A THEORY OF THE ELECTRIC DISCHARGE THROUGH GASES

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Abstract

Three general differential equations are set up which determine the average behavior of a discharge of electricity through a gas. Approximate solutions, giving the electric field E and the concentration of electrons and positive ions, n_1 and n_2, at any distance x from the cathode, are found for several ranges of value of E.

When E is large, the solution corresponds to the conditions in the *cathode* and *anode fall spaces* in a glow discharge. Equations are obtained for the potential drop V across the fall space; the current density at the electrode divided by the square of the gas pressure, j/p^2; and the thickness of the fall space times the pressure, pd.

These equations indicate that for the cathode fall space there is a certain minimum value of V, called V_n; and for j/p^2, called j_n/p^2; and a corresponding maximum value of pd, pd_n, beyond which values the discharge ceases. These stationary values are shown to be constants, dependent only on the nature of the gas used and of the cathode material, and correspond to the *normal cathode fall space.* The equation determining V_n is shown to be of the right form by comparison with the experimentally determined values. From these values of V_n, values of j_n/p^2 and of pd_n are calculated for four gases and four cathode materials, and the calculated values check with the experimental data. The corresponding equations for the *anode fall space* show why there is no corresponding normal anode fall.

A consideration of the discharge when E is large throughout the distance between electrodes indicates that there is another stationary value of the cathode fall space when the current density at the cathode reaches its maximum possible value. The V in this case is much smaller than the V_n for the glow discharge, and the form of the equations indicate that they describe the conditions in an *electric arc.*

Another approximate equation is obtained when E is constant, which is the case in the *positive column* of a glow discharge. This solution indicates that within certain limits of pressure and current density, small sinusoidal variations about the average value E_p, are possible in E. These correspond to the *striations* sometimes observed in the positive column. The equations determining E_p and those determining the distance between striations check with the known empirical laws relating these amounts to the pressure, the radius of the discharge tube and the critical potentials of the gas used. A general discussion is given of the *Faraday dark space* and reasons are given why it should be near the cathode rather than the anode.

WHEN electricity passes through a gas from one plane electrode to another parallel to it, the phenomena taking place are many and complicated. A certain number of positive ions falling on the cathode per second will, by some undetermined mechanism set free an electron which will travel to the anode, ionizing and exciting the gas molecules in its path. In order to maintain the discharge in a steady state, this electron must set free just enough ions in a second to cause another electron to be ejected from the cathode; or else, if any excess over this requisite number is created, this excess must disappear by recombination before striking the cathode.

It seems probable that this ratio between the number of positive ions striking the cathode per second and the number of electrons ejected per second (i.e., the ratio between the positive ion current to the cathode and the electron current from it) is dependent on the nature of the gas in the tube, on the material of the cathode, and on the velocity of the positive ions striking the cathode. This, of course is not true if the cathode is emitting photo- or thermo-electrons independent of any positive ion bombardment, as in the electric arc or with a heated cathode.

At any point in the body of the discharge, ions are being produced by electron impact, are disappearing by recombination, are diffusing away, and are being carried away by the electric field. This whole set of reactions is limited by Poisson's relation between the concentration of ions and the variation of the electric field.

The complete solution of this phenomenon is, of course, impossible, primarily because the gas is a discontinuous material, and we have no means of determining the exact velocity of any ion at any place under any field. However, average values for this velocity have been developed which have considerable accuracy, and likewise statistical relations between the average number of ions caused by an electron and the electric field are known.

If we assume that these relations for mobility and for ionization account for all these actual random motions, we can consider that the gas is a continuous fluid, and that each ion has a definite terminal velocity at any point in the tube, and the problem becomes much simpler. This statistical method of solution is probably quite accurate, if the smallest lengths considered are of a larger order than the length of a mean free path, and if the concentration of ions be fairly large. The shortest distance we shall find necessary to consider is of the order of 50 to 100 mean free path lengths, and the smallest concentration of ions is about 10^7, so the statistical method is quite justifiable.

If n_1 and n_2 are the concentration of electrons and positive ions at any point distant x from the cathode, v_1 and v_2 their respective average drift velocities, and E is the field strength at that point, then the following relations can be set up for a stable discharge in a tube of unit cross section:

$$dE/dx = 4\pi e(n_1 - n_2) \tag{1}$$

$$n_1 v_1 \alpha - R + D_1 d^2 n_1/dx^2 - d(n_1 v_1)/dx = 0 \tag{2}$$

$$n_1 v_1 \alpha - R + D_2 d^2 n_2/dx^2 - d(n_2 v_2)/dx = 0 \tag{3}$$

The first equation is that of Poisson with the sign changed since we are reckoning the direction from cathode to anode as positive. The last two equations represent the fact that for equilibrium, the number of ions formed per second per cc, the number disappearing by recombination, the number diffusing away, and the number being pulled away by the electric field, must add up to zero.

