INTERNAL FRICTION IN SOLIDS

By A. L. Kimball and D. E. Lovell

Abstract

It is shown from tests on eighteen different solids, including several metals, glass, celluloid, rubber and maple wood, that the internal friction for strains below the elastic limit does not obey the liquid viscosity law, as is usually assumed, according to which the frictional force depends upon the velocity of strain, but that the internal friction is entirely independent of strain velocity, so far as can be observed. It was found to depend upon the amplitude of strain during the strain cycles and approximately to obey the law: Energy loss per cycle per unit volume equals ξf_m^2. In this expression f_m is the maximum value of the stress during the stress cycle and ξ a proportionality factor, which may be called the internal friction constant. The method used was to measure the transverse deflections of the end of a rod, about a meter long, of the material being studied, which transverse deflections were produced during rotation of the rod when its end was deflected downwards by suitable loads on it. The experiments differ from most previous work in that relatively large masses of material were employed, tending to reduce surface effects, which are likely to enter in the case of vibration decrement experiments on wires and on thin strips. A table of the internal friction constants obtained is given, and also a table of similar internal friction constants calculated from data of previous investigators. A reasonable agreement is found.

FOUR years ago one of the authors of this paper briefly described how internal friction within a revolving shaft supported in bearings produced a disturbing force which should cause the shaft to whirl or whip at its natural vibration period, and demonstrated the phenomenon by a small model.[1] An attempt was subsequently made to evaluate the coefficient of viscosity of a nickel steel shaft, in order to obtain quantitative information about this disturbing force, but it was unexpectedly found that the internal frictional forces were not like those of a viscous fluid where the forces are greater the more rapid the deformation, but that the dissipative forces were the same whatever the speed of deformation. A search of the literature on the subject revealed the fact that internal friction in solids has been and is at the present time treated as a viscous friction by the great majority of writers on this subject.[2] There are a few, however, who appreciate that this law does not hold in the range of ordinary low period vibration frequencies.[3]

[1] A. L. Kimball, Phys. Rev. **21,** 703 (June 1923).

[2] See Honda and Konno, Phil. Mag. **42,** 115 (1921); Iokibe and Sakai, Phil. Mag. **42,** 397 (1921); Lesch, Zeits. angew. Math. und Mech. **4,** 124 (1924); C. E. Guye, Jour. d. Physique 620 (1912). See also Ibbetson, Mathematical Theory of Elasticity, p. 175 (Macmillan, 1887); Prescott, Applied Elasticity, p. 44 (Longmans, 1924).

[3] See, for instance, Hopkinson and Williams, the Elastic Hysteresis of Steel, Roy. Soc. Proc. **A87,** pp. 502–511 (1912); F. E. Rowett, Elastic Hysteresis in Steel, Roy. Soc. Proc. **A89,** 528–543 (1914); K. Bennewitz, Phys. Zeits. **21,** 703 (1920); K. Bennewitz, Phys. Zeits. **25,** 417–431 (1924); H. Jordan, Deutsch. Phys. **18,** 423 (1915). See also results of Lindsay, Phys. Rev. **3,** 397–438 (1914).

In this paper it is shown that over a considerable frequency range and stress amplitude range for a number of solids of very different physical properties the frictional loss per cycle of stress at a point in the solid is independent of the frequency of performance of the stress cycles, like magnetic hysteresis. In fact, this law has not been found to fail thus far for any solids tested by the writers of this paper, except for very low frequencies of the order of one cycle in several minutes, where the frictional loss per cycle is found to *increase*.

Method of Experiment

The method used was a refinement of that used by Wöhler in studies of the fatigue strength of metals made by him many years ago. Recently this method was used by Mason and analyzed by him.[4] An article on this subject by Inglis has also recently appeared.[5] In the experiments described in this paper, however, the internal friction during stress cycles below the elastic limit only was studied.

