THE POLARIZATION FACTOR IN X-RAY REFLECTION

BY PAUL KIRKPATRICK

ABSTRACT

The polarization factor for the reflection of polarized x-ray radiation. An x-ray beam consisting of a multiplicity of superposed polarized radiations with non-coincident planes of polarization will exhibit the same scattering distribution in a plane normal to the beam as will a beam consisting of two plane polarized components, suitably intense, whose planes of polarization are mutually perpendicular. The customary polarization factor $\frac{1}{2}(1+\cos^2 2\theta)$ does not apply to reflection of polarized radiation and should be replaced by

$$\frac{\sin^2\alpha+P\cos^2\alpha+(P\sin^2\alpha+\cos^2\alpha)\cos^2 2\theta}{1+P}$$

In this quantity P is the primary polarization ratio and α is an angle defining the orientation of the tube with respect to the plane of reflection. For the orientation most often employed the new polarization factor reduces to $(P+\cos^2 2\theta)/(1+P)$. Radiation from a tube so oriented that α is 45° will be reflected with the same intensity as an equally intense unpolarized radiation, the polarization factor assuming the familiar form $\frac{1}{2}(1+\cos^2 2\theta)$. This orientation is recommended for future investigations, since it necessitates no knowledge of the primary polarization. A *new method for measuring the polarization of homogeneous x-rays* is presented.

INTRODUCTION

THE so-called polarization factor which appears in formulas expressing the intensity of x-rays reflected from crystals was first developed by Sir J. J. Thomson,[1] and serves to account for the decrease of intensity which the change in direction incurs. The factor expresses the proportional reduction of intensity involved in this process, and is written $\frac{1}{2}(1+\cos^2 2\theta)$. The factor may also be written $(\frac{1}{2}+\frac{1}{2}\cos^2 2\theta)$, thus exhibiting the fact that the intensity of the deviated portion of one-half of the incident (unpolarized) radiation is constant with the angle of deviation, while the intensity of the deviated portion of the other half varies as $\cos^2 2\theta$. The electric vectorial components of that half of the incident radiation which gives rise to the constant deviated portion lie in a direction at right angles to the plane of reflection while the components of the half which gives rise to the variable portion lie in the plane of reflection.

Although the polarization factor was designed to apply only to an unpolarized incident beam, those who have made use of it have not always been mindful of this limitation. Indeed the writer has not been able to discover a single case in the literature of crystal reflection where this restriction has been recognized, though the polarization factor has entered into the numerical computations of numerous papers.[2] The continuous spectral

[1] Thomson, Conduction of Electricity Through Gases.

[2] A. H. Compton, Phys. Rev. **7**, 658 (1916); Bragg, James, and Bosanquet, Phil. Mag. **41**, 308 (1921), and later papers; Harris, Bates, and MacInnes, Phys. Rev. **28**, 235 (1926). These papers are representative of a much larger number which might be cited.

radiation from primary x-ray sources is never unpolarized, and Bishop's[3] results indicate that the characteristic radiations may not be assumed to be free from polarization either. It is desirable to develop a polarization factor which shall apply to reflection of x-radiation initially partially or completely polarized.

An Equivalence Theorem

As a preliminary step it will be well to simplify our method of representing the beam emitted from the x-ray tube. The beam as actually emitted is complex, and must be supposed to be constituted of a large number of elementary emissions having their planes of polarization oriented in many different directions. It will now be shown that this beam scatters identically with a much simpler beam, consisting of two components polarized in mutually perpendicular planes. It is well known that the predominant direction

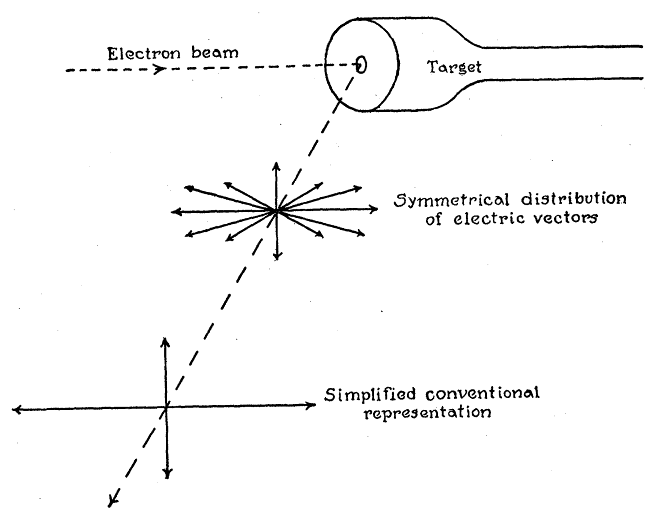

Fig. 1. Complex x-ray beam made up of a multiplicity of independent emissions. The electric vectors of the separate emissions are symmetrically disposed with respect to their prevailing direction, and in the present consideration may be replaced by two mutually perpendicular vectors.