The term α in Eqs. (2) and (3) is the number of ions formed per cm path by an electron travelling under a field E, and with a corresponding velocity v_1. A quite accurate relation between α and E has been developed by Townsend,[1] and is that

[1] Townsend, "Electricity in Gases," Chapter VIII.

$$\alpha = pN\epsilon^{-pNv_0/E} = pNz \tag{4}$$

D_1 and D_2 are the diffusion constants for positive and negative ions, and are proportional to their mobilities, μ_1 and μ_2 of the respective ions[2] according to

$$D_1 = D\mu_1 \; ; \; D_2 = D\mu_2 \tag{5}$$

At low field strengths $D = kT/e$, but for high fields, D is probably dependent on the gas in the tube.

R is the number of ions which recombine or disappear in some way from the discharge per cc. There are a certain number recombining by collision, but it has been shown[3] that the greater number of recombinations take place at the walls of the tube. However as R is small it need only be considered when dealing with the positive column or the Faraday dark space, so its discussion will be postponed until then.

The velocities will equal the mobilities times the field strength. Inasmuch as E will in most cases be large, the most accurate value[4] for these mobilities will be

$$\mu_2 = 0.815\left(\frac{e\lambda_2}{Ep(M^2/1.134)^{1/2}}\right)^{1/2} = \frac{u}{(Ep)^{1/2}}$$

$$\mu_1 = 0.815\left(\frac{e\lambda_1}{Ep(Mm/1.134)^{1/2}}\right)^{1/2} = \frac{au}{(Ep)^{1/2}}\,. \tag{6}$$

In the case of the Faraday dark space, however, as we shall see later, E is very small. In this case, the most accurate values of the mobilities will be

$$\mu_2' = 0.815\frac{e\lambda^1}{(2MKT)^{1/2}} = \frac{u'}{p}\; ; \quad \mu_1' = a'u'/p \tag{7}$$

λ_1 and λ_2 are the mean free paths of the electron and of the positive ion at 0°C and 1 mm pressure, and p is the pressure in mm of mercury. The terms a and u are dependent only on the gas used; u is the constant factor in the mobility of the electron, which is fairly accurately known; a is the ratio between the mobility of the electron and of the positive ion. It is not necessary to know this accurately, since we shall see later that if it is large enough that $a+1$ or $a-1$ are practically equal to a, this factor cancels out.

Making these substitutions, Eqs. (2) and (3) become

$$aNpzn_1E^{1/2} - Rp^{1/2} + \frac{aD}{E^{1/2}}\frac{d^2n_1}{dx^2} - a\frac{d(n_1E^{1/2})}{dx} = 0 \tag{8}$$

$$aNpzn_1E^{1/2} - Rp^{1/2} + \frac{D}{E^{1/2}}\frac{d^2n_2}{dx^2} + \frac{d(n_2E^{1/2})}{dx} = 0 \tag{9}$$

[2] J. J. Thompson, Conduction of Electricity Through Gases, Second Edition, pp. 42 and 43.
[3] W. Schottky, Phys. Zeits. **25**, pp. 635 (1924).
[4] K. T. Compton, Phys. Rev. **22**, 333 (1923).

The two, with Eq. (1), will completely determine n_1, n_2 and E for any x and for any boundary conditions.

A complete solution of these equations is impracticable. However, the three most interesting parts of the glow discharge (the cathode and anode falls, the Faraday dark space, and the positive column), and the whole of such discharges as the electric arc, are really special cases of the general solution. In the fall spaces in the glow discharge and in the arc, E is quite large; in the Faraday dark space E is very small; and in the positive column E, n_1 and n_2 are comparatively constant.

Case I. When E is large Eq. (8) becomes $d(n_1E^{1/2})/dx = Npzn_1E^{1/2}$, or $n_1E^{1/2} = -c\epsilon^{pN\int zdx} = -c\epsilon^{pNx}$, since z is practically unity for large values of E. From Eq. (9) $n_2E^{1/2} = ac\epsilon^{pNx}$. From these two equations and from Eq. (1)

$$E = c'\epsilon^{2pNx/3} + c_0 \tag{10}$$

This would be the form of the field distribution at the anode. By turning the tube around, solving, and then returning to our original coordinates, with $x=0$ at the cathode, after solving we obtain,

$$E = c''\epsilon^{-2pNx/3} + c_0' \tag{11}$$

as the form of the field distribution near the cathode. This same form of distribution has been obtained by other methods by Compton and Morse.[5] They obtained as the exponential factor $kpNx$, where k was about 0.6. The two methods thus check each other.