Figs. 1 and 2 show a photograph and drawings of the apparatus used. The materials tested were made into the form of rods, usually 1.27 cm in

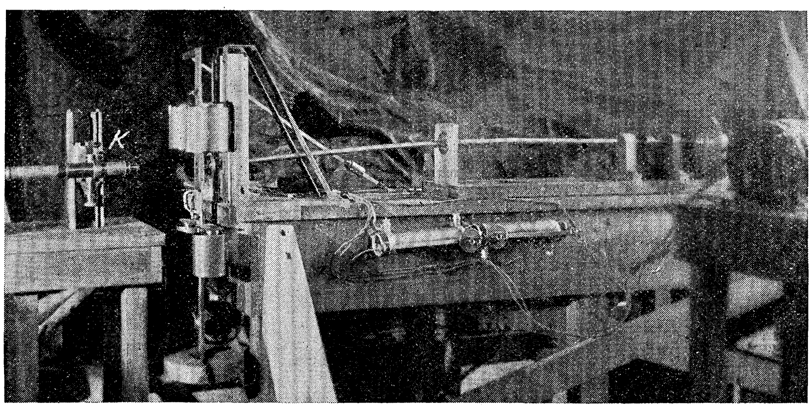

Fig. 1. Photograph of apparatus.

diameter and about 1 meter long. The rod was supported in two ball bearings B_1 and B_2, and revolved by means of an electric motor through the pulley P which was supported in the bearings marked B, the torque being transmitted to the rod by means of a small universal coupling C. The bearing B_2 was located so that more than half of the rod overhung. On the projecting end of the rod was another ball bearing B_3, on which was hung the frame U, which carried a weight W on the pan at its bottom. The rod was thus deflected downwards, as shown in the photograph of Fig. 1, and at the same time it was free to revolve. The frame U carries four plungers which dip into

[4] See article on this subject by Mason, Engineering **15**, 698–9 (1923).

[5] N. P. Inglis, Hysteresis and Fatigue of the Wöhler Rotating Cantilever Specimen, The Metallurgist, Feb. 1927.

four rigidly supported cups containing a suitable damping fluid. This proved an effective means of steadying the end of the rod during revolution, so that the amount of its deflection could be observed.

Fig. 3 shows how the end of the rod was deflected in these experiments. The deflection was not exactly downwards but was displaced by an angle ϕ from the vertical, because of the internal friction in the revolving rod. As

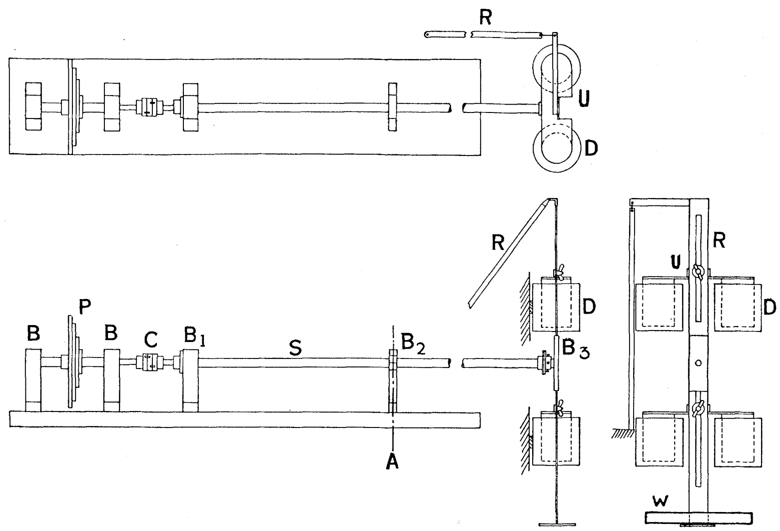

Fig. 2. Plan and elevations of apparatus.

the rod revolves, the fibers parallel to its axis are carried around its axis, alternately up on one side and down on the other. As they move up they stretch, and as they move down they shorten. Any frictional resistance to the process of lengthening results in tension, and frictional resistance to the process of shortening results in compression. Consequently, in this case the upward moving fibers are in frictional tension, T_F, and the downward moving ones are in frictional compression, C_F. For the direction of rotation shown in Fig. 3, the left hand fibers are therefore in tension due to this cause and the right hand ones are in compression. These frictional stresses are superimposed upon the elastic stresses, T_E and C_E, and just as the elastic stresses by themselves produce an upward reaction, R_E, so the frictional stresses by themselves produce a transverse reaction. The forces, R_E and R_F, are components of the force which balances W exerted by the weight on the end of the shaft (Fig. 3). If the weight is removed, R_E and R_F vanish (neglecting the weight of the rod) so that both the elastic and frictional couples must vanish. If $\phi=0$, $R_F=0$, that is, no frictional force is present. This is true when the rod has the weight on its end, but is not revolving. It will be shown later that the transverse deflection produced by R_F gives a means of measuring the amount of internal friction in the rod, and how the internal friction varies with stress amplitude and frequency.

