NOTES ON QUANTUM THEORY.

A THEORY OF ULTIMATE RATIONAL UNITS; NUMERICAL RELATIONS BETWEEN ELEMENTARY CHARGE, WIRKUNGSQUANTUM, CONSTANT OF STEFAN'S LAW.

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THE importance to science of a simple and established set of units has led to the almost universal adoption of the system based upon the centimeter, the gram and the second. These units themselves, however, have no rational significance. The centimeter was intended to be a certain simple fraction of the earth's circumference; the gram to be the weight of one cubic centimeter of water at the temperature of its maximum density, and the second is a certain fraction of the mean solar day. But neither the size of the earth nor its velocity of rotation is unique or even constant; nor does water in any notable way stand by itself among substances.

It has sometimes been assumed that there is some peculiar appropriateness in choosing three and only three fundamental units. This is by no means the case. It happens that the branch of physics which was first developed was the mechanics of rigid bodies. That science may be represented by a manifold of five dimensions, in which the three dimensions of space, one of time and one of mass,—or density,—are the five independent variables. In general for the treatment of such a manifold, five fundamental units would be necessary, but owing to the symmetry of the space of Euclidean geometry it is possible and convenient to adopt a single unit—that of length—in the discussion of all problems in three-dimensional geometry. The number of units required by the mechanics of rigid bodies is thus reduced to three. When, however, we consider other branches of physics, other units are necessary. Thus in the theory of heat it is customary to introduce new units, such as the unit of temperature, without attempting in the first instance to determine the dimensions of these units in terms of the units of mechanics. On the other hand, when important relations between existing units are discovered, the number of units which are to be regarded as fundamental is reduced, as we have pointed out in the case of the units of geometry.

Since it was, in fact, in the science of geometry that such a reduction in the number of fundamental units first occurred, it may be well to consider in this simple case the nature of the assumptions made.

In the first place, owing to the existence of rigid and movable measures of length, it was assumed that distances in all directions can be expressed in terms of a common measure. But fixing the unit of length does not determine immediately the unit of area, or of volume. The units actually chosen are the area of a square of unit side and the volume of a cube of unit edge, but many different units might have been proposed. Unit area might have been defined as the area of a circle of unit radius, or diameter, or circumference.[1] These units would be to the one actually chosen in the ratios π, $\pi/4$ and $1/4\pi$ respectively. There are numerous reasons for preferring the units actually in use, not the least important of which is the fact that an area or volume may be built up of squares or cubes, but not of circles or spheres. Admitting, then, that the choice is a wise one, it must nevertheless be emphasized that it is a choice of a somewhat arbitrary character.

Having made this choice, however, it is important to make a similar choice wherever a similar occasion arises, and thus to adopt consistently a system of "square" units.[2] In the case of angles the choice of unit has not been unanimous. In one system a simple fraction or multiple of a right angle is the unit, in the other the unit is the angle subtended by a circular arc equal to the radius of the circle and in this latter system a right angle is $\pi/2$. In our sense we shall speak of the former unit as a "square" unit just as the latter is commonly known as the "circular." We shall have occasion to notice how frequently the choice between two fundamental units involves the introduction or omission of some simple function of π.

When we add to the three dimensions of space the one dimension of time we obtain the four-dimensional manifold of kinematics. The science of kinematics involves geometry and a further degree of extension in time. Kinematics, therefore, in the broadest sense, includes all of

[1] The second of these units is actually in use in stating the cross section of circular wire, which is said to be one circular mil when the diameter is one mil.

[2] We might make an even more fundamental classification, for the axioms and propositions of geometry may be divided into those which depend upon parallel translation and those which depend upon rotation [cf. Wilson and Lewis, "The Space-Time Manifold of Relativity," Proc. Amer. Acad., *48*, 389 (1912)]. The parallel geometry is more general and is common to Euclidean geometry and to the non-Euclidean geometry of relativity. In this parallel geometry we may divide a given plane into unit parallelograms, and such conceptions as density can be readily introduced; but a field possessing radial symmetry about a point could not be adequately treated. For the most part, the square units we shall adopt will be identical as far as this is possible with those which could be obtained from the parallel geometry alone.