The Cathode Fall

Observations have shown that the greatest fall in potential across the discharge occurs close to the cathode. At the end of this fall the electric field comes to its minimum value, and the gas becomes luminous. Up to certain currents this space assumes an equilibrium state; i.e., the potential drop across the space is a constant independent of the pressure, its thickness is inversely proportional to the pressure, and the current density at the cathode is proportional to the square of the pressure, the current concentrating on a patch of this density. This state is called the *normal cathode fall.*

As the total potential across the tube is increased, the patch of cathode receiving current increases, but otherwise conditions across the fall remain the same, until the patch has covered the whole cathode. A further increase in applied potential increases the current density, increases the drop in potential across the space, and decreases its thickness slightly. This state is called the *abnormal cathode fall.*

If V_n is the normal fall in potential across the space, V_a the abnormal, j_n and j_a the normal and abnormal current densities, and d_n and d_a the respective thicknesses of the space, then:

[5] K. T. Compton and P. M. Morse, Phys. Rev. **30**, 305 (1927).

$$V_n = \text{constant} \; ; \; V_a > V_n \tag{12}$$

$$j_n/p^2 = \text{constant} \; ; \; j_a > j_n \tag{13}$$

$$pd_n = \text{constant} \; ; \; d_a < d_n \tag{14}$$

where these constants are determined by the gas and the cathode material. These empirical relations will serve to check the solution.

If we assume values for n_1, n_2 and E from Eq. (11), and substitute these in some modification of Eqs. (8) and (9), values of the constants c should be obtained which will hold approximately throughout the cathode fall. Assume that $n_1 = A - (A - n_1^0)\epsilon^{-2pNx/3}$, $n_2 = A + a(A - n_1^0)\epsilon^{-2pNx/3}$ where n_1^0 is the electron density just outside the cathode. From Poisson's equation [Eq. (1)], and from these two

$$\frac{dE}{dx} = -4\pi e(a+1)(A - n_1^0)\epsilon^{-2pNx/3}$$

Thus our assumption fits the requirements of Eq. (11), and also fits the conditions required at the cathode. Substituting these in the equation formed by subtracting Eq. (9) from Eq. (8), it is found that

$$A = \frac{4}{9\pi} \frac{Dp^2N^2}{e(a+1)} \tag{15}$$

Therefore if we can determine the value of D, N and a and n_1^0 the ion concentrations and field can be determined at any point in the cathode fall. The electron concentration at the cathode n_1^0, is naturally of the same dimensions as A, so we can let it equal Ac where c is some constant representing the electron emission of the cathode, and probably dependent only on the cathode material. Then the three equations become:

$$n_1 = A\left[1 - (1-c)\epsilon^{-2pNx/3}\right] \tag{16}$$

$$n_2 = A\left[1 + a(1-c)\epsilon^{-2pNx/3}\right] \tag{17}$$

$$E = (8/3)DpN(1-c)\left[\epsilon^{-2pNx/3} + E_a\right] \tag{18}$$

where E_a is a constant of integration to be determined later.

The current density of electrons from the cathode will be $e\mu_1 E n_1$, evaluated at the cathode.

$$j_a^- = \frac{4}{9\pi}\left(\frac{8}{3}\right)^{1/2} D^{3/2}N^{5/2}up^2c(1-c)^{1/2}(1+E_a)^{1/2} = Bp^2c(1-c)^{1/2}(1+E_a)^{1/2} \tag{19}$$

where B depends only on the nature of the gas. Similarly:

$$j_a^+ = Bp^2(1-c)^{3/2}(1+E_a)^{1/2} \tag{20}$$

The ratio between j_a^+ and j_a^- is the number of positive ions required to strike the cathode for each electron emitted. This probably is a constant dependent on the cathode material and on the velocity of the positive ions striking the cathode. This velocity is

$$u[8DN(1-c)/3]^{1/2}(1+E_a)^{1/2}$$

a constant independent of the pressure and dependent on E_a and the gas and cathode material. The ratio, from Eqs. (19) and (20), is $(1-c)/c$. The reciprocal of this, which is the "fraction" of an electron ejected for each positive ion striking the cathode, is therefore independent of the pressure, and will probably increase as the positive ion velocity increases. That is,

$$c/(1-c)=2H(1-c)^{1/2}(1+E_a)^{1/2}/9 \text{ or } 1-c=1-H(1+E_a)^{1/2} \tag{21}$$

where H is very small. This relation is not accurately true, but in any case $1-c$ will decrease slightly as E_a increases.

The total current to the cathode is

$$j_a=Bp^2(1-c)^{1/2}(1+E_a)^{1/2} \tag{22}$$

which corresponds to the requirements of the first part of Eq. (14).

