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A STUDY OF THE LONGITUDINAL VIBRATION OF WIRES.

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I. INTRODUCTION.

THE elastic behavior of wires has been very carefully investigated by means of torsional vibrations. This method lends itself most readily, perhaps, to the study of internal friction, for the period may easily be regulated to any desired length, the amplitude may be made large, and, by suitable adjustment of the suspended mass, the free vibration may be continued for a comparatively long time.

Weber[1] was the first to observe that a vibrating body is damped in a vacuum, thus showing with certainty the existence of an internal cause of damping, and O. E. Meyer[2] made the first quantitative study of the phenomenon, by observing the damping of torsional vibrations.

Voight[3] measured the damping of flexural, as well as torsional vibrations, with the view of determining whether, according to Boltzmann's theory, the logarithmic decrement is independent of the period, as the latter shows it should be if damping is caused by the elastic after-effect; or whether there is really internal friction, in which case the logarithmic decrement should depend on the period. The results of his work were not entirely in favor of either theory. For flexular vibrations he found that, for copper and some other metals, the logarithmic decrement varied inversely as the period, thus supporting the internal friction theory, while with aluminum, cast iron, and cadmium the logarithmic decrement was much more nearly constant. In a few cases the decrement actually increased with increasing period, which is at variance with both theories. His conclusion was that in some cases the preponderant factor in damping is internal friction, in other cases it is the after-effect. Voight also found, as Schmidt[4] had done earlier, that the logarithmic decrement varies

[1] Pogg. Ann., 34, p. 247, 1835.
[2] Ibid., 113, pp. 77, 193, 1861.
[3] Abh. der Königl. Gesell. der Wiss. zu Göttingen, 38, 1892.
[4] Wied. Ann., 2, p. 48, 1877.

approximately as the square of the amplitude. Streintz,[1] however, found it to be independent of the amplitude in torsional vibrations, and also of the period, if the period were varied by changing the moment of inertia of the suspended mass. Guthe and Sieg[2] found certain platinum-iridium wires, when vibrated torsionally, to give a maximum value of the logarithmic decrement, and the position of the maximum on the decrement-amplitude curve depended on the initial amplitude. This work will be referred to more in detail later in comparing with the damping of longitudinal vibrations. Harris,[3] using bismuth wires, observed a decrease of the logarithmic decrement with decreasing amplitude, but no maximum or minimum values.

Other experiments, notably those of Streintz, Wiechert,[4] and Bouasse[5] and Carriere, have shown that the previous history of the wire very markedly influences the after-effect and the damping. While the after-effect has been carefully observed for longitudinal deformation, as far as the writer has been able to find, little or nothing has been done on the problem of the quantitative measurement of the damping of longitudinal vibrations.

The present investigation is an attempt to measure as accurately as possible the period of longitudinal vibrations of wires, to consider possible causes of difference between this and the theoretical period, and to measure the damping of the vibrations. Both static and dynamic observations will be given which indicate that the modulus of the wires is not exactly a constant with varying extension. An expression for the period involving the amplitude will be derived, which shows that this variation of the modulus cannot account for the observed variation of the period with amplitude. It will also be shown that, with the ordinary assumptions, the variation of the logarithmic decrement with amplitude is not sufficient to explain the variation of the period.

II. Elongation of the Wires.

Apparatus.

Four wires were used; one each of copper, steel, phosphor-bronze, and platinum-iridium. The steel wire was new piano-wire; the copper, a piece from a spool of commercial wire. Three of the four wires were about 230 cm. long, while the fourth, the platinum-iridium, was only about

[1] Wien. Ber., 69, II. Abt., p. 337, 1874: 80, II. Abt., p. 397, 1879.

[2] Phys. Rev., Vol. 30, No. 4, 1910.

[3] Phys. Rev., Vol. 35, No. 2, 1912.

[4] Wied. Ann., 50, p. 546, 1893.

[5] Ann. de Fac. des Sci. de Toulouse, II., 431; III., 217; IV., 357. Ann. de Chimie et de Physique, 8me Serie, t. 14, p. 190.

140 cm. long. No attempt was made to take account of previous treatment of the wires, since the adjustment for vibration was rather difficult, and it was usually necessary to vibrate the wire for some time before measurements could be made. Excepting the platinum-iridium, which contained 40 per cent. of iridium, and was prepared by Dr. Heraeus, of Hanau, the composition of the wires was not well known, and it is not to be expected that the results are rigidly characteristic of these metals. In fact, it is known that hardness has a great influence on the internal friction of wires. Annealed wires exhibit greater damping than tempered ones of the same material.

A heavy iron bracket was fixed rigidly to the brick wall of the laboratory. A strap of iron was bolted firmly to the front of the bracket; but this was not used as a clamp, for clamping the wire would flatten it, and weaken it at the point where it entered the clamp, besides rendering uncertain the location of the upper end of the vibrating length. Instead, a small slot, only large enough to admit the wire easily, was sawed in the iron strap. This slot was filled with solder, and the upper end of the wire was soldered in, thus attaching the wire very firmly to the support. In order to keep the temperature uniform and fairly constant, the wire was enclosed in a wooden box, the internal cross-section of which was about 16 $cm.^2$ The upper end of the box was lightly covered with cotton batting to prevent air currents passing through, yet not so as to interfere with the vibrations. Two thermometers were inserted through the side of the box, one near the top and the other near the bottom.

A fan was installed near the box to keep the air in the room in circulation. Without the fan the temperature of the upper part of the box sometimes became as much as 3 degrees higher than the lower end, especially in colder weather, when the room was heated by pipes near the ceiling. When the fan was working the upper and lower parts of the box differed usually by not more than $0^\circ.1$ C. A variation of one or two degrees has very little effect on the period of vibration, since the modulus changes only slightly with the temperature; but in determining statically the elongation due to a certain load, it was very important that the temperature should not change between readings.

The following precautions were taken to straighten the wires before using. The steel and phosphor-bronze wires were suspended and stretched for about one day by ½ the maximum load which they were required to carry. The copper wire was very soft, and was straightened by passing it through a drawplate, using a hole the size of the wire. This straightened the wire without drawing it any smaller. The platinum-iridium wire was moderately loaded and annealed by passing through it an electric current sufficient to raise it to a yellow heat.

Instantaneous Recovery.

In order to make a comparison of static and dynamic moduli, the simple method of measuring the elongation directly was chosen. Consider a wire loaded as in Fig. 1. Let P be the position of the lowest point of the wire when at rest. From the elastic forces called out by the distortion, P has a tendency to return to some point O. This point may not be at all the position of P before the load was imposed. It does not even remain constant when P moves up and down. The variation of O is due to two things: (1) the elastic after-effect; (2) the heating effect of varying the length of the wire. When the wire is at rest, (2) is not to be considered; therefore, disregarding this for the present, let us call $OP = e$ the instantaneous recovery. This is evidently the distance on which the restoring force depends, and this is the distance it is desired to measure.

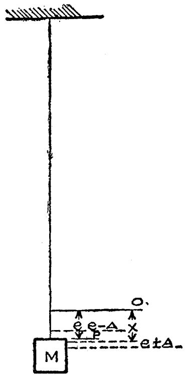

Fig. 1.

Lever Systems for Measuring Recovery.

In order to determine changes occurring very soon after unloading, the device shown in Figs. 2, 3, and 4 was used. Figs. 2 and 3 show two sections at right angles to each other. Fig. 4 shows one of two similar

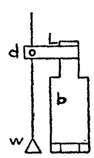

Fig. 2.

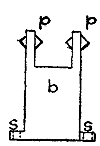

Fig. 3.

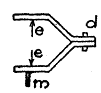

Fig. 4.

levers, made each of two stips of brass bent and soldered together along one half their length. Small conical hollows were made at *ee*, Fig. 4, to receive the points of p in Fig. 3. The arms of the lever were bent just far enough apart so they would hold on the axle p without play, and with little friction. The lever was thus free to move about the axis of p in a vertical circle. L, Fig. 2, is this lever seen from the side. d is a small closely fitting pin to one end of which the wire W was lightly soldered. m is a small mirror with its plane at right angles to the lever, and containing the axis of p. One of the levers was used for the wire under test, the other for a comparison wire hung about 2 cm. from the first. The mirrors on the two levers were placed so close together that by means of a single telescope two scales could be viewed at the same time, one reflected from each mirror.

Wire No. 2 indicated any sagging of the bracket, and change of temperature. It was found that with the fan running and the box in position the temperature change during the readings necessary for a single determination of the elongation was ordinarily too small to be observed. For measuring the elongation the mass of figure 1 was replaced by a scale pan.

A mass was placed on the pan and after some time a reading was taken on both scales, and the mass was immediately removed. Then readings were taken on the wire under test at the end of 5 sec., 15 sec., 1 min., 2 min., and 3 min. Two or three readings on the comparison wire gave corrections to be applied for sagging of the support, and for temperature changes. The comparative rigidity of the lever system allowed a reliable reading to be taken 5 sec. after removal of the mass. The results are shown for the copper wire in figure 5. The numbers in parentheses refer to different masses removed from the pan as shown in the accompanying table.

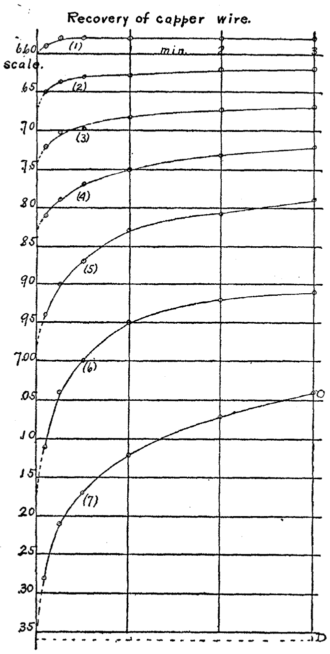

Fig. 5.

Copper.

	Mass, gms.
(1)	461
(2)	930
(3)	1,401
(4)	1,872
(5)	2,343
(6)	2,815
(7)	3,295

Each mass was left on the pan 2 min. The relative positions of the curves in Fig. 5 show how slowly the copper wire recovered from strain. The after-effect and the heating effect are both present in these curves. The curve for a time interval less than 5 sec. is unknown, but producing it as the observed part seems to indicate we would have an intersection on the scale axis as shown by the dotted line.

Owing to the various quantities upon which the computed elongation depends when deduced from the scale readings, and the possibility of error, especially in the motion of the pin d, shown in Fig. 2, it was thought best to depend on the above apparatus only for the small changes taking place in the first minute or two, and to measure the entire recovery of the wire simply by means of a micrometer microscope focused directly on a point of the wire. Then the correction shown by Fig. 5, and that for the sagging of the bracket were applied afterward.

For all except copper it was seen that in one half min. to one min. the wire had entirely recovered from the strain. The microscopic readings for these could therefore be taken soon after removal of the load. With the copper they were all taken long enough after release so that the change in length had nearly ceased, but the time of each reading was noted, and the proper correction applied from the plots.