Euclidean geometry as a special case. In other words, geometry may be regarded as a cross-section of zero extension in time, through the manifold of kinematics.

In classical kinematics the units of length and of time were regarded as independent. The first great step toward the realization of the essential interdependence of these two units was made by Maxwell when he discovered the identity between the velocity of light and the ratio of the electromagnetic to the electrostatic units. Maxwell inferred that this identity was merely an evidence of the electromagnetic nature of light, but the development of the principle of relativity has led us to a more fundamental conclusion, which perhaps can best be stated in the following form: The constant c, known as the velocity of light, is merely a number imposed upon kinematics through the arbitrary choice of the centimeter and the second as units of length and time. In other words, all the equations of physics may be simplified by choosing a unit of time which is to the second in the ratio of 1 to 3×10^{10}. As examples we may cite the Maxwell equations of electricity and magnetism or the equation for the pressure of a beam of light in terms of the energy emitted or received per second, or any of the equations of non-Newtonian mechanics, in all of which cases every term which has the dimensions of time to any power contains the factor c to that power.

Not only has the principle of relativity shown the existence of this relation between the fundamental units of time and space, but it has enabled us to construct for kinematics a four-dimensional geometry which is as satisfactory as the Euclidean geometry of ordinary space. "Suppose that a student of ordinary space, habituated to the interpretation of geometry with the aid of a definite horizontal plane and vertical axis, should suddenly discover that all the essential geometrical properties of interest to him could be expressed by reference to a new plane, inclined to the horizontal, and a new axis inclined to the vertical. Whereas formerly he had attributed special significance to heights on the one hand and to horizontal extension on the other, he would now recognize that these were purely conventional and that the fundamental properties were those such as distance and angle, which remain invariant in the change to a new system of reference."[1]

In the four-dimensional geometry of kinematics an interval of time is a length, and we have the same reason for adopting a single fundamental unit in kinematics as in geometry. We shall use the centimeter provisionally as this unit. The dimensions of any kinematic quantity must therefore be represented by a power of a single dimension. This di-

[1] Wilson and Lewis, l. c.

mension, representing a one-dimensional interval in space or in time, will be denoted by $[I]$. Thus the dimensions of area are $[I]^2$, of volume $[I]^3$; both angle and velocity have zero dimensions; acceleration is $[I]^{-1}$, angular velocity $[I]^{-1}$ and angular acceleration $[I]^{-2}$.

Of the secondary units of kinematics the most important is that of velocity and there can be no dispute as to the desirability of choosing in this system the velocity of light as the unit of velocity, every other velocity is therefore a proper fraction, but in the case of other secondary units the choice is not always obvious. Thus the unit of angular velocity to be chosen depends upon the previous choice of unit angle.

In mechanics a new variable appears, and the choice of mass, energy, density, force, momentum, or any other simple mechanical quantity to serve as this new variable is quite arbitrary. Since it is customary we will regard mass as this primary variable and take, provisionally, the unit of mass, the gram, as independent of the unit of kinematics. All mechanical quantities may therefore be assigned dimensions in terms of interval $[I]$ and mass $[M]$. Thus energy and momentum have the dimensions $[M]$, force is $[M][I]^{-1}$; density and pressure $[M][I]^{-3}$, etc. The secondary units which have been universally adopted for these quantities seem to be unquestionably the best. In our system the gram is the unit of momentum and of energy as well as of mass, and the unit of force is one gram per unit interval, which may be derived from any of the older expressions for force, namely, energy per unit distance, momentum per unit time, or mass times acceleration.