The sideways displacement of the rod was measured by means of a cathetometer (K, Fig. 1). On the end of the shaft was fixed a pin point, the tip of which was polished and illuminated by means of an arc lamp, giving a small bright spot suitable for observation. Set screws weıe provided in order to set the point as nearly on the axis of rotation as possible. Observations were taken as follows: a suitable weight was placed upon the pan W, Fig. 2; the speed of revolution of the rod was brought up to some chosen value, say 1,000 r.p.m.; and the position of the pointer noted on the scale of the cathetometer. In most cases oscillations made it impossible to have the pin point centered for all speeds, so that a small bright circle or ellipse was seen in the cathetometer. The position of its center was found by taking settings on its circumference. The direction of rotation of the rod was then *reversed*, and another observation of the position of the pin point taken.

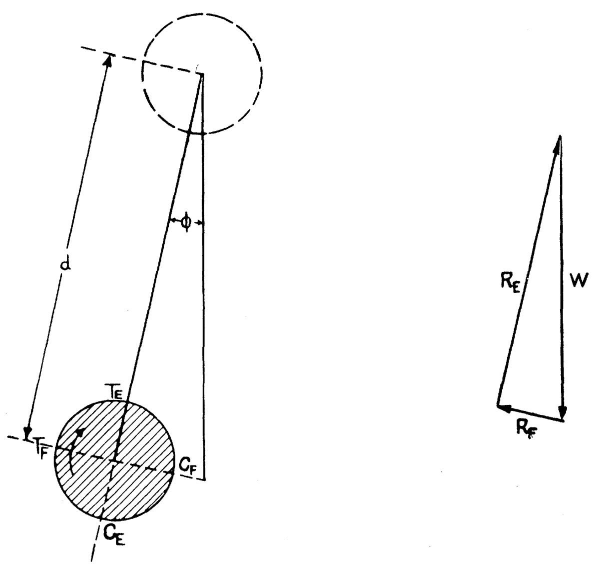

Fig. 3. View of weighted end of shaft showing sideways deflection due to internal friction.

In the design of this apparatus great precaution was required to avoid surface friction effects or reactions of any kind which might produce a false reading, such as might come from the couplngs or the bearings. Friction in the bearing journals does not enter as long as all the bearing axes are parallel. An experiment on an exaggerated case of bearing friction was made and this conclusion verified. No rings of any sort should be fitted on the test rod, otherwise surface contact friction may be produced. This point was kept in mind when using the ball bearings.

Internal Friction Independent of Speed of Performance of Strain Cycle

According to the classical theory of viscosity the transverse deflection of the rod should be proportional to the speed of rotation. The first experiment was made on a rod of 3.5 percent nickel steel in order to observe the increase in the transverse deflection with speed, but no such increase in deflection was observed over a speed range of from 2 to 3 up to 200 cycles per second. A definite deflection was present, however, as was found by reversing the direction of rotation of the rod. The amount of the transverse deflection could be increased only by increasing the load on the end of the rod, but was

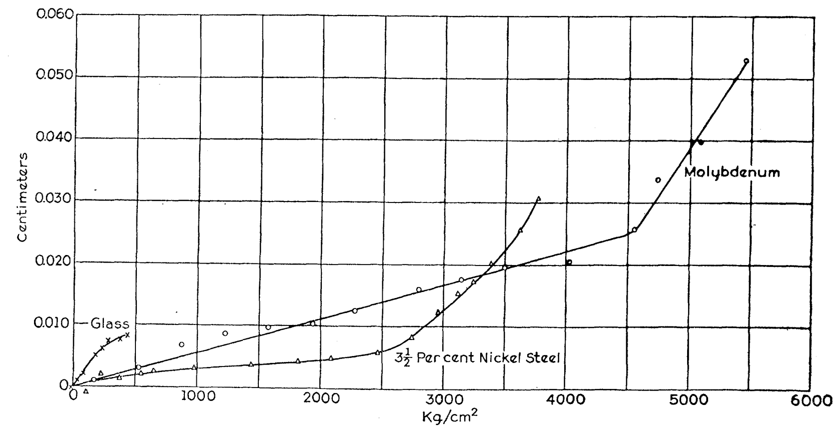

Fig. 4. Curves of sideways deflection at end of shaft plotted against downward deflection and maximum stress intensity.

found for every load to be entirely independent of the rotational speed of the rod. This showed that for nickel steel the forces produced by friction within the rod are independent of the velocity of strain, and that for a given amplitude of strain, the frictional loss per cycle is independent of the speed of performance of the strain cycle.

