# THE LONGITUDINAL VIBRATIONS OF A LIQUID CONTAINED IN A TUBE WITH ELASTIC WALLS

By T. H. Gronwall

Abstract

This paper investigates the most general longitudinal vibrations of an elastic tube partly filled with liquid. From the rather complicated exact solution of the problem, a simple approximate formula is derived for the reduction of the velocity of sound in the liquid due to the vibrations of the wall of the tube.

## 1. Introduction

IN AN experimental investigation of the velocity of sound in liquids now being made by Mr. L. G. Pooler, the following arrangement is used: A vertical steel tube has its upper end free, and its lower end is threaded into a brass holder. The lower end of the tube is closed by a thin steel diaphragm, the edge of which is clamped into the holder, and the lower face of the diaphragm is in contact with an air chamber in which sufficient pressure is maintained to just balance the static pressure of the column of liquid in the tube. The tube stands on the bottom of a tank which may be filled with oil nearly to the top of the tube and the temperature may be held constant by thermostatic control. The diaphragm is now excited electromagnetically to a known frequency, and the tube gradually filled with the liquid to be investigated, and the height of the liquid noted at which resonance occurs. This height is found to vary almost linearly with the reciprocal of the frequency used throughout a wide range. The final readings are taken at the frequency of the free vibrations of the clamped diaphragm.

The velocity of sound $c$ calculated directly from these measurements is appreciably smaller than the velocity $c_0$ in an unlimited body of the liquid, and it has been already pointed out by Helmholtz[1] that this is due to the elastic vibrations of the tube.

In the present paper, the relation between $c$ and $c_0$ is obtained from the general equations of elasticity; as is to be expected, it is extremely complicated and takes the form of an infinite determinant set equal to zero. However, the introduction of suitable approximations reduces this equation to a very simple form, the final result being

$$c_0/c=1+y+3y^2+(12-\pi^2/3)y^3+(55-3\pi^2)y^4+\cdots,$$

where the small quantity $y$ is given by

$$y=\frac{\rho_1 c^2}{E}\left(\frac{b^2+a^2}{b^2-a^2}+\sigma\right);$$

here $a$ and $b$ are the interior and exterior radii of the tube, $E$ and $\sigma$, Young's modulus and Poisson's ratio for the tube, and $\rho_1$ the density of the liquid.

[1] Helmholtz, Gesammelte Abhandlungen, v. 1, p. 246.

This problem has been investigated by Lamb and others[2] with the important restrictions that the liquid fills the tube completely, and that the series expansions of the displacements and stresses are limited to a single term ; in this manner, Lamb obtains a result which leads to the following expansion

$$c_0/c=1+y+3y^2/2+5y^3/2+35y^4/8+\cdots.$$

This formula leads to smaller values of $c_0$ than the one given above, the difference being one-tenth of one percent or more, and therefore not negligible.

## 2. The Longitudinal Vibrations of the Tube

Let $a$ be the interior, $b$ the exterior radius, and $l$ the length of the tube. Using cylindrical coordinates $r$, $\theta$, $z$, the $z$-axis being directed vertically upwards along the axis of the tube, and the origin situated at its lower end, the longitudinal vibrations of the tube are obtained by setting the displacement $u_\theta$ equal to zero and assuming the displacements $u_r=u$ and $u_z=w$ to be independent of $\theta$. Writing

$$\Delta=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}, \qquad 2\varpi=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}, \tag{1}$$

and denoting the density of the tube by $\rho$, the equations of vibration are[3]

$$\begin{aligned}\rho\frac{\partial^2 u}{\partial t^2}&=(\lambda+2\mu)\frac{\partial\Delta}{\partial r}+2\mu\frac{\partial\varpi}{\partial z},\\ \rho\frac{\partial^2 w}{\partial t^2}&=(\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{2\mu}{r}\frac{\partial(r\varpi)}{\partial r}.\end{aligned} \tag{2}$$

The stress components are[4]

$$\begin{aligned}\widehat{rr}&=\lambda\Delta+2\mu(\partial u/\partial r), \qquad \widehat{r\theta}=0, \qquad \widehat{rz}=\mu(\partial u/\partial z+\partial w/\partial r),\\ \widehat{\theta\theta}&=\lambda\Delta+2\mu u/r, \qquad \widehat{\theta z}=0, \qquad \widehat{zz}=\lambda\Delta+2\mu(\partial w/\partial z).\end{aligned} \tag{3}$$

Since the lower end of the tube is fixed, and the upper end free, we have the boundary conditions

$$\begin{aligned}&w=0 \text{ at } z=0,\\ &\widehat{zz}=0, \qquad \widehat{rz}=0 \text{ at } z=l.\end{aligned} \tag{4}$$

On the cylindrical boundaries, the shear $\widehat{rz}$ must vanish :

$$\widehat{rz}=0 \text{ at } r=a \text{ and } r=b. \tag{5}$$

[2] H. Lamb, Memoirs and Proceedings of the Manchester Literary and Philosophical Society, vol. 42 (1898), no. 9, where further references are given. See also H. G. Green, Phil. Mag. **45,** 907 (1923) who, apparently unaware of Lamb's work, obtains a result which is substantially identical with his.