The theory of heat involves at least one new variable. Since the discovery of the "mechanical equivalent of heat" the dimensions of thermal quantities can be expressed in terms of mechanical dimensions and of temperature. Thus it is customary to regard entropy and "heat capacity" as having the dimensions of energy divided by temperature. If now in addition to the "mechanical equivalent of heat" we find another very fundamental relation between thermal and mechanical quantities we shall be justified in eliminating temperature as a fundamental dimension. Such a relation is in fact furnished by the constant of the gas law, a law which is valid for all substances in the gaseous state at a sufficient degree of attenuation. This equation is ordinarily written: $PV = nRT$, where n is the number of mols. Since R is a universal constant we could, by using in place of the Centigrade degree a unit R times as small, simplify the formula to $PV = nT$. Since, however, the molecule seems a more desirable unit than the mol, and since the number of molecules, N, in one mol is now known about as accurately as the molecular weights, we will write for the number of molecules in n mols

$m = nN$, and the gas law becomes $PV = m(R/N)T = mkT$, where k is another universal constant. Now this constant appears not only in the equation for gases, but also in other important formulæ, and always to the same power as T, for example in the so-called Rayleigh equation for the distribution of energy in black radiation, $E_\lambda = c\lambda^{-4}kT$. We may therefore eliminate this constant once for all by writing $\theta = kT$, where the unit of θ is k times the Centigrade degree. The gas law becomes $PV = m\theta$. Any dimensions which k may have possessed are now incorporated in the temperature θ, and temperature therefore has the dimensions of energy or mass.

This process of reducing the number of fundamental units, or dimensions, which has been proposed only recently[1] for the thermal quantities, was adopted at the outset in the case of electrical quantities. If in the first instance electric charge had been measured in some arbitrary unit, then Coulomb's law would have contained a universal constant, like c or k. Charges were, however, so defined as to make this constant unity, and on the assumption that it is also free from dimensions the electrostatic system of units was developed. Another important system of units, the electromagnetic, becomes identical with the electrostatic when we take the velocity of light as unity. The adoption of the electrostatic system led to a number of equations in which the prominence of the factor π led Heaviside to suggest another unit of charge which is to the former as $1 : \sqrt{4\pi}$ and which Heaviside called the rational unit. The difference between the electrostatic and the Heaviside units is like the difference between the circular and square units of area; Coulomb's law expresses the force between two charges, each of which determines a field of radial symmetry while the Heaviside unit is so chosen that the capacity of a unit cubical condenser is unity.

In the preceding discussion of the methods by which geometrical, kinematical, mechanical, thermal and electrical quantities may be expressed in terms of a few fundamental dimensions or units, we have attempted to point out the difficulties attending the choice of secondary or derived units, but also to indicate, in spite of these difficulties, the way in which we may hope to diminish still further the number of fundamental independent dimensions. Having expressed all of the quantities hitherto considered in terms of two arbitrary units of interval and mass, it is necessary only to find *one* more relation of a universal

[1] Planck, Vorlesungen über die Theorie der Wärmestrahlung. First edition, p. 164. We should call attention here to the fact that Planck has suggested a system of units from which the arbitrariness of the ordinary units has been eliminated by the use of a sufficient number of universal constants. Planck did not, however, attempt to make his units rational in the sense in which we use the term.

character in order to express one of these units in terms of the other, or *two* more such relations to determine absolutely both of these units. We shall then have a system of units all of which are in a sense dimensionless. An interesting question now arises. Supposing that instead of finding two such, we found three or more such relations, would we not then have several different sets of rational units between which the choice would be purely arbitrary? Our answer to the question can best be expressed by stating our belief that these different sets of units will be dependent upon one another in a very simple way, and that if in the manner suggested we obtain the ultimate units of interval and of mass by the aid of two universal and fundamental relations, then all universal constants will prove to be pure numbers, involving only integral numbers and π, just as we have seen that in geometry several different units of angle, area and volume may be chosen, which, however, differ only by such a factor. This we shall call the *theory of ultimate rational units*.