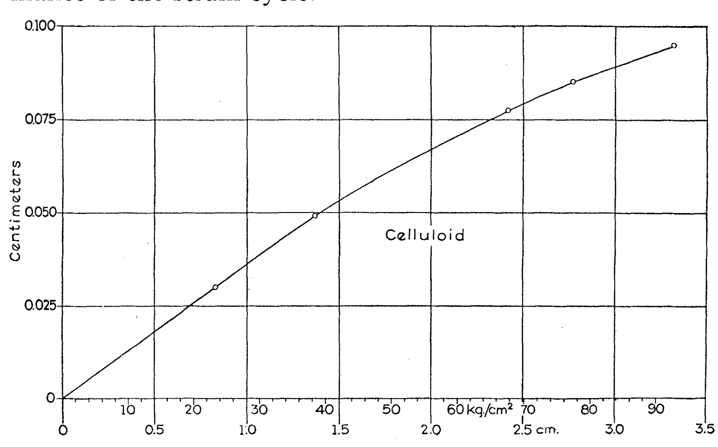

Fig. 5. Same as Fig. 4 for case of celluloid.

A considerable number of metals, alloys and other solids have been tested by the revolving rod method, and every solid thus far examined exhibits an internal friction which is independent of the speed of performance of the strain cycles. This characteristic of internal friction in solids thus appears to be universal.

Figs. 4 and 5 show typical curves of internal friction for molybdenum, 3.5 percent nickel steel, glass and celluloid for varying loads on the rods; that is, for varying bending stresses in the rods, the speed in each case being constant. Figs. 6 and 7 show the same thing but with fixed loads on the rods and varying speed.

Other materials which exhibit a linear relation between internal friction and maximum stress are: swaged quarter-inch-diameter tungsten (up to its

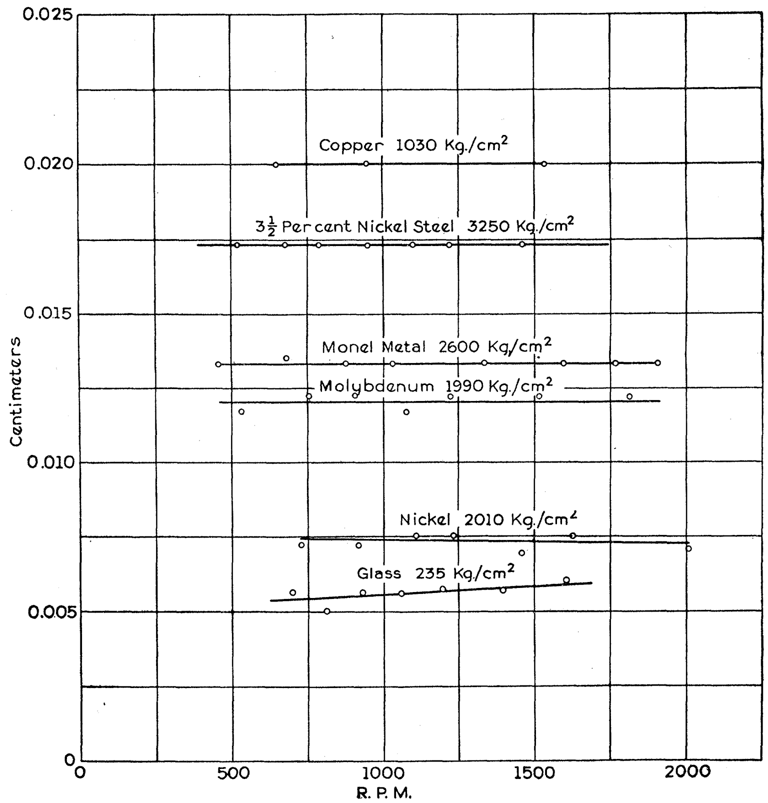

Fig. 6. Curves of sideways deflection at end of shaft plotted against speed of rotation for several different materials.

breaking point), maple wood, zinc, commercial rolled nickel, and copper (up to 7,000 lbs. per sq. in.). In the case of the other materials listed, the linear relation is only an approximation.