Instantaneous Recovery Independent of Period of Deformation.

It was found by means of the lever system that the length of time the load remained on the pan did not influence the instantaneous recovery. The following table shows the instantaneous recovery for various periods of deformation of the copper wire. t is the length of time the weight remained on the pan, e the instantaneous recovery in cm. of the scale. It is the difference between the intersection of the curve with the scale axis (see Fig. 5) and the reading just before release.

TABLE I.

Load = 2 Kg.

t min.	e.
1	11.25
5	11.23
15	11.20
30	11.22
60	11.22
30	11.23
15	11.21
5	11.20
1	11.23

This indicates that although the wire may suffer a gradual change when distorted by a constant force, a given point of the wire has always the same instantaneous recovery.

Since the change observed by aid of the levers and mirrors after the removal of the load is small compared with the total recovery, we may compute the change s in the length of the wire, corresponding to the distance OD shown on the plot of Fig. 5, by the equation

$$s = \frac{1}{2}\frac{r}{a}OD, \tag{1}$$

where r is the radius of the lever arm and a is the distance from mirror to scale. r was determined by measuring, with a microscope, the diameter of the circle in which the pin *d* moves when the lever is rotated about the axis. *r* was thus found to be 2.124 cm.

Temperature Change Due to Loading and Unloading.

The formula for temperature change given by Sir William Thomson[1] is

$$t = \frac{AT\alpha}{\omega c}\Delta p, \qquad (2)$$

where A = the reciprocal of Joule's equivalent.
T = abs. temp.
α = coeff. of linear expansion.
ω = linear density of the wire.
c = specific heat.

Edlund[2] found, experimentally, that this formula gave too high a value for t, and according to his results $A = \frac{1}{68270}$ when Δp is in gm.

When the load is suddenly removed the wire is heated, and as it cools it contracts, which contraction added to the after-effect gives the curves in Fig. 5. To compute the part due to the cooling of the wire we have the change of length due to a change in temperature t,

$$\Delta l = \frac{AT\alpha^2 L}{\omega c}\Delta p. \qquad (3)$$

If we take $L = 200$ cm., and $\Delta p = 3{,}000$ gm., Table II. shows the Δl of equation (3) for each of the four wires. Edlund's value of A is used.

TABLE II.

Wire	Δl cm.
Steel	0.0010
Copper	0.0004
Phos. br.	0.0020
Pt.-ir.	0.0008

The instantaneous recovery of the wires is given by

$$e = e' - s + \Delta l, \qquad (4)$$

where e' is the elongation as measured by the microscope directly. Δl must be determined for the particular length used in obtaining s; then if this is not the same as the length used in measuring e', $s - \Delta l$ must be reduced to that length before subtracting from e'. For the phosphor-bronze and platinum-iridium $s = \Delta l$ as nearly as the plots will show. For steel, s was a little more than twice Δl, and for copper, nearly ten

[1] Math. and Phys. Papers, Vol. 3, p. 66.
[2] Pogg. Ann., 126, p. 539, 1865.

times Δl. This indicates that phosphor-bronze and platinum-iridium have no after-effect, yet this supposition does not agree with the results of timing. Reasons will be given later for supposing that all the wires have an elastic after-effect.

Variation of Instantaneous Recovery with Temperature

To find how the instantaneous recovery varies with the temperature, which is equivalent to finding the variation of the modulus with temperature, the room was kept successively at ten different temperatures ranging from 15°.7 to 26°.4 C., and the instantaneous recovery was determined for each temperature for all the masses concerned in Fig. 5. This test was made for copper only, for the variation of the modulus with temperature is much smaller for the others. According to Pisati it is only one part in 10,000 per degree for steel at temperatures from 0° to 50°. With copper, however, it is large enough to be detected over the range of only 10° employed here, and by plotting instantaneous recoveries and temperatures the equation

$$e_t = e_{23}[1 + 0.0009\ (t - 23)] \qquad (5)$$

was found for the relation between the recovery and the temperature.

Since the word elongation is not to be used in any other sense in this paper, "elongation" will henceforth be used to denote the instantaneous recovery.

Rapidity with Which the Wire Regains Normal Temperature.

The heat evolved by the removal of the load is soon dissipated. A test was made on the copper wire, which, with a diameter of 0.5 mm., was much the largest wire of the four, to see how much time was required for the wire to regain its original temperature. An iron-constantan thermocouple was connected to the wire by separating one of the junctions, passing the separated elements through small holes in the side of the box, and soldering them as lightly as possible to the copper wire, a few cm. apart. The junctions were covered with a little cotton batting so they might cool at most no faster than the remainder of the wire. The other junction of the thermocouple was kept at a constant temperature outside the box. The cross-section of the thermocouple wire was about 1/30 that of the copper wire to which it was attached.

When a mass of 2 kg. was removed from the pan, the deflection of a galvanometer in the thermocouple circuit had practically all disappeared in one minute, showing that in that time the wire had regained its normal temperature. Since the other wires were of less than one half the diameter of the copper wire, the rise in temperature would no doubt disappear in a much shorter time for them.

Results for Elongation.

In Table III. are given the elongations for the various wires. The values of e for copper were all reduced to 23° by formula (5). While the entire set of elongations for copper was taken at 21°, the wires were vibrated at higher temperatures, and it was for comparison with the vibration results that the reduction was made. For all the other wires the variation of modulus with temperature was neglected.

M and e are corresponding masses and elongations, the zero for each being taken with the pan alone suspended.

M includes one half the mass of the wire.

Mass of the pan = 223 gm.

L' = length for which the elongation was measured.

L = vibration length.

e = elongation for the length L.

D = diameter of the wire.

TABLE III.

Copper. L' = 209.7 cm. L = 232.1 cm. D = 0.586 mm. Temp. 21°.0

Steel. L' = 154.1 cm. L = 231.1 cm. D = 0.217 mm. Temp. 24°.2

M Gm.	e Cm.	e_{23} Cm.	M/e.	M Gm.	e Cm.	M/e.
957	0.0821	0.0822	11,640	1,000	0.2874	3,478
1,387	0.1182	0.1184	11,710	1,500	0.4309	3,480
1,818	0.1560	0.1563	11,630	2,000	0.5766	3,467
2,248	0.1941	0.1945	11,560	2,500	0.7213	3,465
2,679	0.2315	0.2320	11,550	3,000	0.8675	3,457
3,101	0.2690	0.2695	11,510			

Phosphor-bronze. L' = 187.6 cm. D = 0.239 mm. L = 231.5 cm. Temp. = 22°.5.

Platinum-iridium. L' = 140.5 cm. D = 0.206 mm. L = 144.6 cm. Temp. = 22°.5.

M Gm.	e Cm.	M/e.	M Gm.	e Cm.	M/e.
953	0.4138	2,303	953	0.1461	6,523
1,383	0.6023	2,296	1,383	0.2119	6,527
1,814	0.7912	2,293	1,814	0.2780	6,525
2,244	0.9816	2,286	2,244	0.3448	6,508
2,675	1.1742	2,278	2,675	0.4137	6,466
			3,097	0.4812	6,436

M/e has no significance for the material of the wire, since it contains the length and diameter, but as long as these are constant it serves to indicate whether Young's modulus is constant or not. Owing to the shortness of the wires used, no great degree of absolute accuracy is claimed for the above elongations, and also, since the computation of the period of vibration does not demand it Young's modulus has not been computed. Its absolute value is of little importance here, but its variation is of great importance.

III. Determination of Frequency of the Vibrations.

Apparatus.

The pan was removed from the wire, and in its place was soldered a brass disk 6½ cm. in diameter, provided with threads on the circumference, so that over it might be screwed a hollow cylindrical shell about 14 cm. long. Inside this shell could be attached five other solid disks, making six different masses which could be used in vibrations. The shell was closed at the bottom, so that for all masses used, the same surface was exposed to the air and the damping due to the air friction was, therefore, constant.

From the lower end of the cylinder a brass rod 8 mm. in diameter projected vertically downward an additional 10 cm., and carried on its lower end a small soft iron armature. The sections of the mass separated from

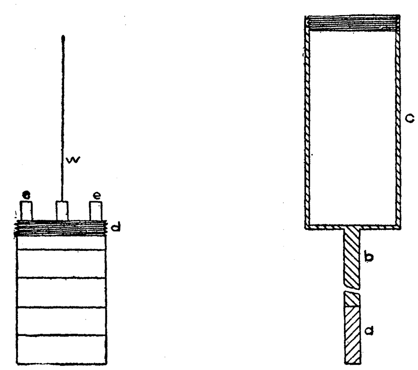

Fig. 6.

a, soft iron armature; *c*, cylindrical shell; *d*, upper disk which screws into the shell.

the cylinder are shown in Fig. 6. The electric circuit was arranged, as in Fig. 7, so that the wire was self-driven after the manner of the electrically driven tuning fork. The switch *L* closed both circuits at once. The vibration could be started by raising or lowering the adjustable cup *g* until, when the proper height was reached, the vibration would start and the amplitude could be controlled by the rheostat.

A tuning fork supplied with adjustable masses for regulating the period, and carrying a lens on one prong, was arranged to vibrate horizontally in front of the wire, so that when a drop of mercury was placed on the wire a Lissajous' figure could be viewed through the vibrating lens by means of a microscope. A phonic wheel was included in the circuit with the tuning fork, so that the fork could be accurately rated while the wire was vibrating. The vibration was so slow that the time required for an elastic pulse to travel the length of the wire was very short compared with the period of the wire, therefore we may consider the wire to be uniformly strained throughout its entire length at all times.

Measuring the Free Period.

This, of course, was a forced vibration of the wire, and the period was therefore slightly less than that of the free vibration. It was found, however, that by adjusting so that the fork was slightly too slow, then

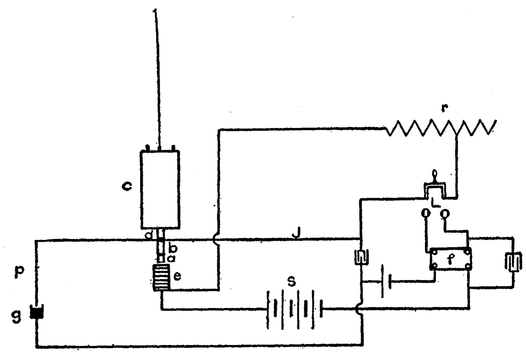

Fig. 7.

e, electromagnet actuated by battery *s*; *f*, relay; *g*, mercury cup; *j*, fine wire soldered to moving system at *d*; *p*, stiff platinum-tipped wire for contact with mercury at *g*.

shutting off the current driving the wire, the free period could be obtained with considerable accuracy.

