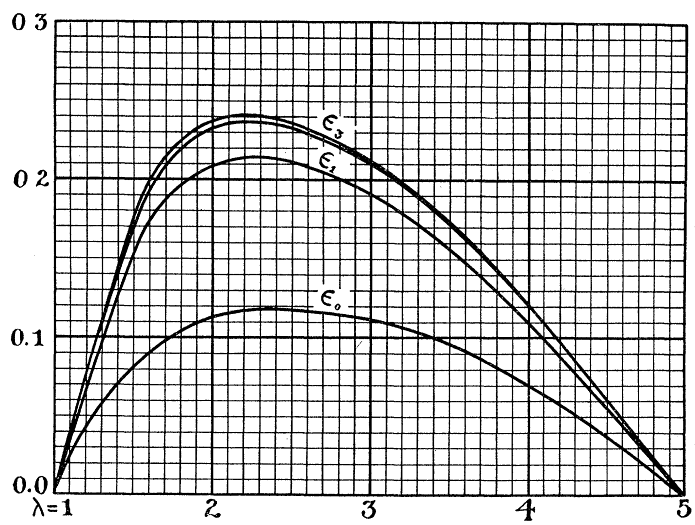

Fig. 2. From the values of ϵ_i given on this figure, the values of B_i' (see equation 36) can be computed from the formulas:

$$B_0'=-0.099\lambda+0.644-\epsilon_0$$
$$B_1'=-0.146\lambda+0.894-\epsilon_1$$
$$B_2'=-0.157\lambda+0.946-\epsilon_2$$
$$B_3'=-0.159\lambda+0.958-\epsilon_3$$

The values of B_i', where $i>3$ can all be computed from the formula,

$$B_i'=-0.162\lambda+0.972-\epsilon_3$$

to represent these values more accurately on a figure of reasonable dimensions, the values of B_i' are not shown directly, but rather the amounts ϵ_i by which the values of B_i' differ from a linear dependence on λ.

Eq. (28) may now be used to check these computed values of β_i and B_i. In fact, for the special case under consideration, (28) demands that

$$\sum(\pi^2B_i'/4z_i^2)=\tfrac{1}{2}.$$

If one uses the values of z_i and B_i' given above, for $i<4$, and, for $i>4$, the values $4z_i^2/\pi^2=(2i+1)^2$, $B_i'=8/\pi^2$, the left member will be found to check the right member to the fourth decimal place.

Eq. (27), for the concentration in the liquid, now reads

$$v=\tfrac{1}{2}-[0.327e^{-4.117T}+0.0766e^{-24.14T}+0.0306e^{-63.68T}+0.0160e^{-123T}+\cdots], \quad (39)$$

where $T=kt/a^2$.

The diffusion problem, namely, to determine the coefficient of diffusion k from the experimental time-concentration curve for the liquid, is now easily solved. The coefficient k fortunately enters Eq. (46) only as a multiplier of t.[8] Thus, if (39) be plotted and the values of T and t be observed corresponding to the same value v on the graph of (36) and on the experimental curve respectively, then $k=a^2T/t$.

In Fig. 3 is shown a graph of (39). The circles are values observed in the diffusion of urea from a gel into a layer of water. The experimental research from which these data were taken will be published elsewhere. The points are included here only to show that the theoretical result obtained is in close agreement with the facts.

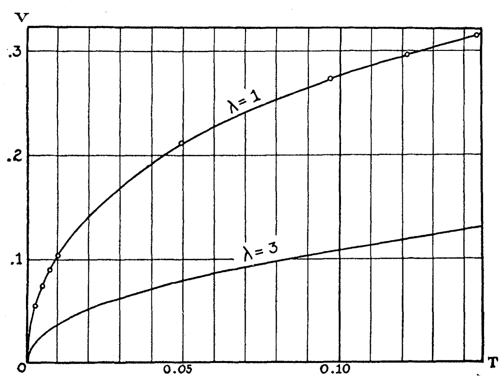

Fig. 3.

The particularly simple determination of the coefficient of diffusion from the concentration-time curve for the liquid suggests that a convenient method of determining coefficients of thermal diffusion could be based on such a method. The required data, namely, the temperature of the stirred liquid as a function of time, could be obtained much more easily than is the case with the data for many methods for such determinations. The boundary conditions on the face $x=a$ could, moreover, be met with a high degree of accuracy, the thermal convection in the liquid aiding materially in the stirring process in the layer of the liquid next to the solid.

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April 1, 1928.

[8] That this would be the case is obvious from equations (1) and (5).