Internal friction is known to be a very variable thing, depending upon the past thermal and mechanical history of the specimen. The material often appears to be in an unstable state which is shown by a considerable change in the internal friction, with the first few cycles. After a few hundred cycles a so-called "cyclical state" is reached in which the internal friction remains about constant as long as stress and heat conditions are not changed too much.

The materials whose friction coefficients are compared in Table I were in a cyclical state, and the values of these coefficients afford a fairly good comparison between the internal friction of these various materials.

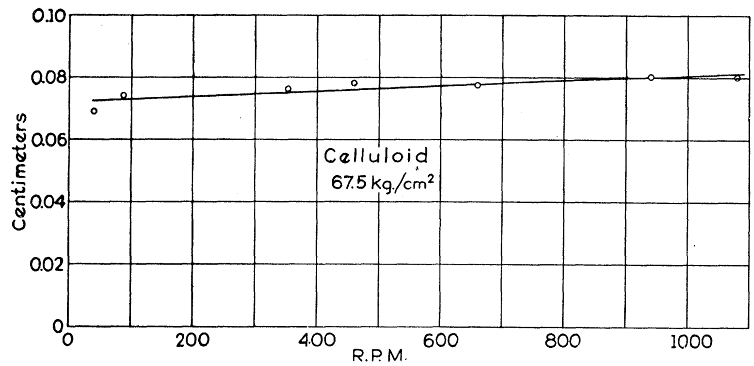

Fig. 7. Same as Fig. 6 for celluloid.

Definition of Coefficient of Internal Friction

A question which arises at this point is how the coefficient of internal friction is best defined. To have a definite meaning the coefficient of internal friction must be based upon a definite law of internal friction. From a study of a large number of curves of the revolving rod experiment in connection with the literature on this subject, the law of internal friction which most generally fits the facts and which at the same time is a simple one, is

$$F = \xi f^2_m \tag{1}$$

where F is the frictional loss per unit volume per cycle at a particular point; f_m the amplitude of the stress cycle above and below zero stress at the point in question; and ξ is the internal friction constant, the internal friction loss per unit volume per cycle per unit stress amplitude.

Table I gives a list of all the materials thus far tested. The internal friction constant is expressed in ergs per cm^3 per $dyne/cm^2$ stress amplitude. The materials in this table, with four exceptions, are seen to be metals and alloys. Even for the others, celluloid, glass, maple wood and rubber, the frictional loss per stress cycle is independent of frequency. Glass was somewhat erratic, because of the small stresses used and the difficulty of observation, but there is no doubt as to the frictional loss per cycle being practically independent of frequency. In the case of rubber a vibration test was made, using a circular bar of rubber 2.48 cm diameter and 53.3 cm long. Torsional vibrations were produced using a weight whose moment of inerita was so varied that a 4 to 1 variation in frequency was produced. In this case also, as nearly as could be observed, the frictional loss per cycle was a constant for a given stress amplitude. Of the metals and alloys tested, tin and zinc are seen to have much higher coefficients of internal friction than the others. Monel metal and nickel, though very ductile, have little frictional loss.

Discussion of the Law of Internal Friction

It should be clearly understood that the internal friction studied in this paper is for small stress amplitudes, such as those which arise in a vibrating body. If the stress range is so large that the elastic limit of the solid is approached, the internal friction begins to rise rapidly, and the law of Eq. (1) no longer holds. The plastic yielding characteristic of many solids above the yield point belongs to a class of internal friction phenomena completely outside of the range of, and governed by different laws from, the phenomena here studied.

Table I
Internal friction constant for various materials.

Material	Tan ϕ $\times 10^3$	Elastic Modulus in dynes/cm^2 $\times 10^{-11}$	Internal Friction constant (c.g.s. units) $\times 10^{15}$
Rubber 90% pure*			19,000,000.
Celluloid	14.4	0.214	21,000.
Tin, swaged	40.7	3.1	402.
Maple wood	6.69	1.3	172.
Zinc, swaged	6.30	9.4	20.8
Glass	2.05	6.3	10.2
Aluminum, cold-rolled	1.07	5.8	5.7
Brass, cold-rolled	1.54	8.5	5.6
Copper, cold-rolled	1.59	10.	4.95
Tungsten, swaged	5.24	38.7	4.26
Swedish iron, annealed	2.50	18.9	4.16
Phosphor bronze, annealed	1.01	12.	2.67
Mild steel, cold-rolled	1.57	21.	2.33
Molybdenum, swaged	2.19	34.6	1.95
Nickel, cold-rolled	1.02	21.	1.55
Nickel steel, $3\frac{1}{2}$% swaged	0.73	21.	1.10
Monel, cold-rolled	0.454	17.8	0.79
Phosphor bronze, cold-rolled to maximum hardness	0.117	11.6	0.306

* Vibration test.