So far we have been dealing with conditions near the cathode, where E is large, and z is practically unity. If we wish to investigate the conditions near the end of the cathode fall, just before the equations of Case I break down, we must consider z more carefully. Taking the value of z from Eq. (4), expanding, and then integrating, it is found that, approximately

$$\int z dx=x-G\epsilon^{2pNx/3} \quad \text{where} \quad G=v_0/32D(1-c)$$

which is very small, as we shall see later. Therefore our approximation that $\int zdx$ equals x is legitimate as long as the quantity which contains it is large throughout the fall space. This means that Eqs. (16) and (17) hold fairly well to the end of the fall space. However, E becomes small at the end, so we must include this second term to determine just where E becomes zero, if it does. E becomes

$$E=\frac{8}{3}Dp\,N(1-c)\left[\epsilon^{-2pNx/3}-\frac{pNv_0x}{48D(1-c)}+E_a\right] \tag{23}$$

The integral of E, the potential V, is large at the end of the space, and so this second term can be omitted, giving

$$V=4D(1-c)-4D(1-c)\epsilon^{-2pNx/3}+8DN(1-c)E_a px/3 \tag{24}$$

It is rather difficult to determine exactly where these equations break down entirely, that is, where the cathode fall ends. A reasonable assumption would be that it ends (i.e., $x=d$) when $E=8DpN(1-c)E_a/3$.

A more probable assumption would be to find the number of positive ions formed per second through a distance x from the cathode, and set x equal to d when this integral equals j_a^+. That is we consider the positive ion current to the cathode as being entirely due to those positive ions formed within the fall space, and that those formed outside, a much smaller number, are lost through recombination. This seems plausible, inasmuch as E becomes very small at the end of the fall space, so small that very few of the ions formed in the positive column can get across to the cathode.

However, when the integration is carried out, and the integral set equal to j_a^+, we find that when $x=d$, E must equal $8DpN(1-c)E_a/3$. So the original assumption was correct.

But when this is so

$$\epsilon^{2pNd_a/3}=v_o pNd_a/48D(1-c) \quad \text{or} \quad Npd_a=\frac{3}{2}\log\left[\frac{48D(1-c)}{v_0 Npd_a}\right] \tag{25}$$

This shows that pd_a is a constant, and therefore, by Eq. (24), V at d, *i.e.*, V_a, must be a constant, and therefore the first parts of the empirical equations (12) and (14) are satisfied.

Thus for a given pressure, gas, cathode, and E_a, the three measurable quantities j_a/p^2, pd_a and V_a are definite and determinable. The only constant which is not determined by these physical conditions, but which can be given any value, is E_a. Let us see what happens when it is increased. By Eq. (21) $c/(1-c)$ increases, or $1-c$ decreases slightly. By Eq. (22) j_a increases, by Eq. (24) V_a increases, and by Eqs. (21) and (25), pd_a decreases slightly.

Thus any change in either of the three quantities, say in j_a, produces a definite change in the other two. This must correspond to the conditions of the abnormal fall. Let us see if we can obtain a condition similar to the normal cathode fall.

The Normal Cathode Fall.

As E_a is allowed to vary, we see that its minimum allowable value is zero. For if E_a were a minus quantity, the number of ions formed per second within the fall space will be less than j_a and the discharge will cease.

Then j_a has a definite minimum value

$$j_n=Bp^2(1-c)^{1/2} \tag{26}$$

and V_a has a definite minimum value, given approximately by

$$V_n=4D(1-c) \tag{27}$$

and pd_a has a definite maximum value given by Eq. (25) when $(1-c)$ has its maximum value.

These stationary values are obviously those of the normal cathode drop. If all the constants, N, v_0, D, u and c were known for any particular cathode and gas, it would be possible to calculate V_n, pd_n and j_n/p^2 from them directly. However, the values of D and c are not known.

But since we have shown that V_n is a constant, though its value cannot be calculated with the data available, and since both D and c occur in the expression for V_n, we can use the experimentally determined values of V_n to determine the related values of pd_n and j_n/p^2. Thus

$$pd_n=\frac{3}{2N}\log\left(\frac{12}{Npd_n}\frac{V_n}{v_0}\right) \tag{28}$$

and, approximately:

$$\frac{j_n}{p^2}=\frac{4}{9\pi}\left(\frac{8}{3}\right)^{1/2} N^{5/2}u\left(\frac{V_n}{4}\right)^{3/2}=17.74\times 10^{-7}\left(\frac{L}{M}\right)^{1/2} N^{5/2}V_n{}^{3/2} \tag{29}$$

or less, depending on the particular value of u taken. N and V_0 are the constants in Townsend's equation, and are determined for a number of gases. V_0 is approximately, though not exactly, equal to the ionizing potential of the gas. L is the mean free path of the positive ion, which is known to be approximately equal to the mean free path of the molecule or[6] somewhat larger, so if L were considered as the mean free path of the molecule the values given for j_n would come out somewhat smaller than the true values, but would indicate approximately what the true value should be. M is the molecular weight of the gas in the tube.