The free vibration could be observed easily for ¾ min. with the copper wire, about 4 min. with the platinum-iridium wire, and 5 min. with the steel and phosphor-bronze. In case the period varied with the amplitude, as it did with the copper and platinum-iridium, and also to a slight extent with the phosphor-bronze, the fork was set too slow for small amplitudes, and too fast for large ones. By means of the micrometer microscope the amplitude for which the Lissajous' figure was at rest could be satisfactorily determined. This, however, necessitated changing the rate of the fork, and so rating again by the phonic wheel for every amplitude measured, an expenditure of time which was partly avoided by counting the number of seconds elapsing during one cycle of the Lissajous' figure and taking for the mean amplitude, the average of the amplitudes at the beginning and the end of the count.

Relation of Frequency to Amplitude.

For the steel wire no variation of the frequency with amplitude could be detected. The phosphor-bronze showed a very slight increase in frequency with decreasing amplitude, the variation for platinum-iridium was more marked, and the copper wire showed the greatest variation of all. In all cases the plot of frequency and amplitude was practically a straight line, which, produced to the frequency axis, gave the frequency

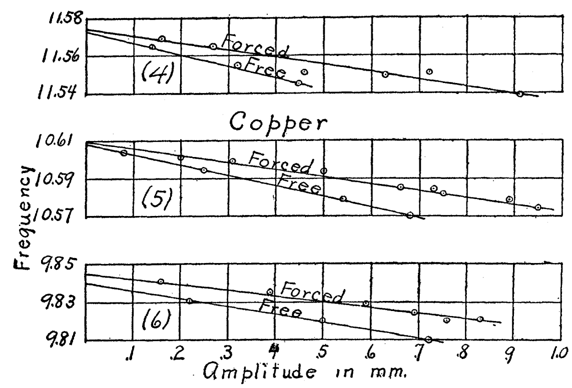

Fig. 8.

for infinitely small vibrations. The equation connecting frequency and amplitude is $n = n_0 - bA$, where b is a positive number. The relation is shown graphically for part of the observations in Figs. 8 and 9. The numbers (4), (5), (6), in parentheses, refer respectively to the largest

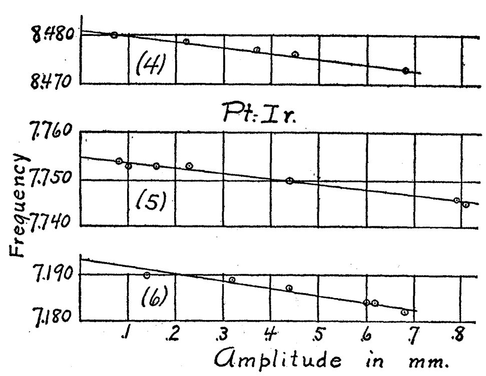

Fig. 9.

three masses given in Tables V. and VI., in order of increasing magnitude. The cause of this variation of period with amplitude will be discussed later.

Free and Forced Vibrations.

In Fig. 8 for copper are included the plots of the forced frequency. It is seen that this also gives a straight line of nearly the same slope as that of the free vibrations, and the forced period differs very little from the free. That there is not a greater difference is due to the fact that the relay included in the circuit aided the induction in delaying the pulse given by the electromagnet. If the pulse were given just at the instant when the armature reached the middle of its downward motion, there would be no difference of the free and forced periods.

Although the force acting on the armature is not simply harmonic with respect to the time yet it may be represented by a Fourier's series, and it may be shown from the difference between the forced and the free periods that for small amplitudes the fundamental of the Fourier's series is in the neighborhood of 65° behind the motion of the armature.

Table IV. is given as an illustration of the manner of timing.

Table IV.

Copper.

M = 2,678 gm. ρ = 5/1. Temp. = 21°.9 C.

N = frequency of fork.
n = frequency of wire.
$\rho = N/n$.

N	Forced.		Free.		Sec. per Cycle.	Wire Slow or Fast.
	Amp. (Mm.)	n	Amp. (Mm.)	n		
			0.08	10.604•		
53.020	0.20	10.602			102	slow
	0.31	10.598			34	slow
			0.25	10.593		
	0.50	10.593			∞	——
52.964	0.66	10.586			31	slow
	0.73	10.583			21	slow
			0.54	10.578		
	0.75	10.582			44	fast
52.888	0.89	10.578			∞	——
	0.95	10.574			50	slow
52.860			0.68	10.572		

The 6th column shows the number of seconds required for the Lissajous' figure to pass through one cycle, while the last indicates whether the wire was faster or slower than 1/5 the rate of the fork.

Tables V. and VI. show in condensed form the free amplitudes and frequencies for the various masses.

Such tables are not given for phosphor-bronze and steel for reasons already stated.

TABLE V.

Copper.

Mass, Gm.	Amp., Mm.	n	p	Temp., C.
956	0.10 .15 .33	17.700 17.693 17.676	5/2	23°.7
1,386	.17 .32 .59	14.715 14.696 14.672	3/1	23°.8
1,817	.19 .24 .40	12.853 12.848 12.839	7/2	24°.0
2,247	.14 .32 .45	11.565 11.555 11.545	4/1	24°.0
2,678	.08 .25 .54 .68	10.604 10.593 10.578 10.572	5/1	21°.7
3,100	.22 .50 .72	9.831 9.818 9.810	5/1	23°.8

Variation of Frequency with Mass.

Since, if Hooke's Law holds the period is proportional to the square root of the mass, we should have the product of mass and the square of the frequency equal to a constant. From Table III., the ratio M/e is not a constant. It decreases with increasing mass, except for the first two masses of the three wires, copper, steel and platinum-iridium. It would not be concluded from this alone that the elongation is not proportional to the weight removed, for there is considerable uncertainty about the point to which the curves of Fig. 5 should be produced before they reach the axis of ordinates. However the results of timing also show that M/e is not a constant, for Mn^2 is not constant, except for the case of copper. Table VII. shows the masses, frequencies, and Mn^2, for each of the four wires.

The n for the copper and the platinum-iridium is obtained from Figs. 8 and 9. It is the n indicated by the straight line, for the amplitude zero. For copper a small correction to n was made to reduce it to 23° temperature, the same temperature as that for the elongations of Table

TABLE VI.

Platinum-iridium.

Mass, Gm.	Amp., Mm.	n	ρ	Temp., C.
	0.08	13.056		
	.20	13.053		
953	.44	13.046	3/1	23°.0
	.58	13.044		
	.80	13.037		
	0.09	10.836		
1,383	.47	10.828	4/1	23°.2
	.75	10.824		
	0.05	9.450		
	.12	9.449		
1,814	.21	9.447	5/1	24°.4
	.66	9.442		
	.98	9.439		
	0.07	8.480		
	.22	8.479		
2,244	.37	8.477	5/1	24°.5
	.45	8.476		
	.68	8.473		
	0.08	7.754		
	.10	7.753		
	.16	7.753		
2,675	.23	7.753	5/1	24°.6
	.44	7.750		
	.79	7.746		
	.81	7.745		
	0.14	7.190		
	.32	7.189		
	.44	7.187		
3,097	.60	7.184	6/1	24°.6
	.62	7.183		
	.68	7.182		
	.78	7.181		

III. Since $1/n \propto \sqrt{e}$, one half the correction in per cent. which was made for e, was made here for n: or we might use a formula similar to equation (5)

$$n_t = n_{23}[1 - 0.00045(t - 23)]. \qquad (6)$$

For the other three wires the correction was neglected for reasons already stated.

TABLE VII.

Copper.						Steel.			
M[1] Gm.	n	Temp.	Corr.	n_{23}.	Mn^2.	M Gm.	n	Temp.	Mn^2.
956	17.710	23°.7	+0.006	17.716	3.001×10^5	954	9.525	21°.5	8.655×10^4
1,386	14.732	23°.8	+0.006	14.738	3.011	1,384	7.908	21°.6	8.655
1,817	12.865	24°.0	+0.006	12.871	3.010	1,815	5.903	21°.7	8.649
2,247	11.573	24°.0	+0.005	11.578	3.012	2,245	6.199	23°.1	8.627
2,678	10.607	21°.7	−0.005	10.602	3.010	2,676	5.675	23°.0	8.618
3,100	9.841	23°.8	+0.004	9.845	3.005×10^5	3,098	5.268	22°.7	8.598×10^4

Phosphor-bronze.				Platinum-iridium.			
M Gm.	n	Temp.	Mn^2.	M Gm.	n	Temp.	Mn^2.
953	7.767	21.3	5.749×10^4	953	13.058	23.0	1.625×10^5
1,383	6.437	21.8	5.731	1,383	10.837	23.2	1.624
1,814	5.611	21.9	5.711	1,814	9.450	24.4	1.620
2,244	5.038	22.0	5.696	2,244	8.481	24.5	1.614
2,675	4.594	21.3	5.646×10^4	2,675	7.755	24.6	1.609
				3,097	7.193	24.6	1.602×10^5

In Table VII. all the wires but the copper show a steady decrease in Mn^2 with increase of M. The decrease is very small, or zero, between the first two masses, and it has been noted that there was, in general, no decrease of M/e in Table III., until the second mass was reached.

Relation between M and e.

For all except the small masses the results of both the direct measurement of elongation and of the timing, indicate for steel, phosphor-bronze, and platinum-iridium a greater elongation than that demanded by Hooke's Law. We will therefore assume the relation

$$M = \alpha e + \beta e^2, \tag{7}$$

and deduce the period which would result from oscillations along the curve represented by this equation. Since M/e decreases with increasing M, β will evidently be negative. It is more logical, perhaps, to write

$$e = aM + bM^2,[2] \tag{8}$$

but for solution of the differential equation given later it is simpler to have M expressed in terms of e, as in (7).

[1] M includes one third the mass of the wire, which addition was negligible for all except copper.

[2] J. O. Thompson found the formula $e = aM + bM^2 + cM^3$ to satisfy his observations in static experiments with wires. See Amer. Jour. Science, Vol. 43, p. 32, 1892.

IV. Derivation of the Period.

If we assume small vibrations of the wire, the period, neglecting the damping term, is known to be

$$T = 2\pi \sqrt{\frac{M}{\frac{dM}{de} g}}. \tag{9}$$

From (7) $dM/de = \alpha + 2e\beta$, whence

$$n = \frac{1}{2\pi} \sqrt{\frac{g(\alpha + 2\beta e)}{M}}. \tag{10}$$

Since the period has been shown by Figs. 8 and 9 to depend on the amplitude, the following derivation is given to show whether vibration along the curve represented by equation (7) causes the period to vary appreciably with the amplitude or not.