The first column of Table I gives the value of tan ϕ which is a direct measure of the ratio of the frictional moment which produces the transverse deflection, to the elastic moment, which balances the weight by its upward reaction (see Fig. 3). It follows from Eq. (1) that this moment ratio is equal to

$$\tan\phi = \xi E/\pi \tag{2}$$

from which it is seen that tan ϕ is independent of the rod dimensions. The derivation of this relation is given in Appendix I. Eq. (2) also shows that tan ϕ is independent of the load W, that is, it requires that the ratio of transverse deflection to downward deflection be constant so that the curves of Figs. 4 and 5 are straight lines.

The values of ξ obtained from Eq. (1), as given in Table I, give a very useful means of comparing internal frictions in solids, whereas constants based on the old idea that the viscosity of a solid is like that of a liquid are completely misleading.

It is to be noted that this law gives a constant logarithmic decrement of vibration amplitudes, because the frictional loss per cycle is proportional to the square of the amplitude.

The theory presented is established only within the range of the experiments outlined in this paper, but the authors are of the opinion that a law of the type represented by Eq. (1) will be found to hold over a very wide

TABLE II

Internal friction constants as calculated from results of previous investigators.

Material	Treatment	Investigator	Youngs modulus $E\times10^{-11}$	Rigidity modulus $G\times10^{-11}$	Logarithmic decrements Normal δ_η	Tangential δ_τ	Internal friction constants $\times10^{15}$ Normal ξ_η	Tangential ξ_τ
Tin		Voigt	3.2	*1.6	.0129	.0110	40.3	69.
Zinc		"	9.4	*3.9	.00605	.00581	6.4	14.9
Aluminum		"	5.8	*3.35	.000820	.00603	1.41	1.80
Steel		"	21.	*8.29	.00239	.00193	1.14	2.32
Bronze		"	11.8	*4.4	.001006	.000308	0.85	0.70
Nickel		"	21.	*7.82	.00177	.00109	0.84	1.39
Brass		"	8.5	*3.6	.000674	.000310	0.795	0.86
Copper		"	10.	†5.5	.000715	.000420	.715	0.764
Copper		Thompson		†5.5		.00365		6.63
Aluminum		"		*3.35		.00670		20.0
Copper	Annealed	Honda	10.		.0075		7.5	
Copper	Rolled	"	10.		.00177		1.77	
Aluminum	Rolled	"	5.9		.00055		.93	
Zinc	Rolled	"	9.6		.00501		5.2	
Zinc	Annealed	"	9.6		.0017		1.77	
Nickel	Rolled	"	21.		.00169		.805	
60–40 Brass		"	8.66		.00192		2.2	
Copper		Guye		5.5		.001		1.8
Glass		"		*2.54		.00015		0.59
Zinc		Iokibe & Sakai		3.0		.0587		195.
Aluminum		"		2.5		.0055		22.
Copper		"		5.5		.00076		1.4
Nickel		"		7.2		.000106		0.147
Iron		"		7.8		.00128		1.64
Tungsten		"		13.25		.00035		0.264
.55% C Steel		"		7.65		.00119		1.55
.9% C Steel		"		7.7		.00086		1.1
1.3% C Steel		"		7.6		.00089		1.2
Glass	Plate glass	Quimby					0.205	
Aluminum	Rolled	"					0.605	
Copper	Hard drawn	"					1.08	
Copper	Wire	Lindsay	10.		.00169		1.69	
Steel	"	"	21.		.00072		.342	
Phosphor bronze	"	"	11.8		.00037		.314	

* From Smithsonian Tables (1920).
† From Iokibe & Sakai.

range of frequencies. The only case to the authors' knowledge where measurements have been made for very high frequencies is that of the experiments of Quimby.[6] He uses Stokes liquid-viscosity law in interpreting his results and obtains constants smaller than those of previous investigators in approximately the same ratio as his frequencies are higher than theirs. On the basis of the authors' law, however, Quimby's results and those of previous

[6] S. L. Quimby, Phys. Rev. **25**, 528–573 (1925).

investigators give constants of the same order of magnitude, although Quimby's constants are somewhat smaller. This is perhaps because these constants were obtained from stress cycles of much smaller amplitude than used in this investigation.