Thus the constants in Eqs. (28) and (29) are all known more or less approximately, and the values of pd_n and j_n/p^2 must check the empirical results without the use of any adjustable constant.

The numerical data available[7] to check these equations are rather fragmentary and inaccurate. It is possible, however, to make numerical checks on all three Eqs. (27), (28) and (29).

From Eq. (27), V_n depends on D, which depends only on the gas, and on $1-c$, which depends mainly on the cathode material used, though some gases might affect its value. However it is to be expected that the ratio between the V_n's for the same gas but for two different cathodes will be approximately equal to the ratio between the V_n's for a different gas and for the same two cathodes. And similarly for the ratios between the V_n's for the same cathode and for two different gases.

These ratios for observed V_n's are given in Table I.

TABLE I.

V_n for N_2 / V_n for He	for Fe	Al	Mg	Pt	Ag	Cu
	1.56	1.57	1.60	1.50	1.56	1.50
V_n for A / V_n for H_2	for Fe	Al	Mg	Pt	Ag	Cu
	.60	.58	.64	.55	.57	.57
V_n for Fe / V_n for Al	for H_2	N_2	O_2	He	Ne	A
	1.21	1.14	1.13	1.14	1.20	1.22
V_n for Mg / V_n for Pt	for H_2	N_2	O_2	He	Ne	A
	.63	.83	.81	.78	.72	.73

This indicates that Eq. (27) is fairly correct, even though we have no means of determining the actual values of D or $1-c$.

[6] R. B. Kennard, Phys. Rev. **31**, 423 (1928).

[7] Bär, Handbuch der Physik (Springer) XIV, pp. 190–210.

Using the observed values of V_n, and the known values of the other constants, we can calculate pd_n and $j_n^{1/2}/p$, as given in Table II. In this table

TABLE II. *Values of pd_n and $j_n^{1/2}/p$.*

Cathode		Fe		Al		Mg		Pt	
Gas		Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
H_2	pd_n	0.94	0.90	0.85	0.73	0.85	0.70	1.02	1.03
	$j_n^{1/2}/p$	.264	—	.23	.25	.22	—	.32	.30
N_2	pd_n	.38	.42	.36	.30	.36	—	.38	—
	$j_n^{1/2}/p$	.407	—	.37	.37	.35	—	.40	.59
He	pd_n	1.82	1.66	1.77	1.32	1.72	—	1.82	—
	$j_n^{1/2}/p$	.091	—	.082	—	.076	—	.091	.10
A	pd_n	.338	.35	.31	.29	.32	—	.35	—
	$j_n^{1/2}/p$	.26	—	.22	—	.23	—	.29	.35

p is in mm Hg; d in cms; j_n in milliamperes/cm². These values check within the limits of accuracy of the measurements, and within the range of variation of results from different observers. We can therefore conclude that our approximations have not been too inexact, and that Eqs. (28), (29) and (30) represent, to sufficient accuracy, the normal cathode fall space.

It is of interest to note the similarity of these results with those obtained by disregarding Poisson's equation and considering that the potential fall distributes itself across the fall space so as to produce the greatest current possible,[5] as in the previous paper by Compton and Morse. The equation for j_n/p^2 in that paper is practically identical with Eq. (30) above, and the similarity between the exponents has been noted earlier in this paper. The expression for pd_n is different, but in that paper the distance d_n was more or less arbitrarily chosen, whereas in the present case pd_n is more or less naturally defined. Moreover the formulas in the present case fit the data somewhat better, and show that V_n *must* be a constant, whereas in the earlier paper this had to be assumed constant.

However, it is interesting to see that in this case a distribution of potential so as to give maximum current is equivalent to a distribution of potential so as to satisfy Poisson's equation.

THE ANODE FALL

By a similar process, the drop in potential at the anode can be calculated. Since no positive ions are emitted by the anode, n_2 at the anode must be zero. The equations become, if L is the distance from cathode to anode

$$n_1=\frac{A}{a+1}\left[a+\epsilon^{2pN(x-L)/3}\right]; \qquad n_2=\frac{A}{a+1}\left[a-a\epsilon^{2pN(x-L)/3}\right] \tag{30.5}$$

where A has the same value as that given in Eq. (15).