There are a number of important constants from which to choose the two which are to determine the ultimate units: the constant of gravitation, the charge of one electron, the mass of an isolated electron, the constant of Stefan's law, the constant h of the Planck radiation equation, and several others which may posses universal significance. Indeed, the properties of any other atom are as universal, presumably, as those of the electron, but in the present state of our knowledge the electron appears to be unique in a sense in which the other atoms are not. Thus if it were decided to choose the mass of some atom as unit mass the mass of the electron would be the most promising choice. While each atom has a different mass, there is however one property of the electron which is in every sense unique, and that is its charge. Unlike the mass of the electron, the charge of the electron is constant under all circumstances. Because of these facts, and because of the success which has attended the various attempts to regard other branches of physics as parts of the science of electricity, we have without hesitation decided to take the electron charge as the first of the two fundamental constants needed for the final determination of the ultimate units.

The dimensions of electric charge become in our system $[M]^{\frac{1}{2}}[I]^{\frac{1}{2}}$ and it is interesting to note that two of the other constants mentioned above have, as dimensions, powers of the dimensions of electric charge, for Planck's h has the dimensions $[M][I]$ and the constant a of Stefan's Law, $[M]^{-3}[I]^{-3}$. We must expect, therefore, that the square of the electron charge, Planck's h, and the reciprocal cube root of a, will differ only by simple numerical factors.

The charge of one electron in ordinary electrostatic units we may call ϵ.

If we call the same charge in Heaviside electrostatic units ϵ', then $\epsilon' = \sqrt{4\pi}\epsilon$. Numerous other systems of electrical units might be invented, and we shall refer to one other in which the electron charge is $\epsilon'' = 4\pi\epsilon$, and which has advantages in certain cases. For example, the energy emitted per second by a simple harmonic oscillator, which has played so large a part in the modern theory of radiation, is in the three units, repectively

$$-\frac{dE}{dt} = \frac{8\pi^2}{3}\frac{E}{M}\epsilon^2\nu^2 = \frac{2\pi}{3}\frac{E}{M}\epsilon'^2\nu^2 = \frac{1}{6}\frac{E}{M}\epsilon''^2\nu^2,$$

where E is the energy, M the mass, and ϵ the charge of the oscillator of frequency ν. Our opinion as to the relative desirability of these units may be modified as the properties of the electron become better understood. Formerly, when electricity was regarded as a continuum, the Heaviside system was unquestionably superior. We have seen that in geometry the unit square is a more satisfactory measure of area than the unit circle, and that in general we must attempt to choose consistently units similar to the unit square and the unit cube. Thus we regard a uniform vector field as simpler than one which possesses radial, that is to say spherical, symmetry. The Heaviside system was based upon the properties of the uniform electric field between the plates of an infinite plane condenser. Whether our present recognition of the atomic nature of electricity deprives the Heaviside system of its advantages is a question which merits the most careful scrutiny.

As ordinarily expressed the dimensions of ϵ^2 are energy times length. According to the choice of the electrical unit it will be natural to choose ϵ^2, $4\pi\epsilon^2$ or $(4\pi\epsilon)^2$ as the unit of energy times interval with the dimensions $[M][I]$. For convenience in the following calculation we shall continue to express the energy in ergs. In terms of this unit we shall expect to find the constants a and h to be simple numbers.

Let us consider first the constant a of Stefan's law, which reads

$$E = aVT^4,$$

where E is the energy (ergs) in the volume V (c.c.) at the absolute temperature T (deg. Cent.). We may put this equation in the form

$$E = ak^{-4}V(kT)^4 = ak^{-4}V\theta^4. \qquad (1)$$

Both E and θ are now expressed in ergs.

Hence $a = k^4(E/V\theta^4)$, where the quantity in parenthesis is measured in reciprocal (erg cm.)^3. If we choose now a new unit of energy times length, namely, ϵ^2, ϵ'^2 or ϵ''^2, we may express the factor in parenthesis in these units by writing:

$$a = \frac{k^4}{\epsilon^6}\left(\frac{E\epsilon^6}{V\theta^4}\right) = \frac{k^4}{\epsilon'^6}\left(\frac{E\epsilon'^6}{V\theta^4}\right) = \frac{k^4}{\epsilon''^6}\left(\frac{E\epsilon''^6}{V\theta^4}\right). \quad (2)$$