Table II gives a list of values of ξ which have been calculated from the results of previous investigators, using their logarithmic decrements as a means to calculate ξ. The value of ξ can be found by the very simple relation

$$\xi=\delta/E \tag{3}$$

where δ is the logarithmic decrement and E Young's modulus. The proof of Eq. (3) is given in Appendix II.

It must be kept in mind, however, that ξ as determined by (1), (2) and (3) may not always mean the same thing. It depends upon the kind of stress in the stress cycle. It is known from the theory of elasticity that the stress at any point in any body, however stressed, may be resolved into a pure dilatation and a pure shear. In bending vibrations, where normal stresses are produced, they are combined dilatations and shears, but for torsional oscillations, the stresses are almost pure shears. When (3) is applied to torsional vibrations, E should be replaced by the rigidity modulus. In the revolving rod experiments, the cyclical stresses are normal in character. Table II contains values of ξ obtained from logarithmic decrements produced both by normal and by tangential stresses, the former being specified by ξ_η and the latter by ξ_τ.

The complete analysis of internal friction phenomena in solids requires a separation of that produced by pure dilatation and that produced by pure shear, and a knowledge of the effect of their combination in various ratios.

Attention should be called to the fact that for frequencies of the order of those in bells and in bodies which respond with a prolonged ring when struck, the law of Eq. (1) shows that such a ring is perfectly possible, whereas on the basis of the old liquid-viscosity law, using the constants of previous investigators for that law (except Quimby's constants), no ring is possible. A bell would be aperiodic, as if made of putty. This is mentioned by Quimby in his paper.

In conclusion we wish again to point out that the law of Eq. (1) is not to be considered as complete or exact, as regards the loss being proportional to the stress amplitude squared. There are divergencies from this in the experiments of others as well as our own. Rowett[3] finds that for annealed iron the cube law fits his results best. Bennewitz[3] uses the square law, however. The static tests of Rowett show a cube law. Some results of static tests on armco iron for eccentric stress cycles by G. H. Keulegan of the Bureau of Standards have recently appeared,[7] which the cube law fits. These latter results were based on stress cycles of quite small amplitude, going up to less than 1/10 the elastic limit of the material.

The thing which we particularly wish to bring out is that for stress cycles of frequency of two to three a minute, up to fifty a second (that is, within

[7] G. H. Keulegan, U. S. Bur. Stds. Tech. paper No. 332, Nov. 1926.

the range of the experiments) the frictional loss per cycle is strictly independent of the frequency of performance of the cycles for every solid thus far tested.

We are unable by the use of the revolving rod method to check results of certain investigators, whose results do not show this independent frictional loss per cycle upon frequency, notably some of Voigt's[8] results and recent results obtained by Subrahmaniam.[9] The discrepancy may lie in the fact that in the present experiments relatively large masses of material were used which would tend to minimize surface effects. Further study is required to settle this point.

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APPENDIX I

Proof that in the rotating rod experiments the internal friction constant $\xi=\pi\eta/E$ where $\eta=\tan\phi$, Fig. 3, and E = Young's modulus. In Fig. 3 let the downward elastic deflection at the end of the rod from the undeflected position of its axis be d. Then using the symbols of Fig. 3, the potential energy of downward elastic deflection is given by

$$W_E=(1/2)R_E d\ ; \tag{4}$$

and the work per cycle due to internal friction by

$$W_F=2\pi R_F d \tag{5}$$

From (4) and (5)

$$W_F/W_E=4\pi R_F/R_E=4\pi\tan\phi=4\pi\eta \tag{6}$$

ξ is found by evaluating W_F and W_E and substituting in (6).