If the anode area is equal to the cathode area, and the current is such that the whole of the cathode is covered with current, *i.e.*, the cathode space is abnormal, then the current densities at the two electrodes must be equal, and

$$E=\frac{8}{3}DpN\left[(1-c)(1+E_a)+\frac{\epsilon^{2pN(x-L)/3}}{a+1}-\frac{1}{a+1}\right] \tag{31}$$

This indicates that E at the anode equals E at the cathode.

This must be so, for

$$\int_0^L (dE/dx)dx=4\pi e\left[\int_0^L n_1dx-\int_0^L n_2dx\right]$$

$$=E_L-E_0=4\pi e[(\text{total no. of electrons})-(\text{total no. of pos. ions})]$$

But for the tube as a whole to be neutral the total number of electrons must equal the total number of positive ions, and therefore E at the anode must equal E at the cathode, as long as the discharge covers the same area at the anode as at the cathode.

These equations indicate that the concentration of positive ions at the cathode is a times the concentration of electrons at the anode.

Since there is no emission of positive ions from the anode, and no necessity for their emission, since they do not create ions, the ratio between bombardment and emission, which fixed the minimum, or normal values of the cathode drop, is not present at the anode, and so there is no analogous normal drop here. The field at the anode therefore reduces continuously as the current is diminished. This means that when the cathode fall is normal, E at the anode is less than E at the cathode. This is not contrary to the previous statements, for in the normal case the current does not cover the whole of the cathode, and we must integrate over a different cross section at the cathode than at the anode. This makes the ratio of the number of lines of force per square cm. (*i.e.*, the field) at the cathode to that at the anode equal to the ratio of the area of the anode to the area of the patch of the cathode carrying current.

Thus the anode drop is not at all as important as the cathode drop. The fall in potential across it is much smaller than the cathode fall, and the concentration of electrons at the anode surface is only $1/(1+a)$ of the concentration of positive ions at the cathode.

The above analysis gives an approximate solution for the two portions of the usual glow discharge tube where the field is large, when electrons are emitted from the cathode by positive ion bombardment.

If electrons are emitted from the cathode by other means, there no longer exists the constancy of the ratio $c/(1-c)$, and, except in the case of very slight electron emission, there is no longer a normal cathode fall space.

For large pressures, and values of E large enough to make z equal to unity throughout the whole discharge, the equations become

$$n_1=A+\frac{A}{a}\epsilon^{2pN(x-L)/3}-A(1-c)\epsilon^{-2pNx/3}$$

$$n_2=A-A\epsilon^{2pN(x-L)/3}+aA(1-c)\epsilon^{-2pNx/3}$$

where c is proportional to the electron current from the cathode. The current will be J times as great as the normal current density, where, if I is the actual current density

$$I=\frac{4}{9\pi}\left(\frac{8}{3}\right)^{1/2}D^{3/2}p^2N^{5/2}uJ$$

from Eqs. (27) and (29). Since we know n_1 at the anode, and the current density at the anode, we can find the field at the anode: $E=(8/3)DpNJ^2$ This determines the constant of integration, in the general equation for E, and V across the whole discharge is

$$V=4D(1-c)+\frac{8}{3}pNDLJ^2+\frac{4D}{1+a}-\frac{8}{3}\;\frac{pNDL}{a+1} \tag{32}$$

In a gas filled tube containing a filament emitting enough electrons so that positive ion bombardment would not appreciably affect the electron emission, c will remain constant, and V will increase as J increases, until an arc is struck.

In the ordinary electric arc, c is dependent on the current density, represented by J, and will in general increase rather rapidly as J increases. This is obvious from consideration of any of the suggested mechanisms of electron emission from the arc cathode. Therefore as the current is increased V will decrease, until a certain point is reached where further increase in current density causes no further increase in c, either because this excess energy is all radiated away or because the material melts, or for some other reason. At this point a condition very analogous to the normal cathode drop will be in effect, with a constant current density, and a constant potential drop across this space, which, of course, will be much smaller than the normal potential drop for glow discharges. Any further increase in the total current simply increases the area of the "hot spot."

This is exactly what happens in striking an electric arc by passing beyond the condition of a glow discharge. As the current increases from zero, the voltage drop across the discharge decreases, until a "hot spot" forms. Here there is a discontinuity in the curve, and after that the voltage drop at the cathode is independent of the total current. Thus the conditions in the hot spot of an arc, which represent the maximum possible current density in a gas discharge, are quite similar to the conditions in the normal cathode fall in a glow discharge, which represent the minimum possible current density; and this similarity illustrates the essential identity of the equations for arc discharge and for glow discharge.

The Positive Column

The most of the length of the discharge tube, from the anode out to near the cathode, is filled with a luminous portion of the discharge, called the *positive column*. Throughout its length the field has a comparatively constant value E_p. Between certain limits of pressure and current the column is divided into alternate striations of nearly equal distance apart, and having their greatest intensity near the cathode. When this is the case E_p has imposed upon it a slight sinusoidal variation about its average value, having a wave-length l equal to the distance between striations.