of the electric vectors of the real beam lies in a plane containing the cathode stream. The symmetrical scattering of the beam on either side of this direction shows furthermore that the distribution of the planes of polarization of all the wave trains is a statistically symmetrical one with respect to the favored direction. The state of affairs is conventionally depicted in Fig. 1, where for every electric vector of amplitude E and inclination δ to the predominant direction there is represented also a similar electric vector symmetrically inclined upon the other side of the predominant direction. Fig. 1 also shows an equivalent system of two mutually perpendicular vectors, which will be shown to be a permissible substitution.

[3] Bishop, Phys. Rev. **28**, 625 (1926).

When such a beam is incident upon a scattering body of low atomic number the intensity of radiation scattered in any direction perpendicular to the beam is proportional to the sum of the squares of the amplitude components perpendicular to that direction. If the intensities of radiation thus scattered in the directions of minimum and of maximum scattering be compared experimentally, as has frequently been done, a ratio is obtained which characterizes the state of polarization of the beam as a whole, and which may be expressed

$$P=\frac{\sum(E\sin\delta)^2}{\sum(E\cos\delta)^2} \tag{1}$$

Eq. (1) may also be stated in two other forms which will presently be found useful. They are

$$\Sigma(E^2\cos^2\delta)=\Sigma E^2/(P+1) \quad \text{and} \quad \Sigma(E^2\sin^2\delta)=\Sigma E^2/(1+1/P) \tag{2}$$

Consider now the intensities of radiation scattered in two other mutually perpendicular directions, also normal to the direction of propagation of the beam. One of these directions of scattering, A, makes an angle ϵ with the

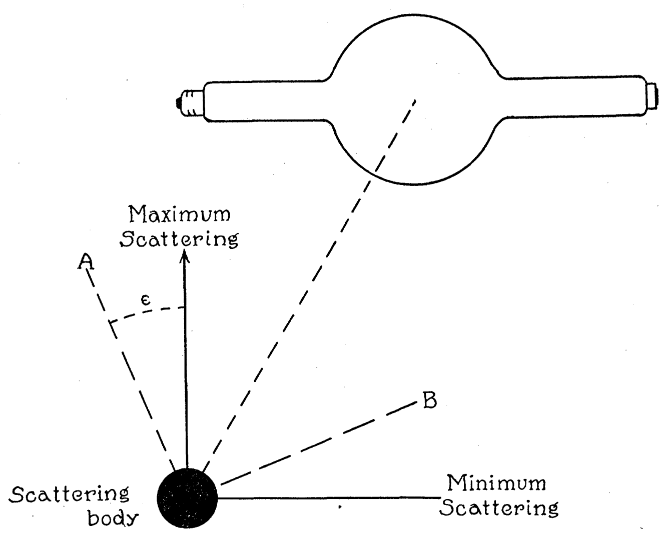

Fig. 2. X-ray scattering in a plane normal to the direction of propagation.

direction of maximum scattering. The ratio of the intensity of radiation thus scattered in the direction of B to that scattered in the direction of A, Fig. 2, may be formulated thus.

$$R=\frac{\sum(E\sin[\delta-\epsilon])^2+\sum(E\sin[\delta+\epsilon])^2}{\sum(E\cos[\delta-\epsilon])^2+\sum(E\cos[\delta+\epsilon])^2} \tag{3}$$

The double summations in numerator and denominator are necessary to take care of both members of the symmetrical pairs of electric vectors.