Returning to Fig. 1, let x be the distance from O to P at the time t. Let M' be the mass which by equation (7) corresponds to the elongation x. Let F' represent the force of restitution at the time t. M' will be less than M when x is less than e, and greater than M if x is greater than e. Then from equation (7)

$$F' = M'g = Mg\frac{M'}{M} = Mg\frac{\alpha x + \beta x^2}{\alpha e + \beta e^2}. \tag{11}$$

In equation (3) let $h = \frac{\Delta l}{\Delta p} = \frac{AT\alpha^2 L}{\omega c}$

$$\Delta l = \Delta x$$
$$\Delta p = M' - M = \alpha(x - e) + \beta(x^2 - e^2).$$

Δx is the increase in length due to the shifting of the position of O by temperature changes in the wire, the temperature changes being caused by the vibration. Then

$$\Delta x = h\,\Delta p = h[\alpha(x - e) + \beta(x^2 - e^2)]. \tag{12}$$

Since equation (7) does not contain this temperature effect, we must add Δx to the x of that equation, and, instead of x, write $x + h[\alpha(x - e) + \beta(x^2 - e^2)]$, Δx is negative when $x < e$. $\Delta x = 0$ when $x = e$. Thus we take the temperature of the wire to be that of the room when $x = e$. We will also suppose the vibrations to be so rapid that there is no time for the wire to receive or to give out heat, but the vibrations take place under adiabatic conditions. Then we have, for the elastic force acting on the mass M, when the distance OP is x

$$F' = Mg\frac{\alpha\{x + h[\alpha(x-e) + \beta(x^2 - e^2)]\} + \beta\{x + h[\alpha(x-e) + \beta(x^2 - e^2)]\}^2}{\alpha e + \beta e^2}. \tag{13}$$

Since the effect of the weight of the wire itself on the vibrations is equivalent to the effect of one third such weight suspended from the lower end, in what follows, M will denote the suspended mass plus one third the mass of the wire. Except for the copper wire, however, this addition was less than ½ gram, and was neglected.

Let

$$F = F' - Mg. \tag{14}$$

Then if we neglect terms of the order of $h\alpha\beta(x^2 - e^2)$ and $h\alpha\beta x(x - e)$, which are small compared with αx and $\alpha^2 h(x - e)$ we shall have from (13) and (14)

$$F = Mg\frac{\alpha(1 + h\alpha)(x - e) + \beta(x^2 - e^2)}{\alpha e + \beta e^2} = g[\alpha(1 + h\alpha)(x - e) + \beta(x^2 - e^2)]. \tag{15}$$

The external and the internal friction are each assumed proportional to the velocity, and the two terms representing them may be combined thus,

$$\mu\frac{dx}{dt} + \eta\frac{dx}{dt} = \sigma\frac{dx}{dt}, \tag{16}$$

where μ is the external damping coefficient, and η the internal.

As the mass M moves up and down, O (Fig. 1), the point to which the lower end of the wire tends to return, will also oscillate, because of the after-effect. This motion of O will doubtless be somewhat behind the motion of P in phase. The complete vibration requiring only about 1/5 sec. for the lowest frequency observed, there would be only a short time for the wire to recover between vibrations, but immediately after release the motion of O due to the after-effect is comparatively rapid, so it is not at all certain that this motion may be disregarded; in fact, a later discussion will indicate that this is perhaps an important factor in determining the period.

Kohlrausch proposed the formula $-dy/dt = \alpha(y/t)$, or $y = c/t^\alpha$, where y is the deformation due to the after-effect at the time t after the distorting force has ceased to act, and α is a constant. If this were approximately true for small t, we should have a very large dy/dt, but all his observations were taken after 10 sec. and as far as the writer has observed, practically nothing is known of the behavior of the after-effect during the first second after removal of the distorting force. Since an assumption would be worth little without the support of observational data and also since it would doubtless greatly complicate the equation, this effect will for the present be neglected.

The differential equation of motion may then be written

$$\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + g\frac{\beta x^2}{M} + \frac{g\alpha(1 + h\alpha)}{M}x - \frac{g\alpha^2 he + gM}{M} = 0. \quad (17)$$

If we neglect h and β, then $M = \alpha e$, and (17) becomes

$$\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + \frac{g}{e}(x - e) = 0, \quad (18)$$

which is the well-known equation representing damped vibrations, the solution being

$$x = e + Ae^{-\sigma t \div 2}\cos\sqrt{\frac{g}{e} - \frac{\sigma^2}{4}} \cdot t, \quad (19)$$

where the time is counted from the lowest point of the vibration, at which point $x = e + A$, A being the amplitude. The period of this motion is

$$T = \frac{2\pi}{\sqrt{\frac{g}{e} - \frac{\sigma^2}{4}}}. \quad (20)$$

If β is neglected, but h is not, we have

$$T = \frac{2\pi}{\sqrt{\frac{g(1 + h\alpha)}{e} - \frac{\sigma^2}{4}}}. \quad (21)$$

From measurements of σ discussed later it will appear that for the largest value of σ found, $\sigma^2/4$ compares with g/e about as 1 with 10^6, showing that under the assumption that the frictional resistance varies as the velocity, the damping has no appreciable effect on the period. Hence, if in (17) we neglect $\sigma(dx/dt)$, we have left

$$\frac{d^2x}{dt^2} + \frac{g}{M}[\beta x^2 + \alpha(1 + h\alpha)x - (\alpha^2 he + M)] = 0. \quad (22)$$

This equation represents undamped vibrations, but since we desire only the period that fact is of little importance.

Solution of the Equation.

Let

$$\frac{\beta}{3} = \gamma$$

$$\frac{\alpha(1 + h\alpha)}{2} = \delta$$

$$M + \alpha^2 he = \zeta$$

and (22) becomes

$$\frac{d^2x}{dt^2} + \frac{g}{M}(3\gamma x^2 + 2\delta x - \zeta) = 0. \quad (23)$$

By a first integration we obtain

$$\frac{p^2}{2} = -\frac{g}{M}\gamma\left(x^3 + \frac{\delta}{\gamma}x^2 - \frac{\zeta}{\gamma}x + D\right), \tag{24}$$

where $p = dx/dt$, and D is the constant of integration.

$$p = \frac{dx}{dt} = \sqrt{\frac{-2g\beta}{3M}}\sqrt{x^3 + \frac{\delta}{\gamma}x^2 - \frac{\zeta}{\gamma}x + D}. \tag{25}$$

D may be determined from the condition, that when $x = e - A$, $p = 0$.

If we let $f(x) = x^3 + (\delta/\gamma)x^2 - (\zeta/\gamma)x + D$ then $f(x) = 0$ when $p = 0$,

$$D = -(e - A)^3 - \frac{\delta}{\gamma}(e - A)^2 + \frac{\zeta}{\gamma}(e - A), \tag{26}$$

and is thus a function of the amplitude.

If $A = e$, which is the greatest amplitude possible, $D = 0$.

This amplitude is always impracticable on account of the tendency of the wire to vibrate transversely when this point is nearly reached.

Equation (25), solved for t, is

$$t = \sqrt{\frac{3M}{-2g\beta}}\int_{x_0}^{x}\frac{dx}{\sqrt{x^3 + \frac{\delta}{\gamma}x^2 - \frac{\zeta}{\gamma}x + D}}. \tag{27}$$

Equation (27) gives the time for any part of the vibration, according to the limits assigned to x. If the extreme positions of P are taken as limits the value of t will be $\frac{1}{2}T$.

To reduce the elliptic integral to the standard form of the first class,

$$F(k, \varphi) = \int_0^{\phi}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \int_0^{x}\frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}},$$

we must find the roots of $f(x) = 0$.

Since the 4th power of x is lacking, we have at once that one root is ∞, and D has been chosen so that another root is $e - A$. If we substitute the value of D from equation (26) in $f(x) \doteq 0$, and divide by $x - (e - A)$, we obtain

$$x^2 + x\left(e - A + \frac{\delta}{\gamma}\right) + (e - A)^2 + \frac{\delta}{\gamma}(e - A) - \frac{\zeta}{\gamma} = 0, \tag{28}$$

whence

$$x = -\frac{e - A + \frac{\delta}{\gamma}}{2} \pm \frac{1}{2}\sqrt{(e - A)^2 + 2\frac{\delta}{\gamma}(e - A) + \frac{\delta^2}{\gamma^2} - 4(e-A)^2 - 4\frac{\delta}{\gamma}(e - A) + 4\frac{\zeta}{\gamma}}. \tag{29}$$

ζ may be expressed in terms of δ and γ.

$$\begin{aligned}\zeta = M + \alpha^2 he &= \alpha e + \beta e^2 + \alpha^2 he \\ &= \alpha e(1 + h\alpha) + \beta e^2 \\ &= 2\delta e + 3\gamma e^2\end{aligned}$$

$$4\frac{\zeta}{\gamma} = 8\frac{\delta e}{\gamma} + 12e^2. \tag{30}$$

Substitute this value of $4\frac{\zeta}{\gamma}$ in (29) and the equation becomes

$$x = \frac{1}{2}\left(-e + A - \frac{\delta}{\gamma} \pm \sqrt{9e^2 + A^2 + \frac{\delta^2}{\gamma^2} + 6eA + 6\frac{\delta}{\gamma}e + 2\frac{\delta}{\gamma}A - 4A^2}\right). \tag{31}$$

The expression under the radical differs from the square of $3e + A + \delta/\gamma$ only by $4A^2$. For the largest value of A observed

$$\frac{4A^2}{\frac{(\delta)^2}{(\gamma)^2}} = \frac{1}{90000} \text{ for platinum-iridium,}$$

$$= \frac{1}{250{,}000} \text{ for phosphor-bronze,}$$

$$= \frac{1}{640{,}000} \text{ for steel.}$$

For very small amplitudes the ratio is much smaller. Hence $4A^2$ may be neglected and we may write

$$x = \tfrac{1}{2}\left[-e + A - \frac{\delta}{\gamma} \pm \left(3e + A + \frac{\delta}{\gamma}\right)\right]. \tag{32}$$

$$\therefore \left\{\begin{aligned} x_3 &= e + A \\ x_4 &= -2e - \frac{\delta}{\gamma}\end{aligned}\right\} \tag{33}$$

are the remaining two roots of $f(x) = 0$.

It was to be expected that if $e - A$ is one root, $e + A$ is very nearly another, which simply means that in the vibration P moves very nearly as far below the point $x = e$ as it does above that point.