While the data on this portion of the discharge are nearly all qualitative, there are some approximate empirical relationships which have been found. E_p is directly proportional to the pressure and inversely proportional to r the radius of the discharge tube, if the tube is approximately cylindrical. It decreases slightly with increase of total current through the tube. Similarly l varies inversely as p, directly as r, and decreases slightly with increasing current. The potential drop from one striation to the next is proportional to some critical potential of the gas. For the noble gases and the monatomic vapors it apparently equals the first radiating potential, while for molecular gases it is proportional to the ionizing potential.

Examining Eqs. (8) and (9), we see that for E to be a constant, n_1 must, on the average, equal n_2 and this value, call it n_p, must also be a constant. Therefore just as many ions are created per cm according to the first term, as disappear, according to the second term. That is:

$$aNpzn_1E_p^{1/2}=Rp^{1/2}$$

But for small values of E, z becomes approximately $E/\epsilon pNV_r$ where ϵ is the base of the Naperian logarithms. Then

$$\frac{a}{\epsilon V_r}n_1E_p^{3/2}=Rp^{1/2} \tag{33}$$

R is the number of ions disappearing per second per cc. This is probably due principally to recombination of the ions at the walls of the tube. Then the total current going to the walls per cm length of tube is: πr^2eR. Call it J_r. Schottky[3] has found that

$$J_r=i\cdot\frac{2.4}{r}\left(\frac{\chi}{V_r}\right)^{1/2}D_a^{1/2}$$

Here i is the total current flowing along the column, χ is a constant, having values between .4 and .1 for most tubes, and D_a is a sort of diffusion constant proportional to the product of the actual speeds of the electrons and positive ions, and therefore inversely proportional to the product of their respective mobilities. r is the radius of the discharge itself, which equals the radius of the tube, unless the tube is too large; that is, r equals the radius of the tube until this radius reaches a certain maximum value, after which r is constant, no matter what greater radius the tube may have.

Obtaining a value for i in terms of n_1, and substituting all this in Eq. (33).

$$E_p=2.4\epsilon\,(\chi V_rD_a)^{1/2}$$

But Schottky, by other means, obtained

$$E_p=\frac{2.4}{r}\,(V_rD_a/\chi)^{1/2}$$

This means that $\chi = 1/\epsilon r$. For tubes usually used in experimental work r varies from 1 to 4. Therefore χ would vary from .37 to .09, which is the range of variation which has been experimentally determined. D_a, we have seen above, is inversely proportional to the square of the mobility, or by Eq. (6) $D_a = GpE_p$, where G is a constant. Substituting this, and solving for E_p, we find

$$E_p = 5.76\epsilon V_r Gp/r \tag{34}$$

V_r was defined by Schottky as the ionization potential of the gas, but if cumulative ionization takes place, especially if metastable atoms are easily formed, V_r may well be the first critical potential, or one near it.

This value of E_p satisfies the empirical conditions for its dependence on r and p. With increase of current the chances for the same proportion of this current to reach the walls and recombine is probably less, and therefore the value of E_p required to replace this lost proportion of current will be slightly less. This explains why E_p decreases slightly with increase of current.

While E_p and n_p must be, on the average, constant, it may be that they can vary slightly about these as mean values. Set $n_1 = n_p - n$; $n_2 = n_p + n$, where n is small compared to n_p. Solving Eq. (8) for n, assuming that the first two terms cancel each other,

$$n = B\epsilon^{-E_p/2D} \sin\left[\left(\frac{8\pi e n_p}{D} - \frac{E_p^2}{4D^2}\right)^{1/2} x\right] \tag{35}$$

where B is small compared to n_p, and $E_p/2D$ is very small.

This substituted in Poisson's Equation can be integrated to give a similar expression for the variation of E about E_p. It can be seen that this fluctuation is only possible between certain limits of pressure and current. Also, due to the exponential term, the fluctuations are greatest near the cathode, and diminish as the anode is approached.

If it is assumed that the first term under the radical is much larger than the second, then the distance between striations will be $l = 2\pi(D/8\pi e n_p)^{1/2}$. The value n_p is probably proportional to the n_1 or n_2 at the beginning of the anode drop. However, the discharge has now spread out to fill the tube, consequently n_p is also inversely proportional to the square of r, and from Eqs. (30.5) and (15)

$$n_p = \frac{4kDp^2N^2}{9\pi r^2 e(a+1)}$$

where k is a constant dependent on the gas, and perhaps on the particular tube. When this value of n_p is substituted in the equation for l, it becomes

$$l = \frac{3\pi r}{2pN}\left(\frac{a+1}{2K}\right)^{1/2}$$

The potential drop between striations is

$$lE_p = 8.64\pi\epsilon G[(a+1)/2k]^{1/2}V_r = KV_r \tag{37}$$

The coefficient K is independent of p and r and of current, over a considerable range of values, and it may quite possibly be unity for the noble gases. For other gases K assumes other values; while evidently V_r is the first critical potential for monatomic gases, and the ionizing potential for other gases. This would then correspond to the empirical facts.