And we may expect to find that the three quantities in parenthesis are simple numbers. In fact, it will be seen presently that the third of these numbers is unity, that is,

$$\frac{a\epsilon''^6}{k^4} = \frac{a(4\pi\epsilon)^6}{k^4} = \frac{E\epsilon''^6}{V\theta^4} = 1. \quad (3)$$

The constant a cannot involve π except in so far as this factor is introduced in the unit of energy times length, and therefore only as π^{3p} where p is an integer. The reason for this statement lies in the uniformity of distribution of energy in the hohlraum. There is nothing in the nature of the problem to require the use of any but square units. If Stefan's law were expressed in some other form, for example, if we should write $dE/dt = bT^4$, where dE/dT is the rate of emission of energy from a black sphere of unit radius at the temperature T, b, like a, would be a universal constant, but would contain intrinsically the factor π, since it involves quantities which cannot be completely expressed in square measure. While the constant a therefore does not contain intrinsically the factor π, it might obviously contain some simple coefficient such as 2 or 3/4. This is a more difficult question, but we shall attempt presently from *a priori* considerations to show that no such coefficient is to be expected in the constant a. Meanwhile this will be regarded as an empirical fact on the basis of the following calculations.

The results of numerous experimental determinations of the constant of Stefan's law are extraordinarily discordant. Without considering the earlier measurements, the following results have been published during the last two years for the value of $\sigma \times 10^5$ (where $\sigma = ac/4$): 5.67,[1] 5.80,[2] 5.45,[3] 5.96,[4] 5.54,[5] 5.89,[6] 6.33.[7] The mean of all these values is 5.81, or 5.72 excluding the incomprehensibly high value of Féry and Drecq. Coblentz[8] after a very careful and critical survey of the various determinations gives as the probable value 5.7.

Let us now calculate the value of σ from our assumption (equation 3) that $a = k^4/\epsilon''^6 = k^4/(4\pi\epsilon)^6$. The recent determination of the elementary

[1] Shakespear, Proc. Roy. Soc., *A 86*, 180 (1912).

[2] Gerlach, Ann. d. Phys., *38*, 1 (1912).

[3] Kurlbaum, Verh. d. phys. Gesell., *14*, 576 (1912) (recalculated from older result).

[4] Puccianti, Nuov. Cim., *4* (6), 31 (1912). See also Nuov. Cim., *4* (6), 322 (1912).

[5] Westphal, Verh. d. phys. Gesell., *14*, 987 (1912).

[6] Keene, Proc. Roy. Soc., *A 88*, 49 (1913).

[7] Féry and Drecq, Journ. de Phys. (5), *3*, 380 (1913).

[8] Coblentz, Jahrb. d. Radioakt., *10*, 340 (1913).

charge by Millikan[1] for which he claims an accuracy of 0.2 per cent., a claim which seems to be warranted by the extreme care of his measurements, give $\epsilon = 4.774 \times 10^{-10}$. From the value of ϵ, k can be obtained at once from the equation $k = R\epsilon/F$, where F is the Faraday equivalent in the same units as ϵ. Hence $k = 1.372 \times 10^{-16}$, and since R and F are known with a very high degree of accuracy, the percentage error in k is the same as in ϵ. We thus find $a = 7.60 \times 10^{-15}$ in which the percentage error is twice that in ϵ, as is obvious if we write $a = R^4/(4\pi)^6\epsilon^2F^4$. From this value of a we find $\sigma = 5.70 \times 10^{-5}$.

We believe that the value of a thus obtained is the true constant of Stefan's law, subject only to such uncertainties as are due to the experimental errors in R, F and ϵ.