If we assume $e - A$ and $e + A$ to be two of the roots, the third root comes easily from the relation between the roots and the coefficients, for if ω_1, ω_2 and ω_3 are the roots, and

$$\omega_1 = e - A,$$
$$\omega_2 = e + A,$$

then

$$\Sigma\omega = -\frac{\delta}{\gamma}, \tag{34}$$

and

$$\omega_3 = -2e - \frac{\delta}{\gamma}.$$

To obtain the integral in (27) in the standard form we have to transform the roots

$$\infty,\ e - A,\ e + A,\ -\frac{\delta}{\gamma} - 2e$$

respectively into

$$-\frac{1}{k},\ -1,\ 1,\ \frac{1}{k}.$$

The bilinear transformation is

$$x = \frac{py + r}{qy + s}.$$

Determining the constants, we find

$$\begin{aligned} q &= 1, \\ s &= \frac{1}{k}, \\ p &= e + \frac{A}{k}, \\ r &= \frac{e}{k} + A. \end{aligned} \tag{35}$$

If $A = e$, $p = r$.
If $A = o$, $k = o$, and $s = r = \infty$.

$$\int_{x_0}^{x_1} \frac{dx}{\sqrt{f(x)}} = \frac{ps - qr}{\sqrt{M'}} \int_{y_0}^{y_1} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2y^2)}}.$$

$$M' = \frac{\sqrt{b_0}}{k},\ b_0 = q^4 f\left(\frac{p}{q}\right),\ = f(p),\ \text{for } q = 1;$$

$$\therefore \frac{ps - qr}{\sqrt{M'}} = \frac{e + \frac{A}{k} - kA - e}{\sqrt{f(p)}} = \frac{\frac{A}{k} - kA}{\sqrt{f(p)}}, \tag{36}$$

$$k = \frac{1 - \sqrt{a}}{1 + \sqrt{a}},\ \text{where } a = \frac{\omega_4 - \omega_3}{\omega_4 - \omega_2} \cdot \frac{\omega_1 - \omega_2}{\omega_1 - \omega_3}. \tag{37}$$

The three finite roots are all positive, and the graph of the function appears somewhat as in Fig. 10. The integration takes place from $\omega_2 = e - A$ to $\omega_3 = e + A$, and is therefore real.

D is negative except when $A = e$ in which case it is zero.

$$a = \frac{\frac{\delta}{\gamma} + 3e + A}{\frac{\delta}{\gamma} + 3e - A} \tag{38}$$

Let $c = -(\delta/\gamma) - 3e$, a positive quantity. Then

$$k = \frac{1 - \sqrt{\dfrac{c-A}{c+A}}}{1 + \sqrt{\dfrac{c-A}{c+A}}} = \frac{c - \sqrt{c^2 - A^2}}{A}. \tag{39}$$

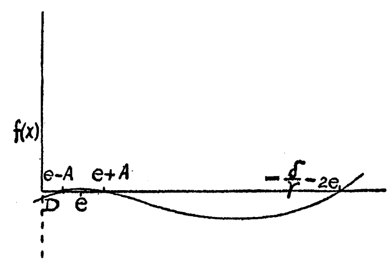

Fig. 10.

When $A = 0$, $k = 0$, and the last member of equations (36) becomes indeterminate. To evaluate it, from (39)

$$\frac{A}{k} = \frac{A}{\dfrac{c - \sqrt{c^2 - A^2}}{A}} = c + \sqrt{c^2 - A^2}, \tag{40}$$

hence,

$$\lim_{A=0} \frac{A}{k} = 2c, \tag{41}$$

also from (39),

$$Ak = c - \sqrt{c^2 - A^2}; \tag{42}$$

$$\therefore \frac{A}{k} - kA = 2\sqrt{c^2 - A^2} \tag{43}$$

and

$$\frac{ps - qr}{\sqrt{M'}} = \frac{2\sqrt{c^2 - A^2}}{\sqrt{f(p)}}; \tag{44}$$

$$\therefore t = \sqrt{\frac{3M}{-2g\beta}} \int_{x_0}^{x_1} \frac{dx}{\sqrt{f(x)}} = \frac{2\sqrt{c^2 - A^2}}{\sqrt{f(p)}} \times \sqrt{\frac{3M}{-2g\beta}} \int_{y_0}^{y_1} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}}. \tag{45}$$

When $x_0 = e - A$, $y_0 = -1$; when $x_1 = e + A$, $y_1 = 1$. Integrating between these limits we have $\frac{1}{2}T$. Hence the period is given by

$$T = \frac{8\sqrt{c^2 - A^2}}{\sqrt{f(p)}} \sqrt{\frac{3M}{-2g\beta}} K, \tag{46}$$

where K is the value of the complete elliptic integral

$$\int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - k^2y^2)}}.$$

k was found to be very small for all values of A really obtained, and it approaches zero with A. Thus if $A = 0.5$ cm. for platinum-iridium, $\sin^{-1} k = 22'$. If $A = 1.0$ cm. for steel $\sin^{-1} k = 7'$. But the largest value of A given in Figs. 11 and 12 is about 0.1 cm. This gives for platinum-iridium $\varphi = \sin^{-1} k = 4' \, 21''$ and for steel $\varphi = 0' \, 42''$. For phosphor-bronze φ is slightly larger than for steel. If $k = 0$,

$$\int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - k^2y^2)}} = \int_0^1 \frac{dy}{\sqrt{1 - y^2}} = \frac{\pi}{2} = 1.5708.$$

If $\sin^{-1} k = 1°$,

$$\int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - k^2y^2)}} = 1.5709.$$

Since k is in no case larger than sin 30′, and for ordinary amplitudes is much smaller, we may use $K = \pi/2$ for all the wires and all amplitudes.

The effect of varying amplitude on the other factors in eq. (46) will be shown after α and β have been determined.

Simplification of the Equation (46) for Small Amplitudes.

When $A = 0$, (46) becomes

$$T = 2\pi \frac{2c}{\sqrt{f(p)}} \sqrt{\frac{3M}{-2g\beta}}. \tag{47}$$

This may be further simplified,

$$D = -f(e), \text{ when } A = 0, \tag{48}$$

$$\frac{\zeta}{\gamma} = 2\frac{\delta}{\gamma}e + 3e^2,$$

$$\begin{aligned} f(p) &= p^3 - e^3 + \frac{\delta}{\gamma}(p^2 - e^2) - \frac{\zeta}{\gamma}(p - e) \\ &= (p - e)\left[p^2 + pe + e^2 + \frac{\delta}{\gamma}p + \frac{\delta}{\gamma}e - 2\frac{\delta}{\gamma}e - 3e^2\right] \\ &= (p - e)\left[p^2 + pe - 2e^2 + \frac{\delta}{\gamma}p - \frac{\delta}{\gamma}e\right], \end{aligned}$$

$$f(p) = (p - e)^2\left(p + 2e + \frac{\delta}{\gamma}\right). \tag{49}$$

From equation (35)

$$p = e + \frac{A}{k} = e + 2c, \text{ when } A = 0; \tag{50}$$

$$\therefore f(p) = (2c)^2(e + 2c - e - c) = 4c^3. \tag{51}$$

Hence, from (47),

$$T = 2\pi \frac{1}{\sqrt{c}} \sqrt{\frac{3M}{-2g\beta}} = 2\pi \sqrt{\frac{3M}{-2g\beta(-\frac{\delta}{\gamma} - 3e)}},$$

$$T = 2\pi \sqrt{\frac{M}{g[\alpha(1 + h\alpha) + 2\beta e]}}, \tag{52}$$

$$n = \frac{1}{2\pi} \sqrt{\frac{g[\alpha(1 + h\alpha) + 2\beta e]}{M}}. \tag{53}$$

This is the final form for the expression of the frequency for vibrations of small amplitude, in which the motion is along the curve represented by the equation $M = \alpha e + \beta e^2$, except for the small heating effect indicated by the term containing h. The effect of larger amplitudes as shown by (46) will be mentioned later.

Omitting the term $h\alpha$, (53) is the same as (10). If the wire obeyed Hooke's Law, we would have only to set $\beta = 0$, and (53) reduces to

$$n = \frac{1}{2\pi} \sqrt{\frac{g\alpha(1 + h\alpha)}{M}} = \frac{1}{2\pi} \sqrt{\frac{g(1 + h\alpha)}{e}}, \tag{54}$$

which is equivalent to (21) except for the damping term.

Computation of Constants for the Equation $M = \alpha e + \beta e^2$.

The next step is to find the constants α and β of the equation $M = \alpha e + \beta e^2$, which will best satisfy the observed values of the frequency as given by equation (53).

We have from (53)

$$Mn^2 = \frac{1}{4\pi^2} [g\alpha(1 + h\alpha) + 2\beta e].$$

Since $2\beta e$ is small compared with $g\alpha(1 + h\alpha)$ it will be sufficient to write M/α in place of e; for $e = M/\alpha - \beta e^2/\alpha$, and the maximum value of $\beta e^2/\alpha$ is about one per cent. of M/α. This will influence the determination of β only, and one per cent. in the value of β is negligible. Making this substitution for e in (53), the equation becomes

$$Mn^2 = \frac{1}{4\pi^2} \left[g\alpha(1 + h\alpha) + \frac{2\beta}{\alpha} M \right]. \tag{55}$$

If we plot Mn^2 as a function of M, we obtain for steel, phosphor-bronze, and platinum-iridium the results shown in Fig. 11. The slope of each of these lines is $\beta/2\pi^2\alpha$ for that wire, and the intercept on the Mn^2 axis is $g\alpha(1 + h\alpha)/4\pi^2$. $h\alpha$ is of the order 0.001 for all the three wires to which the equation applies, hence for $h\alpha$ we may use the value obtained by neglecting h and β altogether. The values of $1 + h\alpha$ are

Copper	1.0021
Steel	1.0013
Phosphor-bronze	1.0015
Platinum-iridium	1.0011

The heating effect has therefore little influence on the frequency.

It will be noticed on the steel and platinum-iridium plots that when M is small Mn^2 is less than it should be. This may be explained by supposing that the wire obeys Hooke's Law for small elongations. Also, as

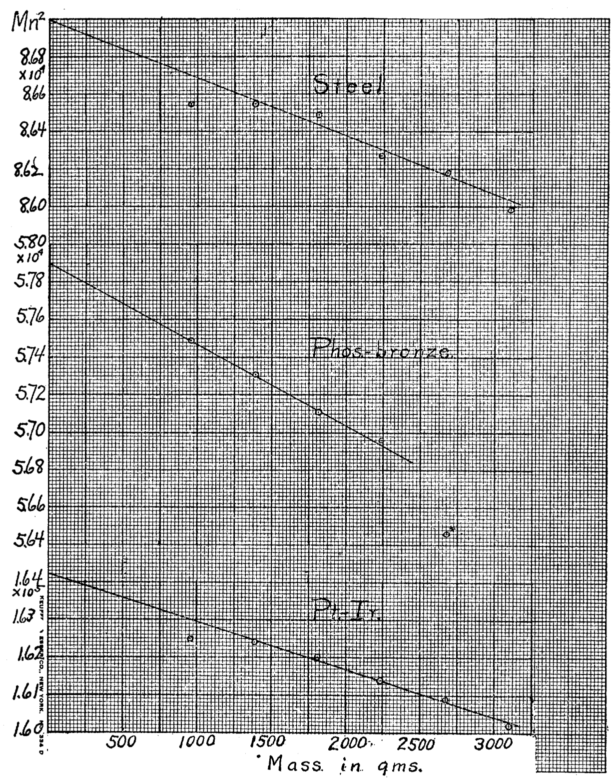

Fig. 11.

will appear later in comparing the results for direct elongation, the wires seem not to be entirely straightened out until the second mass is reached, in spite of the precautions, already noted, which were taken to straighten them. The phosphor-bronze wire does not exhibit this deficiency in Mn^2 because the elongation of that wire was relatively greater, and the first weight took the wire beyond the region where the elongation is proportional to the load. The largest weight used was too great for the phosphor-bronze wire, and although no permanent elongation could be observed when this mass was removed, yet it is plainly evident from the small value of Mn^2 for this mass that the rate of elongation of the wire at this point was greater than that which is given by $M = \alpha e + \beta e^2$.