Thus Eqs. (34), (36) and (37) show that E_p, l and lE_p in their dependence on r, p and V_r, satisfy the empirical facts; hence we can assume that these equations represent approximately the conditions on the positive column.

An explanation of the variation in light can be explained as follows. For each field, the electron will have an average random motion, and an average energy of motion. At a point where E, in its fluctuations about E_p, increases to such a value that the average energy of the electrons is sufficiently near to a radiating energy of the gas molecule, the gas becomes luminous, and some of the excited atoms are ionized. It is not necessary that a great portion of the excited atoms be ionized, for the current through the tube is determined by other means. When E reaches such a value that the electron energy is sufficient to excite another line, this line also appears.

The Faraday Dark Space

At the end of the cathode fall E is zero, or even slightly negative. In this case Eq. (8) can be shown to reduce to:

$$aDd^2n_1/dx^2 = pR'n_1/r = aDb^2n_1$$

for the mobilities used here must be those for a small field as in Eq. (7). This gives

$$n_1 = A\epsilon^{-bx} + n_p\epsilon^{b(x-d)}$$
$$n_2 = aA\epsilon^{-bx} + an_p\epsilon^{b(x-d)} - (a-1)A + (a-1)(A-n_p)x/d$$

where x is zero at the beginning of the dark space, and is d at its end, where the positive column begins. If we solve for E in the usual way, and set E equal to E_p when x equals d, we find that $d = (r/pR')\ (1+k/p)$ where k is dependent on the current density in the positive column. This equation is only very approximately true, for E does not equal zero throughout the dark space, but it serves as an indication of the dependence of d on r, p and the current going through the tube. It checks qualitatively with the experimental data on the width of the dark space.

Conclusion

Thus Eqs. (1), (8) and (9), when solved, account for all the varied phenomena of electric discharge through a gas under different conditions, and what little numerical data is available seems to check these solutions. A set of curves for n_1, n_2 and E for a usual glow discharge, as calculated from the equations given, is shown in Fig. (1).

If we assume that above a certain field E_1 the atoms which are struck by electrons will radiate light, and that above E_2, where E_2 is greater than E_1, the atoms struck will not radiate, but will be ionized, then we can draw two parallel lines across this set of curves, and assume that between these two lines the gas becomes luminous, and above and below them it is not. In this way we can approximately foretell the luminous portions of the tube.

At both ends of the tube a current of many high speed ions is spread more or less evenly over the active portion of the electrode, the ions moving normally to the electrode. As we get further and fuı ther from the electrode, the current tends to be carried by a few, low speed ions which tend to concentrate along the center of the tube.

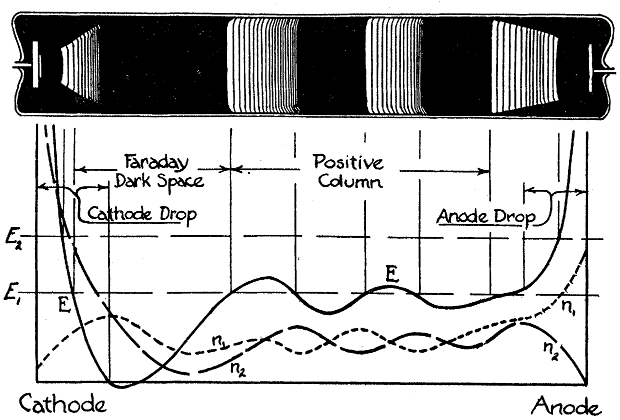

Fig. 1.

The ions required to carry the current to the electrodes are nearly all formed in their respective falls of potential, quite near the electrode for which they are destined. The extra ions of opposite sign are pushed away from each electrode, and travel toward the other electrode. These, the electrons from the cathode fall, and the positive ions from the anode fall, must meet and recombine in some part of the tube. In this place of recombination, the values of E, and the two n's will be the smallest of any part of the discharge. This space must be near the cathode, for even if but few electrons get through the space, these few can increase their number by ionization as the field increases, whereas the positive ions getting through a similar depression near the anode could form no ions, and the discharge would stop.

It is of interest to note that these solutions would not have been greatly changed if μ had been taken according to Eq. (7), instead of according to Eq. (6).

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