W_F, the frictional dissipation per cycle of rotation of the rod is found as follows: First the frictional loss ΔW_F in a slice of the rod of thickness Δx will be found. From the law of equation (1)

$$\Delta W_F=\int_0^{2\pi}\int_0^a \xi f^2{}_m r\ d\theta\ dr\ \Delta x \tag{7}$$

In this expression, f_m, the maximum stress per cycle, is, according elastic theory, a linear function of the radial distance r and may be expressed in terms of F_m, the maximum stress per cycle at the edge of this slice, and the radius a as follows:

$$f_m=F_m r/a \tag{8}$$

Substituting (8) in (7) and integrating,

$$\Delta W_F=(\pi/2)\xi F_m{}^2 a^2\Delta x \tag{9}$$

The total work per cycle W_F is found by integrating (9) along the entire length of the rod l after expressing F_m in terms of F_M the maximum stress in the entire rod, from the relation $F_M/F_m=l/x$

This gives

$$W_F=(\pi/6)a^2 l\xi F^2{}_M \tag{10}$$

[8] W. Voigt, Ann. d. Physik **47,** 671–693 (1892).

[9] G. Subrahmaniam, Phil. Mag. p. 711 (1925); p. 716 (1925); p. 1074 (1926); p. 854 (1927).

W_E, the total potential energy of elastic deflection of the rod is easily found in terms of the maximum stress F_M to be

$$W_E = \pi a^2 l F^2{}_M / 24E \tag{11}$$

Substituting (10) and (11) in equation (6),

$$W_F / W_E = 4\xi E = 4\pi\eta \; ; \; \xi = \pi\eta / E \tag{12}$$

which is the required relation.

In obtaining (11) the approximate beam theory was used, neglecting shearing stresses which are relatively small. The expression (10) can also be obtained by assuming some form of stress-strain hysteresis loop and summing the work done. The result comes out independent of the form of the hysteresis loop assumed, as of course it must, the only requirement being that the area of the loop conform to the energy requirement of equation (1).

Appendix II

Proof that $\xi = \delta/E$ where δ is the logarithmic decrement and E the elastic modulus. For any unit volume in the vibrating body, the potential energy P_m is given by

$$P_m = \tfrac{1}{2} E \epsilon_m{}^2 \tag{13}$$

where ϵ_m is the maximum value of strain at that point. Now dP_m/dt is equal to the frictional dissipation per sec.* in the unit volume in question. Therefore $E\epsilon_m d\epsilon_m/dt = -\xi f_m{}^2 n$ where n is the number of stress cycles per sec. Solving for ξ, and substituting for f_m its value $E\epsilon_m$,

$$\xi = -\frac{1}{En}\left(\frac{d\epsilon_m/dt}{\epsilon_m}\right) = -\frac{1}{En}\left(\frac{dy/dt}{y}\right) \tag{14}$$

where y is the vibration amplitude of the body at some convenient point. The above substitution of y for ϵ_m follows from the elastic theory, by which strains are proportional to vibration amplitude for *all* points in a body vibrating in a given mode. The exponential decrement curve of vibration amplitude is given by $y = y_0 e^{-\alpha t}$ from which $(dy/dt)/y = -\alpha$. But if T is the vibration period, the logarithmic decrement

$$\delta = \log e \frac{y_0 e^{-\alpha t}}{y_0 e^{-\alpha(t+T)}} = +\alpha T = \alpha / n = -\frac{1}{n}\left(\frac{dy}{dt}/y\right) \tag{15}$$

Substituting (15) in (14) the required relation $\xi = \delta/E$ is obtained.

* Actually the maximum value of P occurs at one instant only in each cycle, but P_m is here regarded as a continuous function represented by a curve drawn through the successive maxima.

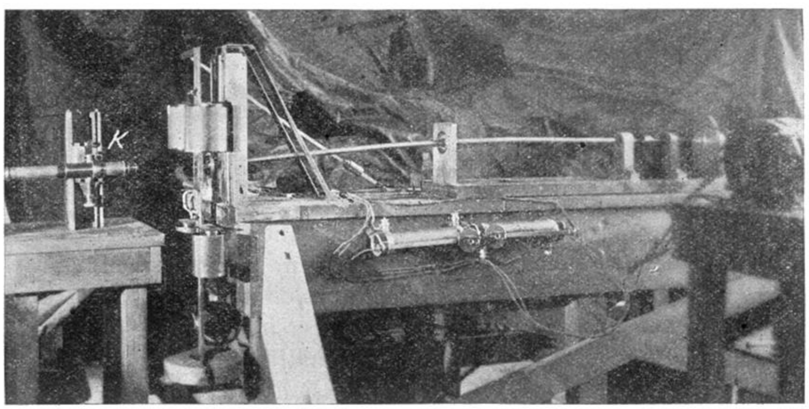

Fig. 1. Photograph of apparatus.