For the copper wire, Mn^2 is a constant within the range of error, but it is to be noted that the greatest elongation is only about the same as the range for which Hooke's Law holds in the case of the steel and platinum-iridium wires. If there is any range for which equation (7) represents the behavior of the copper wire, that range is very limited, for the product of Mn^2 is a constant as far as observed, and the wire could sustain only a little more than the heaviest weight without passing the elastic limit.

From Fig. 11 the following equations were obtained.

$$\text{Steel: } M = 3{,}498\, e - 22.0\, e^2. \tag{56}$$

$$\text{Phosphor-bronze: } M = 2{,}327\, e - 19.6\, e^2. \tag{57}$$

$$\text{Platinum-iridium: } M = 6{,}602\, e - 166\, e^2. \tag{58}$$

To show with what degree of accuracy these represent the observed frequencies, we write eq. (53) in the form

$$n = \frac{1}{2\pi}\sqrt{\frac{g\alpha(1 + h\alpha)}{M} + 2\frac{\beta}{\alpha}g}, \tag{59}$$

and compute n for each mass, comparing the observed with the computed values.

Table VIII.

Steel.				Phosphor-bronze.				Platinum-iridium.			
M Gm.	n_0.	n_c.	n_0-n_c.	M Gm.	n_0.	n_c.	n_0-n_c.	M Gm.	n_0.	n_c.	n_0-n_c.
954	9.525	9.531	−0.006	953	7.767	7.766	+0.001	953	13.058	13.076	−0.018
1,384	7.908	7.908	.000	1,383	6.437	6.436	+0.001	1,383	10.837	10.836	+0.001
1,815	6.903	6.900	+ .003	1,814	5.611	5.611	.000	1,814	9.450	9.449	+0.001
2,245	6.199	6.199	.000	2,244	5.038	5.037	+ .001	2,244	8.481	8.479	+0.002
2,676	5.675	5.673	+ .002	2,675	4.594	4.606	− .012	2,675	7.755	7.753	+0.002
3,098	5.268	5.269	−0.001					3,097	7.193	7.193	0.000

n_0 = observed value of the frequency for infinitely small vibrations, taken from Fig. 9 for platinum-iridium.

n_c = frequency computed from equation (59).

The above residuals show that the frequency is very well represented by equation (59) except for the first masses of steel and platinum-iridium and the last of phosphor-bronze. For the smallest masses it is not certain that the wires were fully straightened, but on examining the results for all the wires including the copper we must conclude that for small distortions, the elongation is proportional to the distorting force, beyond that there is a region where the wire very closely obeys the law expressed by equation (7), then the elongation increases more rapidly

TABLE IX.

Copper.					Phosphor-bronze.					Platinum-iridium.				
M[1] Gm.	e_0 Cm.	n_0.	n_c.	$n_0 - n_c$.	M gm.	e_0 Cm.	n_0.	n_c.	$n_0 - n_c$.	M Gm.	e_0 Cm.	n_0.	n_c.	$n_0 - n_c$.
956	0.0822	17.716	17.397	+0.319	953	0.4138	7.767	7.752	+0.015	953	0.1461	13.058	13.044	+0.014
1,386	0.1184	14.738	14.496	.242	1,383	.6023	6.437	6.426	+0.011	1,383	.2119	10.837	10.831	+0.006
1,817	0.1563	12.871	12.617	.254	1,814	.7912	5.611	5.606	+0.005	1,814	.2780	9.450	9.456	−0.006
2,247	0.1945	11.578	11.310	.268	2,244	.9816	5.038	5.033	+0.005	2,244	.3448	8.481	8.491	−0.010
2,678	0.2320	10.602	10.356	.246	2,675	1.1742	4.594	4.602	−0.008	2,675	.4137	7.755	7.752	+0.003
3,100	0.2695	9.845	9.609	+0.236						3,097	.4812	7.193	7.187	+0.006

[1] M does not include the mass of the scale-pan, which was 223 gm. e_0 is the change in elongation due to M. For the vibrations, M is the tota[l] mass suspended.

even than this formula demands, and finally the strain becomes so great that the wire gives way. The weaker parts of the wire pass through these stages in advance of the stronger parts, and their cross-section being lessened, they elongate more and more rapidly, until rupture takes place at one of these points.

Computation of Frequency for Wires Obeying Hooke's Law.

Formula (21) gives the period if the wire obeys Hooke's Law and if we measure the elongation directly. For small elongations we might expect this formula to give a period agreeing well with the observed, but it does not. In all cases the e observed directly and substituted in (21) gives too small values for n, indicating that the observed e is too large.

Table IX gives the values of n_c computed from formula (21), using the values of e observed directly. The masses are the same as those used in the vibrations, except for steel. For this reason steel is given separately in Table X.

At first sight it might be supposed that these results for phosphor, bronze and platinum-iridium are satisfactory, with the exception-perhaps, of the smallest mass; and that formula (21) gives the period more closely than (59): but it must be remembered that formula (21) assumes that $e \propto M$. Therefore we have no right to substitute in it the values of e_0 in Table IX, because the ratio e_0/M is not constant, and we

Table X.

Steel.

M Gm.	$\frac{e}{M}$.	M Gm.	e Cm.	n_0.	n_c.	$n_0 - n_c$.
1,000	2.874×10^{-4}	954	0.2754	9.525	9.501	+0.024
1,500	2.873	1,384	.3996	7.908	7.888	+0.020
2,000	2.883	1,815	.5240	6.903	6.888	+ .015
2,500	2.885	2,245	.6481	6.199	6.193	+ .006
3,000	2.892	2,676	.7726	5.675	5.673	+ .002
Mean...	2.887×10^{-4}	3,098	.8944	5.268	5.272	− .004

need to use a different ratio of e_0/M for each one, which is contrary to the hypothesis on which (21) is derived.

Columns 3 and 4, Table X, give the vibrating masses and the corresponding elongations, which were computed from the mean e/M by multiplying by the mass used in vibration. n_c is the frequency from

$$n = \frac{1}{2\pi}\sqrt{\frac{g(1 + h\alpha)}{e}},$$

using the e of column 4. Table X. shows, as the values of Mn^2 of Table VII. also do, that no constant value of M/e will satisfy the observed frequencies.

Comparison of e from Static and Dynamic Observations.

Finally the elongations corresponding to the observed frequencies were computed for all the wires. For the copper wire the formula, $e = g(1 + h\alpha)/4\pi^2 n^2$ was used. For the others formulæ (56), (57), and (58) were inverted, giving e in the form of equation (8).

$$\text{Steel: } e = 2.859 \times 10^{-4}M + 5.1 \times 10^{-10}M^2. \qquad (60)$$

$$\text{Phosphor-bronze: } e = 4.297 \times 10^{-4}M + 1.6 \times 10^{-9}M^2. \qquad (61)$$

$$\text{Platinum-iridium: } e = 1.515 \times 10^{-4}M + 5.9 \times 10^{-10}M^2. \qquad (62)$$

For these three wires e_c was computed for the given load plus the pan, then for the pan alone, and the difference compared with the elongation, e_0, observed directly. In the case of copper since the product of Mn^2 was a constant no computation was made for the initial load.

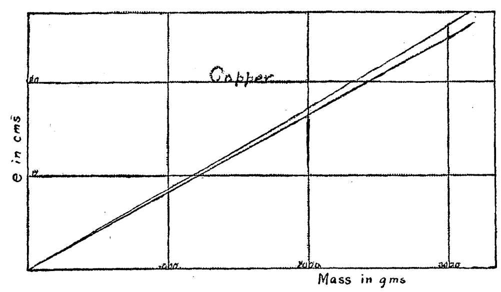

Fig. 12.

Fig. 12 shows e_0 and e_c for copper as ordinates with M on the axis of abscissas. e_c and M give a straight line, while e_0 and M do not. For the other wires neither curve is a straight line, and they lie so near together that they have not been plotted. $e_0 - e_c$ is always positive, however, for all the wires, and increases with M.

V. Discussion of Results.

The dynamic modulus of metals has been found by other methods to be greater than the static modulus. The same is true for these longitudinal vibrations, as shown in the last paragraph. No attempt has been made to determine the absolute value of either modulus, it being outside the purpose of this investigation.

If we consider the modulus to be proportional to dM/de, we have from equation (7) $dM/de = \alpha + 2e\beta$, which may easily be computed by equations (56), (57) and (58). For copper, since $\beta = 0$, we have $M/e = 4\pi^2 Mn^2/g(1 + h\alpha)$.

Possible After-effect in All the Wires.

A large after-effect is shown in Fig. 5 for copper, and some is also exhibited by similar curves for steel. Since the recovery is rapid at first, it seems quite possible that the point of instantaneous recovery may be far beyond that indicated by the dotted continuation of the curves. The greater part of the recovery may be during the first second or two following release, or even while the load is being removed. The curves for copper unquestionably show that it is rapid at first, for very little of the effect there indicated comes from the heating of the wire. For phosphor-bronze and platinum-iridium there is *apparently* no after-effect, but the recovery may be so rapid that it is complete before an observation can be taken. This theory would explain the discrepancy between the observed elongation and that computed from the results of vibrations, which is equivalent to saying it would account for the difference between the static and dynamic moduli.

We may suppose, then, that the observed values of the elongation are all too large. The deformation due to the after-effect is much larger than would be supposed from usual observation, for it decreases very rapidly when the load is first removed, and by the time an observation can be made a great part of it has disappeared. Thus there may be considerable after-effect in wires which apparently show none; the wire may have entirely recovered by the time the first observation can be made.

Influence of the After-effect on the Period.

Considering again Fig. 1, as the point P oscillates, O will also oscillate with a certain amplitude depending on the period, the distance OP, and the material of the wire, as well as its condition as to hardness, temperature, etc. The phase difference between the motions of O and P will depend on the period and the manner in which the wire recovers after removal of the load. This last is an important element, and also an uncertain one. If the phase difference were zero, then P and O would both move downward through their mean positions at the same instant. In the half of the period when P is below its mean position, O would also be below its mean position. As a result the force of restitution would be less than if O remained at rest, and the period would be increased. When P and O were above the mean positions there would be a greater elastic tension on account of the displacement of O, and this would also increase the period, so that we have a decrease in frequency due to the motion of O. It may be noted that this is just opposite to the effect of the motion of O due to the heating and cooling of the wire. (Comp. the term $h\alpha$ in equation (53).)

But we must suppose that O is somewhat behind P in phase. Then when the two points are on opposite sides of their respective positions of rest the frequency would tend to increase, when they are on the same side it would tend to decrease. Unless the motion of O became 90° or more behind that of P, the frequency would probably be decreased by this motion of O.