This result may be presented in a different way. If the equation of the perfect gas be written in the form $PV = m\theta$ and Stefan's law in the form $E = V\theta^4$, these two equations suffice to determine the value of a unit of energy times length, and this unit would then prove to be, within the limits of experimental error, equal to the unit which we have called ϵ''^2. It may be asked why the coefficient of the energy equation representing Stefan's law should be simpler than the coefficient of the corresponding entropy equation, $S = \beta V\theta^3$, where S is the entropy of a hohlraum of volume V at temperature θ, and β is another universal constant, namely $\beta = 4a/3$. Why, then, should we expect a to be unity rather than β? In the units we have chosen, the energy of a unit volume of a hohlraum at unit temperature is equal to two thirds of the energy of one molecule (possessing three degrees of freedom), of any perfect gas at unit temperature. No statement of similar simplicity can be made with respect to the entropy of the hohlraum and that of a molecule. While the average energy at a given temperature is the same for all molecules, and independent of all other conditions, the average entropy of a molecule depends upon the concentration and the chemical nature of the molecule as well. For this reason we should expect a simplicity in the energy equations which the entropy equations do not possess.

We may next proceed to a discussion of the constant h, which appears in the Planck equation for the distribution of energy in the spectrum of a black body, and which must appear in any radiation formula of which the Wien equation is a limiting case. If the Wien equation is in fact true in the limit, the constant h, the "Wirkungsquantum," must be a significant universal constant, irrespective of the validity of any more general expression, such as that of Planck.

If the Planck equation is correct it is possible to calculate h directly

[1] Millikan, PHYS. REV. (2), 2, 79 (1913).

from a, by means of the equation obtained by integrating the Planck equation:

$$a = \frac{8\pi^5 k^4}{15c^3h^3}. \tag{4}$$

If, now, we write, as before,

$$a = \frac{k^4}{\epsilon''^6}$$

then $15c^3h^3 = 8\pi^5\epsilon''^6$, or

$$h = \frac{\epsilon''^2}{c}\sqrt[3]{\frac{8\pi^5}{15}} \tag{5}$$

and substituting we find $h = 6.56 \times 10^{-27}$.

Now the constant of the Wien equation, $c_2 = ch/k$ has been determined with considerable accuracy. The latest value obtained by Coblentz[1] as the mean of several determinations in the Bureau of Standards at Washington is $1.4456 \pm .0004$, while Warburg, Leithäuser, Hupka and Müller[2] in the Reichsanstalt at Berlin obtain as a mean $1.437 \pm .004$. The value of c_2 calculated from h as given above is 1.434.

If one admits that the Planck formula is correct, the close agreement between this value and those obtained experimentally affords strong evidence of the correctness of our calculation of the constant of Stefan's law. On the other hand the agreement does not demonstrate the validity of the Planck formula. Numerous substitutes for that formula might be suggested which would agree equally well with the experimental facts and which would not only satisfy the Wien displacement law, but also approach at the two limits the Wien and the Rayleigh radiation formulæ. Indeed it seems hardly likely that if h is a quantity of really fundamental significance it is represented by so complicated a formula as (5).

We hope to revert to this question of the true radiation formula and the significance of the constant h, in a later publication. At present it need only be pointed out that the fundamental quantum in the theory of electricity and of radiation is the quantum of electricity, and that the so-called "Wirkungsquantum" is merely the square of this fundamental quantum with a simple numerical coefficient, depending upon the units chosen.

We shall not attempt at this point to decide finally as to the choice of the ultimate rational units, but the two experimental data by which these units will be determined will be the charge of the electron and the mass of the isolated electron. It is to be hoped that a further study of

[1] Jahrb. d. Radioakt., *10*, 340 (1913).

[2] Ber. Berl. Akad., *2*, 35 (1913).

the law of gravitation will soon enable us to obtain the constant of that law in terms of the chosen ultimate units, as a pure number.

We have attempted in this paper to show the relation between the so-called fundamental and derived units of geometry and physics, and have outlined a theory of ultimate rational units in accordance with which we have obtained relations between several important universal constants. In particular we have derived from this theory a value for the constant of Stefan's law which we believe to be far more accurate than any of the values of this quantity obtained by direct experiment. We have shown that the fundamental quantum in physics is the charge of the electron.

In a following paper we shall attempt to show that the numerous theories of the so-called energy quantum which have recently been advanced are unnecessary for the explanation of the phenomena with which they deal.

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