Upon expansion and simplification of Eq. (3) there is obtained

$$R=\frac{\cos^2\epsilon\sum E^2\sin^2\delta+\sin^2\epsilon\sum E^2\cos^2\delta}{\cos^2\epsilon\sum E^2\cos^2\delta+\sin^2\epsilon\sum E^2\sin^2\delta} \tag{4}$$

Substituting from Eq. (2) this becomes

$$R=\frac{P\cos^2\epsilon+\sin^2\epsilon}{\cos^2\epsilon+P\sin^2\epsilon} \tag{5}$$

It is apparent from this result that R is a function simply of P and ϵ, and does not require any special distribution of the individual δ's. It is therefore permissible as far as R is concerned to assume any desired distribution for the planes of the elementary emissions. The simplest and most convenient assumption is that the emitted beam is made up of two plane polarized components with their electric vectors mutually perpendicular, the direction of one of them being parallel to the cathode stream. The relative amplitudes of these components may be chosen so as to give any value of P which a complex real beam might possess. Since the real beam may be replaced by the duplex beam without affecting R it may be concluded that any real beam will exhibit the same scattering distribution in a plane normal to the beam as will a beam consisting of two plane polarized components, suitably intense, whose planes of polarization are mutually perpendicular.

Generalization of the Polarization Factor

In the beam of x-rays incident upon the reflecting crystal we need consider only such portion as is of the proper wave-length for reflection at the existing grazing angle, θ. Let this portion be regarded as consisting of two superposed plane polarized beams of the kind just discussed. The amplitudes may be designated by L and T, the greater one being L. The magnitudes of these amplitudes are governed by the requirements that $T^2+L^2=$ total intensity, and $(T/L)^2=P$.

For simple visualization let it be supposed that the plane of reflection is horizontal and that the x-ray tube is capable of rotation about the incident beam as axis. The orientation of the tube at any time may be specified by the angle α, between the plane of reflection and a plane defined by the incident beam and the cathode stream. Resolving the amplitudes T and L horizontally and vertically we have as vertical components $L \sin \alpha$ and $T \cos \alpha$, and as horizontal components $T \sin \alpha$ and $L \cos \alpha$. The reflected intensity due to the vertical components is constant with the angle of deviation, so that the intensity of this reflection portion is proportional to $(L \sin \alpha)^2+(T \cos \alpha)$.[2] The reflected intensity due to the horizontal components varies as $\cos^2 2\theta$, so that the intensity of this portion is proportional to $[(T \sin \alpha)^2+(L \cos \alpha)^2] \cos^2 2\theta$. We therefore have

$$\text{Polarization factor}=\frac{\sin^2\alpha+P\cos^2\alpha+(P\sin^2\alpha+\cos^2\alpha)\cos^2 2\theta}{1+P} \tag{6}$$

Discussion

When the incident radiation is unpolarized P has the value unity, and Eq. (6) assumes the familiar form $\frac{1}{2}(1+\cos^2 2\theta)$. The tube orientation most often employed is probably that defined by $\alpha=0$. Investigators seldom if ever state the values of α employed in their work, but the diagrams accompanying all of the papers cited in footnote 2, indicate that the zero value was used in these cases. For this orientation we have

$$\text{Polarization factor } (\alpha=0)=(P+\cos^2 2\theta)/(1+P)$$

Uncertainty as to the correct values of P makes it impossible to apply exact corrections to work done in the past. A computation based upon values regarded as plausible has shown however that some of the numerical results given in the papers cited should be modified by as much as ten percent.

Lacking precise information as to the values of P it is by all means desirable that future investigations involving comparisons between theoretical and observed intensities of reflection should be carried out with the x-ray tube oriented so that $\alpha=45°$, that is, with the plane determined by the cathode stream and the incident ray inclined at an angle of 45° with the plane of reflection. In this case the right member of Eq. (6) becomes $\frac{1}{2}(1+\cos^2 2\theta)$, and P need not be known. Radiation received from a tube thus inclined will be reflected with the same intensity as unpolarized radiation, regardless of what its actual state of polarization may be.

The determination of P for x-rays of different wave-length and different conditions of exictation has been attempted by the writer[4] and by Ross,[5] using methods in which the intensity of radiation scattered by non-crystalline solids was measured. The results contained in the present paper furnish us with a simple and apparently superior method for the determination of this quantity. Let x-rays be reflected from a crystal with the x-ray tube in the $\alpha=0$ position, and again with the tube in the $\alpha=90°$ position. Measure the intensities of reflection by an ionization method and call their ratio K. Then K is the ratio of the polarization factors for the two positions, and it follows quite simply that for the wave-length reflected

$$P=\frac{K-\cos^2 2\theta}{1-K\cos^2 2\theta}$$

Determinations by this method are now in preparation.

Department of Physics,
University of Hawaii,
Honolulu, T.H.,
February 10, 1927.

[4] Kirkpatrick, Phys. Rev. **22**, 226 (1923).
[5] Ross, Bull. Am. Phys. Soc. Vol. 1, No. 10.