That this does not account for the variation in Mn^2 for steel, phosphor-bronze, and platinum-iridium, appears from the fact that Mn^2 did not vary appreciably in the case of copper, which is just the one in which the after-effect was greatest. It is true that the recovery of copper is not so rapid as that of the other three metals, and hence the amplitude of O may not be as great, but we should expect at least a noticeable variation in Mn^2.

VI. The Damping of Longitudinal Vibrations.

Apparatus.

The amplitude of the vibrations when the wire is only about two meters long cannot be very great; consequently the accurate measurement of the damping becomes a matter of some difficulty. The following method has yielded fairly satisfactory results.

A fiber of glass was attached to the wire, and allowed to project about 1 cm. at right angles to it. The fiber pointed towards a specially constructed camera. The lens, which was one used in an ordinary galvanometer telescope, and had a focal length of about 15 cm., was stopped down until the aperture was only about 5 mm. At the back of the camera was fixed a drum, 40 cm. in circumference, rotating about a vertical axis. The axis was a screw, so that as the drum revolved, it moved parallel to the axis. A small drop of mercury was placed on the end of the glass strip next to the camera, the drop was brightly illuminated by the light from an electric arc, and the camera was so placed that the image of the drop fell on the drum. Owing to the great curvature of the small mercury drop, a sharp spot of light was obtained. A rapid photographic film was attached to the drum, and rotated as the wire vibrated, thus obtaining a record of the vibration. The mercury drop could not be placed directly on the wire, because of the fogging of the film by light reflected from the wire. The glass fiber was colored red, so that any light reflected therefrom would not affect the record. Fig. 13 shows a section of a record for steel.

Since the purpose was only to measure the amplitude, uniformity of motion of the drum was unnecessary. It was therefore turned by hand, a handle outside the camera box serving that purpose. The mass was set vibrating and the drum started. Then the current driving the wire

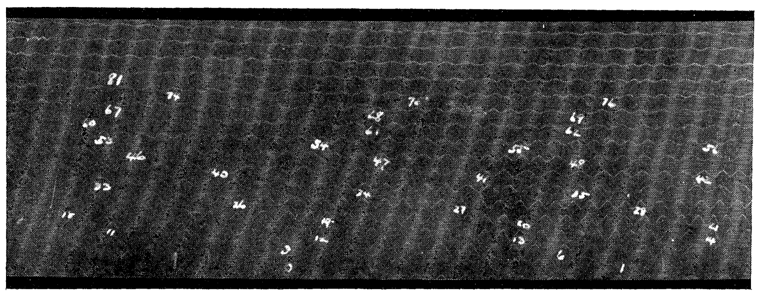

Fig. 13. Steel.

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was shut off and at the same time the drum was stopped for an instant. The several vibrations thus heaped up together indicated where the damping began.

The measurements of the amplitude were made on a measuring engine of the Detroit Observatory, the plate-holder of the engine being mounted to revolve about an axis parallel to the axis of the microscope. This permitted an easy adjustment of the cross hair tangent to two adjacent waves. For all the wires except copper, 1,000 vibrations could easily be measured. Beyond that number they were so small that a relatively large error in the logarithm of the amplitude was introduced. The lens was so placed that the amplitude on the film was about equal to the actual amplitude of the point on the wire. The motion might easily

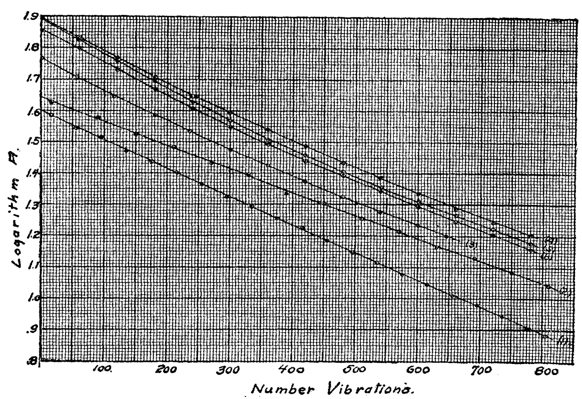

Fig. 14.

Decrement curves for steel.

have been magnified, but what was gained in this way was lost in definition of the trace. The largest double amplitude measured was about 4 mm. on the film, the smallest about 0.3 mm.. Ordinarily the measurements ran from 3 mm. to 0.5 mm. The measuring engine read directly to 0.005 mm.

Two records were made for each mass suspended from the copper wire, because the rapid damping allowed room for two on the film. For the other metals the damping was so small that only one record was made on each film. Every 30th vibration was measured and the logarithms of these were averaged by pairs, so that the plots show only the logarithms for every 60th vibration. Figs. 14 and 15 represent graphically the decrements for the masses given in Table XI. in order of increasing magnitude. Similar curves were drawn for copper, and the logarithmic decrements of Table XI. were obtained from these curves.

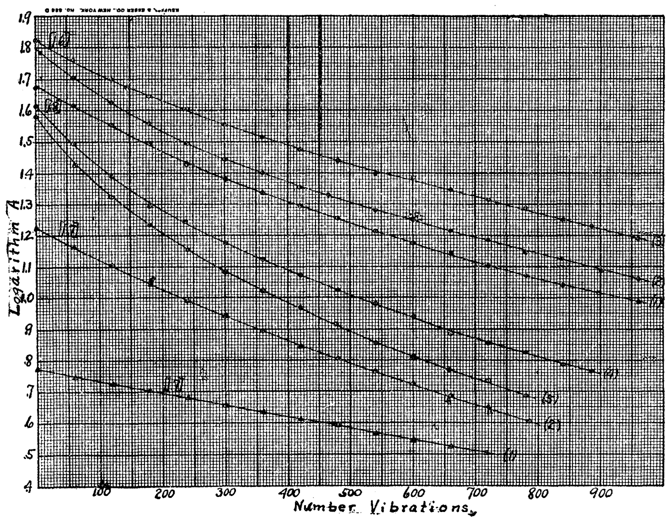

Fig. 15.

Decrement curves, ⊙, for platinum-iridium; △, for phosphor bronze. The scale is the same for all; but to avoid confusion the origin of ordinates has been shifted up or down for certain curves, as indicated by the logarithms in brackets.

Variation of the Logarithmic Decrement.

It was found by Schmidt[1] for torsional, and later also by Voight[2] for flexural and torsional vibrations, that the logarithmic decrement varied with the amplitude. They expressed the relation in the form

$$\lambda = \lambda_0 + bA^2,$$

where λ_0 is the logarithmic decrement for very small vibrations.

The values of Table XI. for copper are plotted in Fig. 16. Although the rate of increase is different for different masses, the increase is nearly proportional to the increase in amplitude[3] in all cases, and

$$\lambda = \lambda_0 + bA$$

represents the relation. The second mass gave an abnormally large decrement for the larger amplitudes, as seen from Table XI. There is no apparent reason for this. Both records taken of this mass show the same peculiarity.

While with the other wires, the logarithmic decrement for a rough approximation, increased in proportion to the increase of amplitude, the increase was much less than with copper, and less regular. The logarith-

[1] Loc. cit.

[2] Loc. cit.

[3] Bouasse and Carriere found $\lambda = bA$ for torsional vibrations of copper. (See Ann, de Chim. et de Physique, 8me Serie, t. 14, p. 208.)

TABLE XI.

Log. Decr.—Amplitude. Copper.

λ in common logarithms.

Mass. Gm.	Log. A.		1.8	1.7	1.6	1.5	1.4	1.3°	1.2	1.1	1.0	.9	.8	Temp. C.	n_0
	A_0	A	6.31	5.01	3.98	3.16	2.51	1.99	1.58	1.26	1.00	.79	.63		
956	2.30	λ						.00274	247	224	206	179		21°.9	17.716
	1.55									.00222	198	181	169		
1,386	2.75							.00475	384	305	259	226		24 .2	14.738
	2.85						.00571	494	377	274	232				
1,817	5.95			.00559	470	394	362	308	277	256	226	219		24 .3	12.871
	6.30			.00546	475	410	367	309	277	240	225				
2,247	3.80					.00354	320	292	252	228	208	203		22 .9	11.578
	3.95					.00342	329	283	254	219	205	198			
2,678	5.20				.00453	404	386	348	323	293	263	250		24 .2	10.602
	5.20				.00454	420	374	339	294	280	264	251			
3,100	4.05					.00408	364	320	296	270	239	221		24 .2	9.845
	6.55			.00562	490	436	370	325	292	259	242	230			

Steel.

M Gm.	A_0	A	6.31	5.01	3.98	3.16	2.51	1.99	1.58	1.26	1.00	Temp. C.	n_0
954	4.00	λ				.00091	90	90	90	88	85	23°.6	9.525
1,384	4.35				.00079	77	74	74	72	70		23 .6	7.908
1,815	6.15			.00103	99	91	84	79				23 .8	6.903
2,246	8.15		.00102	99	92	89	84	80				22 .7	6.199
2,676	8.25		.00114	107	100	92	85	78				22 .5	5.675
3,098	7.70			.00110	107	98	91	85	80	76		22 .3	5.268

TABLE XI.—*Continued.*

Phosphor-bronze.

M Gm.	A_0	A	6.31	5.01	3.98	3.16	2.51	1.99	1.58	1.26	1.00	Temp. C.	n_0
1,383	6.05			.00040	37							21 .7	6.437
2,244	5.60				.00100	87	78	69	64			21 .8	5.038
Platinum-iridium.													
953	4.95				.00107	101	82	71	63	58	45	23 .7	13.058
1,383	6.50			.00153	115	93	72	61	57	49		23 .9	10.837
1,814	4.35				.00108	92	77	65	54	52	50	24 .3	9.450
2,240	7.15			.00188	156	126	102	85	76	70	57	24 .2	8.481
2,675	6.70			.00245	190	157	139	115	96	89	75	24 .2	7.755

The column A_0 contains the largest amplitude measured on each film. The last two columns give the temperatures inside the box when the photographs were taken, and the frequencies for small amplitudes.

mic decrement for steel, especially for the smaller masses, is very nearly constant.