[3] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, 3rd ed. (Cambridge, 1920), p. 291.

[4] Love, l.c., p. 56 and 100.

Moreover,

$$u=u_b \text{ and } \widehat{rr}=-p_b \text{ at } r=b, \tag{6}$$

where $u_b$ is the radial displacement and $p_b$ the pressure of the liquid surrounding the tube, and similarly

$$\begin{aligned} u=u_a \text{ and } \widehat{rr}=-p_a \text{ for } r=a \text{ and } 0\leqq z\leqq h, \\ \widehat{rr}=0 \text{ for } r=a \text{ and } h\leqq z\leqq l, \end{aligned} \tag{7}$$

where $h$ is the height of the liquid in the tube.

Let $\nu$ be the frequency of the vibrations, and write $u=Ue^{2\pi i\nu t}$, $w=We^{2\pi i\nu t}$, where $U$ and $W$ are independent of $t$, and also write

$$\begin{aligned} \Delta &= e^{2\pi i\nu t}(\Delta_1(r)\sin qz+\Delta_2(r)\cos qz), \\ \varpi &= e^{2\pi i\nu t}(\varpi_1(r)\cos qz-\varpi_2(r)\sin qz). \end{aligned} \tag{8}$$

It then follows from (2) that

$$\begin{aligned} -\rho(2\pi\nu)^2U &= [(\lambda+2\mu)(d\Delta_1/dr)-2\mu q\varpi_1]\sin qz+[(\lambda+2\mu)(d\Delta_2/dr)-2\mu q\varpi_2]\cos qz, \\ -\rho(2\mu\nu)^2W &= [(\lambda+2\mu)q\Delta_1-(2\mu/r)(d(r\varpi_1)/dr)]\cos qz-[(\lambda+2\mu)q\Delta_2-(2\mu/r) \\ &\quad \cdot d(r\varpi_2)/dr]\sin qz, \end{aligned} \tag{9}$$

and substituting in (1), it is seen that $\Delta_1$ and $\Delta_2$ must satisfy the differential equation

$$\frac{d^2\Delta}{dr^2}+\frac{1}{r}\frac{d\Delta}{dr}-\left(q^2-\frac{\rho(2\pi\nu)^2}{\lambda+2\mu}\right)\Delta=0 \tag{10}$$

while $\varpi_1$ and $\varpi_2$ satisfy the equation

$$\frac{d^2\varpi}{dr^2}+\frac{1}{r}\frac{\partial\varpi}{dr}-\left(\frac{1}{r^2}+q^2-\frac{\rho(2\pi\nu)^2}{\mu}\right)\tilde{\omega}=0. \tag{11}$$

From the first of (4) and the second of (9), it now follows that

$$\Delta_1=\frac{2\mu}{(\lambda+2\mu)qr}\frac{d(r\tilde{\omega}_1)}{dr};$$

substituting in (10) and reducing by means of (11), it is readily seen that $\varpi_1=0$, so that $\Delta_1=0$. Calculating $zz$ from (3), it follows that in order to make $\widehat{zz}$ vanish at $z=l$, we must have $\cos ql=0$ or

$$q=q_n=\frac{2n+1}{2}\frac{\pi}{l}. \tag{12}$$

Writing

$$\begin{aligned} \alpha_n^2 &= q_n^2-\frac{\rho(2\pi\nu)^2}{\lambda+2\mu}, \\ \beta_n^2 &= q_n^2-\frac{\rho(2\pi\nu)^2}{\mu}, \end{aligned} \tag{13}$$

equations (8) to (11) show that the most general solution of (1) and (2) satisfying the first two boundary conditions (4) are

$$
\begin{aligned}
e^{-2\pi i\nu t}\Delta &= \sum_{n=0}^{n=\infty} [A_n I_0(\alpha_n r)+B_n K_0(\alpha_n r)] \cos q_n z,\\
e^{-2\pi i\nu t}\varpi &= -\sum_{n=0}^{n=\infty} [C_n I_1(\beta_n r)+D_n K_1(\beta_n r)] \sin q_n z,\\
e^{-2\pi i\nu t}\cdot \rho(2\pi\nu)^2 u &= \sum_{n=0}^{n=\infty} \{-(\lambda+2\mu)\alpha_n[A_n I_1(\alpha_n r)-B_n K_1(\alpha_n r)]\\
&\qquad +2\mu q_n[C_n I_1(\beta_n r)+D_n K_1(\beta_n r)]\} \cos q_n z,\\
e^{-2\pi i\nu t}\cdot \rho(2\pi\nu)^2 w &= \sum_{n=0}^{n=\infty} \{(\lambda+2\mu)q_n[A_n I_0(\alpha_n r)+B_n K_0(\alpha_n r)]\\
&\qquad -2\mu\beta_n[C_n I_0(\beta_n r)-D_n K_0(\beta_n r)]\} \sin q_n z.
\end{aligned}
\tag{14}
$$