For phosphor-bronze only two masses were used and there is a remarkable difference in the decrement in these two cases. The decrement for the small mass, 1,383 gm., is only about one third that for the larger mass, 2,244 gm., and that for the smaller mass is very nearly constant. In 700 vibrations this amplitude decreased from 6.04 to 3.27, while for the larger mass in the same number of vibrations it decreased from 5.56 to 1.48. For the other metals there is no apparent dependence of λ

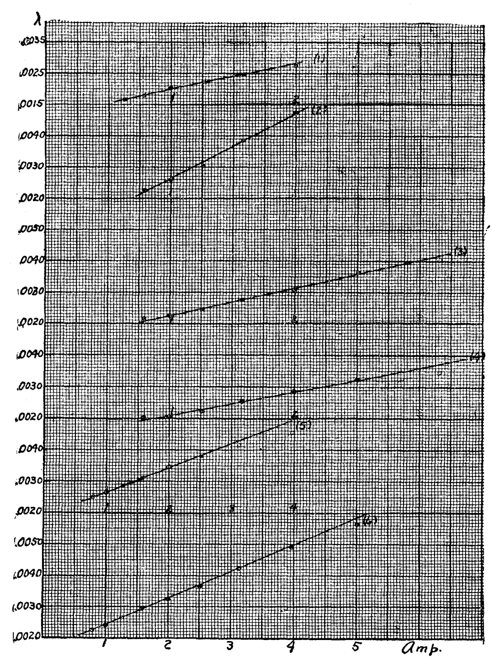

Fig. 16. Copper.

on the period. If we produce the straight lines of Fig. 16, backward to the axis we get for λ_0, omitting No. 2

Mass.	λ_0.
(1)	.00121
(2)	———
(3)	.00136
(4)	.00130
(5)	.00186
(6)	.00160

With the largest mass suspended from the copper wire a single test was made of the effect of starting with a larger amplitude. In the case of the greater initial amplitude the decrement is slightly higher for large amplitudes, but as the amplitudes decrease the decrements in the two cases approach equality.

The temperatures were nearly the same in all cases, so little variation can be expected from this source.

It would be very desirable to test more fully a single wire, carrying a constant load, to determine the effect of change of temperature, and previous treatment of the wire, as well as the variations of other conditions for which there was not opportunity in this limited investigation.

Cause of Damping.

If the damping were caused by internal friction of the particles, and the friction were assumed proportional to the velocity, then we should have $\lambda \propto \frac{T}{M} \propto n$, or $\lambda/n =$ a const. That this is not fulfilled at all by any of the wires is plainly seen by reference to Table XI., and by comparing λ for equal amplitudes and different frequencies. The phosphor-bronze shows a marked variation in the other direction; λ decreases with increase of frequency.

In Boltzmann's theory the decrement is independent of the period, and this seems to be nearer the truth, for longitudinal vibrations. As already stated in the introduction, Voight found for torsional and flexural vibrations that some metals showed, as phosphor-bronze has done in the present instance, an increase of λ with decrease of n. A definite conclusion can not be reached for phosphor-bronze, however, since but two frequencies were used.

Correction for Air Damping.

A correction for the damping due to the friction of the air was made as follows. The cylindrical shell shown in Fig. 6 with the rod attached as when used for vibrations, was suspended in a horizontal position by a bifilar thread, so as to swing in a direction parallel to the axis, thus exposing the same surface and giving it the same motion through the air. The wire in longitudinal vibration presented only a small surface compared with the shell, and its effect would be nearly compensated by the friction of the bifilar threads mentioned above. The logarithmic decrement of this pendulum system was then determined for the different masses.

From equation (16) $\mu = 2\lambda_0/T_0$, where λ_0 and T_0 are respectively the logarithmic decrement, and period of the pendulum vibrations.

$\eta = 2(\lambda/T - \lambda_0/T_0) = 2\lambda_m/T_m$, where λ_m is the logarithmic decrement due to the internal damping alone. T_m is the period as it would be if there were no air damping, and is practically identical with T.

$$\therefore \lambda_m = \lambda - \lambda_0 \frac{T}{T_0}.$$

Thus $\lambda_0(T/T_0)$ is the correction to be subtracted from the measured value of λ to obtain the decrement due to the metal alone.

For small amplitudes the following were observed.

Mass.	λ_0.	T sec.
2,244	0.00027	3.44
3,097	0.00027	3.44

The greatest correction to be applied is for the smallest value of n. The smallest value of n was for phosphor-bronze, where for $M = 2{,}244$ gm., $n = 5.04 \frac{\text{vibs.}}{\text{sec.}}$

There was no mass of 3,097 gm. used with phosphor-bronze, but for steel

$$M = 3{,}097 \text{ gm.}, \quad n = 5.27$$
$$M = 2{,}244 \text{ gm.}, \quad n = 6.20$$

Thus we have for phosphor-bronze,

M gm.	$\frac{\lambda_0 T}{T_0}$.
2,244	0.000019

for steel,

M gm.	$\frac{\lambda_0 T}{T_0}$.
2,244	0.000015
3,097	0.000016

Therefore the maximum correction to be applied to λ given in Table XI. is 0.00002. For the copper it would be about half this value, and hence, when compared with the decrement for internal damping, is small enough to be neglected.

The Platinum-Iridium Wire.

The platinum-iridium wire was chosen because of the peculiar behavior of such wires noticed by Guthe,[1] and later investigated further by the same author and Sieg.[2] They found an excessive damping of torsional vibrations for platinum-iridium wires containing 40 per cent. of iridium, which is the same composition as that of the wire used in the present instance. It was accordingly expected that the damping of the longitudinal vibrations would also be large. The result was a decrement comparable to that of steel and phosphor-bronze, and far below that of the copper. When the wire was later suspended and vibrated torsionally, it was found that neither did this wire exhibit unusually large damping of

[1] Proc. Iowa Ac. Sci., 15, p. 147, 1908.

[2] Phys. Rev., Vol. XXX., No. 4, 1910.

torsional vibrations. The highest decrement observed for the torsional vibrations was 0.00270, when the initial amplitude was 825°. The decrement decreased considerably with the amplitude. Since the length of the wire under discussion was 145 cm. and the one used by Guthe and Sieg was 40 cm., an angle of 825° is about double the twist per unit length for which they observed a logarithmic decrement some 2½ times as large. The diameter of the wire used here was 0.206 mm., as compared with 0.194 mm. for theirs. Their wire showed a difference in period of 2 per cent. between large and small amplitudes. The period of this wire was found to be 18.069 sec. for 11° amp., 18.133 sec. for 550° amp., a difference of only ⅓ per cent. There is, therefore, evidently a difference in the composition of the two wires, or else previous treatment is very influential in determining their behavior.

Variation of Period with Amplitude.

The effect of damping on the period is given by

$$T = \frac{2\pi}{\sqrt{\frac{g}{e} - \frac{\sigma^2}{4}}},$$

where $\sigma = 2\lambda/T$.

For copper, which exhibited the greatest damping, let us take $\lambda = 0.010$ in natural logarithms, which corresponds to the highest λ computed in common logarithms. Also take $T = 0.07$ sec., which is approximately the period for $M = 1{,}386$ gm. Then

$$\sigma = \frac{.02}{.07} = 0.29, \quad \frac{\sigma^2}{4} = 0.020,$$

$$\frac{g}{e} = \frac{980}{.12} = 8170.$$

Hence $\sigma^2/4$ compares with g/e about as 1 with 400,000. The observed variation of T was ⅓ per cent., hence the variation was not caused by the change in λ, provided σ enters into the equation for the period as shown above. The fact that λ varies so greatly with the amplitude shows however, that the internal friction, whatever may be its nature, is *not* proportional to the velocity.

The equation (46) is an expression for the period which involves the amplitude. The largest value of A observed in timing was 0.1 cm. Substituting this in equation (46) we find that for platinum-iridium, $\sqrt{c^2 - A^2}$ differs from c by 1 part in 700,000, and in the denominator, $\sqrt{f(p)}$ differs, when $A = 0$, and when $A = 0.1$ cm. by only 1 part in 3,000,000. It has already been stated that $K[= F(k, \pi/2)]$ is not perceptibly affected by changing A within the limits of the experiment; hence the vibration along the curve represented by (60), (61), or (62) will not account for the variation of T. This also appears by considering that for copper the dynamic modulus was a constant, yet the greatest variation of period with amplitude was observed here.

By comparing elongations from static and dynamic observations as in Fig. 12 it is found that the differences $e_0 - e_c$ per unit length, in the order of magnitude beginning with the largest, place the metals in the following order: Copper, platinum-iridium, phosphor-bronze, steel.

This is the order of variation of period with amplitude; the copper showed the greatest variation, the platinum-iridium considerable, the phosphor-bronze very little and none was detected for steel.

The position of the wires with regard to the magnitude of λ is uncertain for all except copper, which again undoubtedly stands at the head.

The indication is that the wires having the largest after-effect, have also the greatest variation of period with amplitude, and the largest decrement, although it is to be noted that *rapidity of recovery* from after-effect is of as much importance in affecting the period as the total magnitude of the after-effect.

VII. Summary.

1. As a result of the measurement of the frequency of longitudinal vibrations of wires, carrying various loads, it appears in Section IV. that for small loads, the modulus is a constant; for greater loads the modulus decreases with increasing load, although the wire is still far within the elastic limit.

2. These two regions are of different relative extent in different wires. A soft copper wire showed a constant modulus until the elastic limit was very nearly reached (see Table VII.). Steel, phosphor-bronze and platinum-iridium, showed a relatively large range of elongation where the frequencies for small amplitudes are satisfied by an equation of the form

$$n = \frac{1}{2\pi}\sqrt{\frac{g[\alpha(1 + h\alpha) + 2\beta e]}{M}} \quad \text{(see Table VIII.)},$$

where, for a given wire α and β are constants, e is the elongation for the mass M, and e and M are connected by the relation $M = \alpha e + \beta e^2$. h is a small constant depending on temperature changes in the wire as the wire vibrates.

3. The modulus determined dynamically is, as usual, larger than that measured under static conditions (see Fig. 12).

4. This difference in moduli may be explained as the result of a large after-effect, the greater part of which disappears very rapidly after the removal of load (see Sec. V.).

5. The frequency decreases with increase of amplitude, in the case of copper and platinum-iridium. The relation may be represented by $n = n_0 - bA$ where b is a positive quantity. b is greater for large frequencies than for small (Tables V. and VI.). The variation for phosphor-bronze was exceedingly small, and for steel piano-wire n was evidently independent of the amplitude.

6. The wire which showed the greatest variation of period with amplitude also showed the greatest damping and the greatest after-effect.

7. The logarithmic decrement was measured and found to vary with the amplitude (Sec. VI.). For copper, at least, the increase of λ was very nearly proportional to increase of amplitude (Fig. 16). The logarithmic decrement does not vary with different frequencies in such a manner as to indicate that the damping is due to internal friction.

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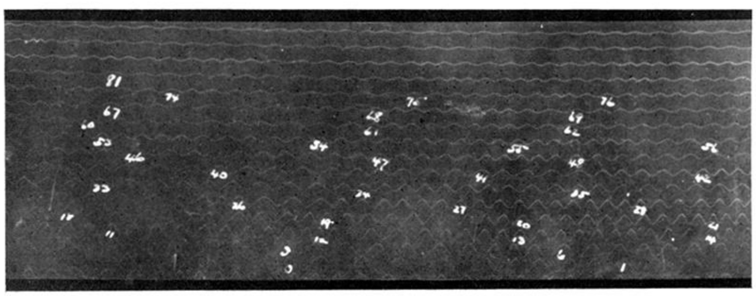

Fig. 13. Steel.