From (3) and (14) we obtain

$$
\begin{aligned}
e^{-2\pi i\nu t}\cdot\frac{\rho(2\pi\nu)^2}{2\mu}\widehat{rz} &= \sum_{n=0}^{n=\infty} \{(\lambda+2\mu)\alpha_n q_n[A_n I_1(\alpha_n r)-B_n K_1(\alpha_n r)]\\
&\qquad -\mu(\beta_n^2+q_n^2)[C_n I_1(\beta_n r)+D_n K_1(\beta_n r)]\} \cos q_n z,
\end{aligned}
$$

and boundary conditions (5) become

$$
\begin{aligned}
&\frac{(\lambda+2\mu)\alpha_n q_n}{\mu(\beta_n^2+q_n^2)}[A_n I_1(\alpha_n a)-B_n K_1(\alpha_n a)]-C_n I_1(\beta_n a)-D_n K_1(\beta_n a)=0,\\
&\frac{(\lambda+2\mu)\alpha_n q_n}{\mu(\beta_n^2+q_n^2)}[A_n I_1(\alpha_n b)-B_n K_1(\alpha_n b)]-C_n I_1(\beta_n b)-D_n K_1(\beta_n b)=0.
\end{aligned}
\tag{15}
$$

From the expression above for $\widehat{rz}$ it is seen that the last boundary condition (4) is not satisfied except at $r=a$, $r=b$; this circumstance is a familiar one in all problems where one or both ends of a cylinder are free, and cannot be avoided.

Using equations (15) to simplify the expression for $u$ in (14) at $r=a$ and $r=b$, we find

$$
e^{-2\pi i\nu t}u = \sum_{n=0}^{n=\infty} \frac{\lambda+2\mu}{\mu}\,\frac{\alpha_n}{\beta_n^2+q_n^2}[A_n I_1(\alpha_n r)-B_n K_1(\alpha_n r)] \cos q_n z, \qquad r=a \text{ or } r=b,
\tag{16}
$$

and treating the expression for $\widehat{rr}$ obtained from (3) and (14) in the same manner,

$$
\begin{aligned}
e^{-2\pi i\nu t}\cdot\rho(2\pi\nu)^2\widehat{rr} &= \sum_{n=0}^{n=\infty} \{-\mu(\lambda+2\mu)(\beta_n^2+q_n^2)[A_n I_0(\alpha_n r)+B_n K_0(\alpha_n r)]\\
&\qquad -2\rho(2\pi\nu)^2\frac{(\lambda+2\mu)\alpha_n}{\beta_n^2+q_n^2}\left[A_n\frac{I_1(\alpha_n r)}{r}-B_n\frac{K_1(\alpha_n r)}{r}\right]\\
&\qquad +4\mu^2\beta_n q_n[C_n I_0(\beta_n r)-D_n K_0(\beta_n r)]\} \cos q_n z,
\end{aligned}
\tag{17}
$$

$r=a$ or $r=b$.

3. *The longitudinal vibrations of the liquid outside the tube.* Let $p$ be the excess of the pressure of the liquid over the hydrostatic pressure, then the density $\rho'$ of the liquid is given by

$$\rho' = \rho_0[1+(p/\kappa_0)], \tag{18}$$

where $\rho_0$ is the hydrostatic density and $\kappa_0$ the cubical elasticity of the liquid. The hydrodynamical equations are[5]

$$-\partial p/\partial r = \rho'(\partial^2 u/\partial t^2), \qquad -\partial p/\partial z = \rho'(\partial^2 w/\partial t^2), \tag{19}$$

and the equation of continuity

$$\frac{\partial \rho'}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho'\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial z}\left(\rho'\frac{\partial w}{\partial t}\right) = 0 \tag{20}$$

Since $p/\kappa_0$ is small, we may replace $\rho'$ by $\rho_0$ in (19) and in the last two terms in (20) ; writing $p = P(r)e^{2\pi i\nu t + qiz}$, $u = U(r)e^{2\pi i\nu t+qiz}$, $w = W(r)e^{2\pi i\nu t+qiz}$, it is seen from (19) that

$$\rho_0(2\pi\nu)^2 U = dP/dr, \qquad \rho_0(2\pi\nu)^2 W = qiP \tag{21}$$

and (20) becomes

$$\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(q^2 - \frac{\rho_0(2\pi\nu)^2}{\kappa_0}\right)P = 0. \tag{22}$$

The boundary conditions are (6) and

$$w=0 \text{ at } z=0, \qquad p=0 \text{ at } z=l, \qquad p \text{ finite as } r\to\infty \;; \tag{23}$$

conditions (23) are satisfied by giving $q$ the values $q_n$ defined by (12), and making

$$e^{-2\pi i\nu t}p = \sum_{n=0}^{n=\infty} F_n K_0(\gamma_n r)\cos q_n z \tag{24}$$

where

$$\gamma_n^2 = q_n^2 - \rho_0(2\pi\nu)^2/\kappa_0. \tag{25}$$

From (24) it follows that

$$e^{-2\pi i\nu t}\cdot\rho_0(2\pi\nu)^2 u = -\sum_{n=0}^{n=\infty}\gamma_n F_n K_1(\gamma_n r)\cos q_n z \tag{26}$$

and using (16) and (17), the boundary conditions (6) become

$$-\gamma_n F_n K_1(\gamma_n b) = \rho_0(2\pi\nu)^2\frac{\lambda+2\mu}{\mu}\,\frac{\alpha_n}{\beta_n^2+q_n^2}[A_n I_1(\alpha_n b) - B_n K_1(\alpha_n b)],$$

$$\rho(2\pi\nu)^2 F_n K_0(\gamma_n b) = -\mu(\lambda+2\mu)(\beta_n^2+q_n^2)[A_n I_0(\alpha_n b) + B_n K_0(\alpha_n b)]$$

[5] See Lamb's Hydrodynamics, 4th ed., chapter I.

$$-2\rho(2\pi\nu)^2\frac{(\lambda+2\mu)\alpha_n}{\beta_n^2+q_n^2}\left[A_n\frac{I_1(\alpha_n b)}{b}-B_n\frac{K_1(\alpha_n b)}{b}\right]$$
$$+4\mu^2\beta_n q_n[C_n I_0(\beta_n b)-D_n K_0(\beta_n b)].$$

Eliminating $F_n$ between the last two equations we obtain

$$\left[-c_n I_0(\alpha_n b)+\left(\frac{\rho_0(2\pi\nu)^2 d_n}{2\mu}\frac{K_0(\gamma_n b)}{\gamma_n K_1(\gamma_n b)}-\frac{d_n}{b}\right)I_1(\alpha_n b)\right]A_n$$
$$+\left[-c_n K_0(\alpha_n b)+\left(\frac{d_n}{b}-\frac{\rho_0(2\pi\nu)^2 d_n}{2\mu}\frac{K_0(\gamma_n b)}{\gamma_n K_1(\alpha_n b)}\right)K_1(\alpha_n b)\right]B_n$$
$$+f_n I_0(\beta_n b)C_n-f_n K_0(\beta_n b)D_n=0 \tag{27}$$

with the following notations

$$c_n=\mu(\lambda+2\mu)(q_n^2+\beta_n^2),\qquad f_n=4\mu^2 q_n\beta_n,$$
$$d_n=2\rho(2\pi\nu)^2\frac{(\lambda+2\mu)\alpha_n}{q_n^2+\beta_n^2},\qquad \Gamma_n=\frac{(\lambda+2\mu)q_n\alpha_n}{\mu(q_n^2+\beta_n^2)}. \tag{28}$$

On account of the smallness of the ratio $(2\pi\nu)^2/\mu$, the terms in (27) containing $\gamma_n$ may be neglected.[6]

From (27), (28) and (15), it is seen that $A_n$, $B_n$, $-C_n$ and $-D_n$ are proportional to the minors of the first row of either of the following determinants

$$\phi_n=\begin{vmatrix} -c_n I_0(\alpha_n a)-\frac{d_n}{a}I_1(\alpha_n a), & -c_n K_0(\alpha_n a)+\frac{d_n}{a}K_1(\alpha_n a), & -f_n I_0(\beta_n a), & f_n K_0(\beta_n a)\\ -c_n I_0(\alpha_n b)-\frac{d_n}{b}I_1(\alpha_n b), & -c_n K_0(\alpha_n b)+\frac{d_n}{b}K_1(\alpha_n b), & -f_n I_0(\beta_n b), & f_n K_0(\beta_n b)\\ \Gamma_n I_1(\alpha_n a), & -\Gamma_n K_1(\alpha_n a), & I_1(\beta_n a), & K_1(\beta_n a)\\ \Gamma_n I_1(\alpha_n b), & -\Gamma_n K_1(\alpha_n b), & I_1(\beta_n b), & K_1(\beta_n b)\end{vmatrix}$$

$$\frac{\mu(q_n^2+\beta_n^2)}{(\lambda+2\mu)\alpha_n}\cdot\psi_n=$$
$$\begin{vmatrix} I_1(\alpha_n a), & -K_1(\alpha_n a), & 0, & 0\\ -c_n I_0(\alpha_n b)-\frac{d_n}{b}I_1(\alpha_n b), & -c_n K_0(\alpha_n b)+\frac{d_n}{b}K_1(\alpha_n b), & -f_n I_0(\beta_n b), & f_n K_0(\beta_n b)\\ \Gamma_n I_1(\alpha_n a), & -\Gamma_n K_1(\alpha_n a), & I_1(\beta_n a), & K_1(\beta_n a)\\ \Gamma_n I_1(\alpha_n b), & -\Gamma_n K_1(\alpha_n b), & I_1(\beta_n b), & K_1(\beta_n b)\end{vmatrix} \tag{29}$$

Writing $G_n$ for the proportionality factor, it follows from (16), (17) and (29) that for $r=a$

[6] Making $n=0$, the largest value of $K_0(\gamma_n b)/b\gamma_n K_1(\gamma_n b)$ occurring in the experimental work is about 12, and the largest value of $\rho_0(2\pi\nu)^2/2\mu$ is $1.5\times10^{-4}$.

$$e^{-2\pi i\nu t}\cdot\rho(2\pi\nu)^2\,\widehat{rr}=\sum_{n=0}^{n=\infty}G_n\phi_n\cos q_nz,$$
$$e^{-2\pi i\nu t}u=\sum_{n=0}^{n=\infty}G_n\psi_n\cos q_nz. \tag{30}$$

4. *The longitudinal vibrations of the liquid inside the tube.* Replacing $\rho_0$ and $\kappa_0$ by the corresponding values $\rho_1$ and $\kappa_1$ for the liquid inside the tube, denoting by $h$ the height of the liquid at which resonance occurs, and observing that when this is the case, no force is transmitted from the diaphragm to the liquid, it is seen that the boundary conditions at the ends of the liquid column are

$$p=0 \text{ at } z=0 \text{ and } z=h. \tag{31}$$

Moreover, $p$ must be finite at $r=0$. It is then seen, exactly as at the beginning of paragraph 3, that $p$ is given by

$$e^{-2\pi i\nu t}p=\sum_{s=1}^{s=\infty}H_sI_0(\delta_sr)\sin(\pi msz/h), \tag{32}$$

where

$$\delta_s^2=\left(\frac{\pi ms}{h}\right)^2-\frac{\rho_1(2\pi\nu)^2}{\kappa_1}, \tag{33}$$

and $m=1$ for the fundamental vibration (corresponding to the smallest value of $h$ for which resonance occurs), while $m=2, 3, \cdots$, for the successive higher harmonics. From (32) and the first of (19), it now follows that

$$e^{-2\pi i\nu t}\cdot\rho_1(2\pi\nu)^2u=\sum_{s=1}^{s=\infty}H_s\delta_sI_1(\delta_sr)\sin(\pi msz/h). \tag{34}$$

There now remain to be satisfied only the boundary conditions (7). Since the functions $(2/l)^{1/2}\cos q_nz$ are orthogonal and normalized with respect to the interval $0\leqq z\leqq l$, it follows from the first of (30) that for $r=a$

$$G_n\phi_n=2/l\cdot\rho(2\pi\nu)^2\int_0^l e^{-2\pi i\nu t}\,\widehat{rr}\cos q_nz\,dz,$$

and since $\widehat{rr}=-p_a$ for $0\leqq z\leqq h$, but $\widehat{rr}=0$ for $h\leqq z\leqq l$ by (7), we obtain by making $r=a$ in (32)

$$G_n\phi_n=-2/l\cdot\rho(2\pi\nu)^2\int_0^h\sum_{s=1}^{s=\infty}H_sI_0(\delta_sa)\sin(\pi msz/h)\cos q_nz\,dz.$$

Writing

$$a_{ns}=\int_0^h\sin(\pi msz/h)\cos q_nz\,dz, \tag{35}$$

the preceding equation becomes

$$G_n\phi_n = -2/l\cdot\rho(2\pi\nu)^2 \sum_{s=1}^{s=\infty} H_s I_0(\delta_s a) a_{ns}. \tag{36}$$

Since the functions sin $\pi msz/h$ are orthogonal in the interval $0 \leqq z \leqq h$, it is seen from (34) that

$$H_s\delta_s I_1(\delta_s a) = 2/h\cdot\rho_1(2\pi\nu)^2 \int_0^h e^{-2\pi i\nu t} u_a \sin(\pi msz/h)\, dz,$$

and according to (7), $u_a$ is to be replaced by the second expression (30); with the aid of (35) this gives, upon replacing the summation subscript $n$ by $n_1$,

$$H_s\delta_s I_1(\delta_s a) = 2/h\cdot\rho_1(2\pi\nu)^2 \sum_{n_1=0}^{n_1=\infty} G_{n_1}\psi_{n_1} a_{n_1 s}. \tag{37}$$

Solving (37) for $H_s$, substituting in (36) and writing $G_n\phi_n = x_n$ we obtain the following infinite system of linear equations in $x_0, x_1, x_2, \cdots$

$$x_n + \sum_{n_1=0}^{n_1=\infty} c_{nn_1} x_{n_1} = 0 \qquad (n = 0, 1, 2, \cdots) \tag{38}$$

where

$$c_{nn_1} = (4/hl)\cdot\rho\rho_1(2\pi\nu)^4(\psi_{n_1}/\phi_{n_1}) \sum_{s=1}^{s=\infty} a_{ns}\cdot a_{n_1 s} I_0(\delta_s a)/\delta_s I_1(\delta_s a). \tag{39}$$

Since all $G_n$ do not vanish, (38) must have a solution where all $x_n$ do not vanish, and consequently the infinite determinant of the system must vanish, or

$$\begin{vmatrix} 1+c_{00}, & c_{01}, & c_{02}, & \cdots \\ c_{10}, & 1+c_{11}, & c_{12}, & \cdots \\ c_{20}, & c_{21}, & 1+c_{22}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots \end{vmatrix} = 0, \tag{40}$$

and this is the desired equation connecting $\nu$ and $h$ in the case of resonance. We shall now introduce successively such approximations as will reduce (40) to a numerically manageable form.

5. *Approximate expressions for the quotient* $\psi_n/\phi_n$. First assume $\alpha_n a$, $\beta_n a$, $\alpha_n b$, $\beta_n b$ to be small, and replace the Bessel functions by the principal parts of their power series expansions;[7] it then follows from (29) that the principal part of $\phi_n$ is

$$\begin{vmatrix} -c_n - \tfrac{1}{2}d_n\alpha_n, & -d_n/a^2\alpha_n, & -f_n, & f_n \log \alpha_n a \\ -c_n - \tfrac{1}{2}d_n\alpha_n, & -d_n/b^2\alpha_n, & -f_n, & f_n \log \alpha_n b \\ \tfrac{1}{2}\Gamma_n\alpha_n a, & -\Gamma_n/a\alpha_n, & \tfrac{1}{2}\beta_n a, & 1/a\beta_n \\ \tfrac{1}{2}\Gamma_n\alpha_n b, & -\Gamma_n/b\alpha_n, & \tfrac{1}{2}\beta_n b, & 1/b\beta_n \end{vmatrix}$$

[7] G. N. Watson, A Treatise on the Theory of Bessel's Functions, Cambridge 1922, pp. 77–80.

$$=ab\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{\Gamma_n f_n \alpha_n}{\beta_n}-c_n-\frac{d_n\alpha_n}{2}\right)\left[\frac{d_n}{2\alpha_n}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\frac{\Gamma_n f_n \beta_n}{2\alpha_n}\log\frac{b}{a}\right].$$

Referring to (28) and observing that for $n$ small, the quotient $2\mu q_n^4/\rho(2\pi\nu)^2$ is small for the values of $l$ and $\nu$ used in the experiments, it is seen that the second term in the square bracket above is negligible in comparison to the first. We may therefore write, approximately,

$$\phi_n=ab\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^2\frac{d_n}{2\alpha_n}\left(\frac{\Gamma_n f_n \alpha_n}{\beta_n}-c_n-\frac{d_n\alpha_n}{2}\right),$$

and treating $\psi_n$ in the same manner,

$$-\psi_n=b\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\frac{\lambda+2\mu}{2\mu(q_n^2+\beta_n^2)}\left[\frac{\Gamma_n f_n \alpha_n}{\beta_n}-c_n+\left(\frac{a^2}{b^2}-1\right)\frac{d_n\alpha_n}{2}\right].$$

Using (28), and introducing Young's modulus $E$ and Poisson's ratio $\sigma$ by the equations

$$E=\mu(3\lambda+2\mu)/(\lambda+\mu), \qquad \sigma=\lambda/2(\lambda+\mu),$$

the preceding expressions for $\phi_n$ and $\psi_n$ give

$$-\rho(2\pi\nu)^2\frac{\psi_n}{\phi_n}=\frac{1}{E}\cdot\frac{a}{\dfrac{b}{a}-\dfrac{a}{b}}\left[\frac{b}{a}+\frac{a}{b}+\sigma\left(\frac{b}{a}-\frac{a}{b}\right)\right.$$
$$\left.+2\sigma^2\frac{a}{b}\left\{\frac{E}{\rho}\left(\frac{q_n}{2\pi\nu}\right)^2-1\right\}^{-1}\right],$$

or neglecting the last term in the bracket, which is small compared to the others,

$$-\rho(2\pi\nu)^2\frac{\psi_n}{\phi_n}=\frac{a}{E}\left(\frac{b^2+a^2}{b^2-a^2}+\sigma\right)\cdot \tag{41}$$

For larger values of $n$, we calculate $\alpha_n$ and $\beta_n$ from (13) by the binomial theorem and retain the first two terms only :

$$\alpha_n=q_n-[\rho(2\pi\nu)^2]/[2q_n(\lambda+2\mu)],$$
$$\beta_n=q_n-[\rho(2\pi\nu)^2]/2q_n\mu.$$

We then expand $I_0(\alpha_n a)$by Taylor's theorem to two terms, so that since $I_0'(x)=I_1(x)$,

$$I_0(\alpha_n a)=I_0(q_n a)-\frac{\rho(2\pi\nu)^2 a}{2q_n(\lambda+2\mu)}\cdot I_1(q_n a),$$

and proceed similarly for the other Bessel functions. Expanding also the constants in (28) in the same manner, introducing all expansions in (29) and discarding terms containing higher powers of $1/q_n$ than the first, a somewhat lengthy but not otherwise difficult calculation gives

$$-\rho(2\pi\nu)^2\frac{\psi_n}{\phi_n}=\frac{2(1-\sigma^2)a}{E}\cdot\frac{\bar{\psi}_n}{\bar{\phi}_n}, \tag{42}$$

where

$$\begin{aligned}\bar{\phi}_n&=[Q(q_na)-Q(q_nb)][R(q_na)-R(q_nb)]-[P(q_na)-P(q_nb)]^2,\\ \bar{\psi}_n&=Q(q_nb)I_1(q_na)^2+2P(q_nb)I_1(q_na)K_1(q_na)+R(q_nb)K_1(q_na)^2-1,\end{aligned} \tag{43}$$

and the functions $P$, $Q$, and $R$ are defined by

$$\begin{aligned}P(x)&=x^2[I_0(x)K_0(x)+I_1(x)K_1(x)]+2(1-\sigma)I_1(x)K_1(x),\\ Q(x)&=x^2[K_0(x)^2-K_1(x)^2]-2(1-\sigma)K_1(x)^2,\\ R(x)&=x^2[I_0(x)^2-I_1(x)^2]-2(1-\sigma)I_1(x)^2,\end{aligned} \tag{44}$$

so that

$$P(x)^2-Q(x)R(x)=x^2+2(1-\sigma). \tag{45}$$

Substituting the leading terms in the power series expansions of the Bessel functions in (43) and (44), it is readily seen that (42) reduces to (41). But it is also found by numerical calculation that the expression (42) varies slowly as $n$ increases; for $n=0$, (42) differs from (41) in the seventh significant figure only, while for $n=5$ and $n=10$, the difference is about a unit in the fourth and third significant figures, respectively. Finally, for $n$ very large, we substitute the asymptotic expansions of the Bessel functions[8] either in (44), (43) and (42) or directly in the determinants (29), and obtain

$$-\rho(2\pi\nu)^2\psi_n/\phi_n=2(1-\sigma^2)/Eq_n. \tag{46}$$

When the tube is thin (the ratio $(b-a)/a$ being less than 0.15) and $n$ is small, the expression (41) for $\psi_n/\phi_n$ is inaccurate. A good approximation is however obtained by expanding the second and fourth rows in the determinant (29) in powers of $b-a$, and limiting the expansion to two terms. After some simple transformations, either determinant reduces to the product of two second order determinants, from which the Bessel functions disappear in consequence of the relation $I_0(x)K_1(x)+I_1(x)K_0(x)=1/x$, and we obtain

$$\begin{aligned}\phi_n=\frac{(b-a)^2}{a^2}\frac{\lambda+2\mu}{(q_n^2+\beta_n^2)^2}\rho^2(2\pi\nu)^4\Big\{\frac{4}{a^2}\Big[(3\lambda+2\mu)q_n^2-\frac{\lambda+\mu}{\mu}\rho(2\pi\nu)^2\Big]\\ -4q_n^2\rho(2\pi\nu)^2\frac{\lambda+\mu}{\mu}+\frac{\lambda+2\mu}{\mu^2}\rho^2(2\pi\nu)^4\Big\},\end{aligned}$$

$$-\psi_n=\frac{b-a}{a^2}\frac{\lambda+2\mu}{(q_n^2+\beta_n^2)^2}\rho(2\pi\nu)^2\Big[4q_n^2\frac{\lambda+\mu}{\mu}-\frac{\lambda+2\mu}{\mu^2}\rho(2\pi\nu)^2\Big],$$

or retaining only the first term in each bracket,

$$-\rho(2\pi\nu)^2\frac{\psi_n}{\phi_n}=\frac{1}{E}\left(\frac{1}{2\sigma}-1\right)\frac{a^2}{b-a}. \tag{41a}$$

[8] Watson, l.c., pp. 202–203.

6. *Reduction of the infinite determinant* (40). Expanding the determinant by Laplace's formula, we find

$$1+\sum_{n=0}^{n=\infty} c_{nn}+\sum_{\substack{n,n_1=0\\ n<n_1}}^{n,n_1=\infty}\begin{vmatrix} c_{nn}, & c_{nn_1} \\ c_{n_1n}, & c_{n_1n_1}\end{vmatrix}$$

$$+\sum_{\substack{n,n_1,n_2=0\\ n<n_1<n_2}}^{n,n_1,n_2=\infty}\begin{vmatrix} c_{nn}, & c_{nn_1}, & c_{nn_2} \\ c_{n_1n}, & c_{n_1n_1}, & c_{n_1n_2} \\ c_{n_2n}, & c_{n_2n_1}, & c_{n_2n_2}\end{vmatrix}+\cdots=0.$$

From (39), (41), (42) and (46) it is seen that every $c_{nn_1}$ contains the small factor $(2\pi\nu)^2/E$. Consequently, the second and following sums above contain the square, cube, $\cdots$ of this factor, and may be neglected in comparison to the first, so that (40) reduces to

$$1+\sum_{n=0}^{n=\infty} c_{nn}=0,$$

or, by (39)

$$1+\sum_{s=1}^{s=\infty}\frac{4\rho_1(2\pi\nu)^2}{hl}\frac{I_0(\delta_s a)}{\delta_s I_1(\delta_s a)}\sum_{n=0}^{n=\infty}\rho(2\pi\nu)^2\frac{\psi_n}{\phi_n}a_{ns}{}^2=0. \tag{47}$$

If we replace $-\rho(2\pi\nu)^2\psi_n/\phi_n$ by the expression (41) which is independent of $n$, it is seen from (46) that this value is too large when $n$ is large, so that the sum with respect to $n$ in (47) becomes too large numerically. We may compensate this error, at least in part, by replacing $I_0(\delta_s a)/\delta_s I_1(\delta_s a)$ by $2/\delta_s{}^2 a$ which is too small when $\delta_s$ is large, the asymptotic value of $I_0(\delta_s a)/\delta_s I_1(\delta_s a)$ being $1/\delta_s$. With these substitutions, (47) becomes

$$1-\frac{8\rho_1(2\pi\nu)^2}{hl}\cdot\frac{1}{E}\left(\frac{b^2+a^2}{b^2-a^2}+\sigma\right)\sum_{s=1}^{s=\infty}\frac{1}{\delta_s{}^2}\sum_{n=0}^{n=\infty}a_{ns}{}^2=0.$$

By (35), $(2/l)^{1/2}a_{ns}$ $(n=0, 1, 2, \cdots)$ are the Fourier coefficients with respect to the orthogonal and normalized functions $(2/l)^{1/2}\cos q_n z$ of a function $f(z)$ which equals $\sin(\pi msz/h)$ for $0\leqq z\leqq h$, but vanishes for $h\leqq z\leqq l$; hence Parseval's theorem gives

$$\begin{aligned}(2/l)\sum_{n=0}^{n=\infty}a_{ns}{}^2&=\int_0^l f(z)^2\,dz\\ &=\int_0^h\sin^2(\pi msz/h)\,dz=h/2,\end{aligned}$$

and the preceding equation becomes

$$1-\frac{2\rho_1(2\pi\nu)^2}{E}\left(\frac{b^2+a^2}{b^2-a^2}+\sigma\right)\sum_{s=1}^{s=\infty}\frac{1}{\delta_s{}^2}=0. \tag{48}$$

To calculate the sum occurring in (48), we observe that the velocity of sound $c_0$ in an unlimited body of the liquid contained in the tube is given by

$$c_0^2 = \kappa_1/\rho_1, \tag{49}$$

while the uncorrected velocity $c$ is given by the experiment as

$$c = 2\nu h/m. \tag{50}$$

From (33), (49) and (50) we find

$$\sum_{s=1}^{s=\infty} \frac{1}{\delta_s^2} = \left(\frac{c}{2\pi\nu}\right)^2 \sum_{s=1}^{s=\infty} \frac{1}{s^2-(c/c_0)^2},$$

and from the well known formula

$$\pi \cot \pi x = 1/x + \sum_{s=1}^{s=\infty} 2x/(x^2-s^2)$$

it follows that

$$\sum_{s=1}^{s=\infty} \frac{1}{\delta_s^2} = \left(\frac{c}{2\pi\nu}\right)^2 \cdot \frac{c_0}{2c}\left[\frac{c_0}{c} - \pi \cot \pi \frac{c}{c_0}\right]. \tag{51}$$

We introduce the notation

$$y = \frac{\delta_1 c^2}{E}\left(\frac{b^2+a^2}{b^2-a^2}+\sigma\right), \tag{52}$$

and write

$$c_0 = c/(1-\epsilon)\ ; \tag{53}$$

by means of (51), (52) and (53) it is now seen that (48) becomes

$$\frac{(1-\epsilon)^2}{1+(1-\epsilon)\pi \cot \pi\epsilon} = y$$

Introducing the expansion $\cot \pi\epsilon = 1/\pi\epsilon - \pi\epsilon/3 - \cdots$, and expanding the left hand member in the equation above in powers of $\epsilon$, we find

$$\epsilon - 2\epsilon^2 + (1+\pi^2/3)\epsilon^3 - \pi^2\epsilon^4 + \cdots = y. \tag{54}$$

Inverting the power series, it is seen that

$$\epsilon = y + 2y^2 + (7-\pi^2/3)y^3 + (30-7\pi^2/3)y^4 + \cdots, \tag{55}$$

and for the correction factor $1/(1-\epsilon)$, by which the measured velocity of sound $c$ has to be multiplied according to (53) in order to give the velocity of sound $c_0$ in an unlimited body of the liquid, (55) now gives

$$\begin{aligned} 1/(1-\epsilon) = 1 + y + 3y^2 + (12-\pi^2/3)y^3 + (55-3\pi^2)y^4 \\ + (1428 - 455\pi^2/3 + 137\pi^4/15)y^5 + \cdots. \end{aligned} \tag{56}$$

In the case of a thin tube, $(b-a)/a < 0.15$, the approximation (41) to $\rho(2\pi\nu)^2\psi_n/\phi_n$ is not valid for small values of $n$, for which it should be replaced by (41a).

For the two thin tubes used in the experimental work, $(b-a)/a=0.148$ and 0.067 respectively, and numerical calculation shows that in both cases, a good approximation to the sum with respect to $n$ in (47) is obtained by using (41a) for $n \leqq n_1$, where $n_1$ is the greatest value of $n$ for which

$$\sum_{n=0}^{n=n_1} a_{ns}{}^2 \leqq \frac{1}{3} \sum_{n=0}^{n=\infty} a_{ns}{}^2 \,;$$

for $n > n_1$, (41) is used as before. It is seen at once that this amounts to replacing (52) by

$$y=\frac{\rho_1 c^2}{E}\left[\frac{2}{3}\left(\frac{b^2+a^2}{b^2-a^2}+\sigma\right)+\frac{1}{3}\left(\frac{1}{2\sigma}-1\right)\frac{a}{b-a}\right]; \qquad (52a)$$

equations (55) and (56) of course remain unchanged.

In the paper quoted in the introduction, Lamb limits all expansions such as (14) to a single term, and his final formula (66) is equivalent to retaining only the first term, $s=1$, in our equation (48). The relation between $y$ and $\epsilon$ then becomes

$$\epsilon-\tfrac{1}{2}\epsilon^2=y,$$

whence

$$1/(1-\epsilon)=1/(1-2y)^{1/2}=1+y+3y^2/2+5y^3/2+35y^4/8+\cdots.$$

Department of Physics,
Columbia University,
February 28, 1927.