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TEMPERATURE DISTRIBUTION ALONG A FILAMENT

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Abstract

A method has been developed for adapting a new integrating machine to the solution of: (1) the integral equation which applies to the distribution of thermionic emission along the central portion of a long filament in an evacuated vessel, the effect of the thermionic emission upon the filament temperature (by changing the heating current as well as by a direct cooling action) being considered; (2) the differential equation which applies to the temperature distribution near the end of a long filament from which the thermionic emission is negligible compared with the filament heating current; (3) The integro-differential equation which applies to the distribution of temperature and thermionic emission along an entire filament heated, in an evacuated vessel, by a direct current. This takes into account the effects of thermionic emission as well as cooling due to thermal conduction. All these solutions fully account for the variation of the following quantities with temperature, graphical relations being used throughout: (a) thermal conductivity; (b) thermionic emission; (c) resistivity; (d) radiation. The method has been applied to various typical cases of tungsten and thoriated tungsten filaments, and in those cases where an experimental check was possible it was found that the results were in good agreement with the measured quantities.

Introduction

THE differential equations which represent physical conditions along an incandescent filament heated by a direct current may be set up easily, and, with certain simplifying assumptions, solved by formal methods. Even with these simplifying assumptions, such solutions become very laborious, and when the empirically determined variations of the parameters are to be taken into account, such formal solutions become impossible.

There has been developed, in the Massachusetts Institute of Technology Electrical Engineering Research Laboratory, an integrating machine which may be adapted to the solution of numerous types of differential and integral equations, and the equations which apply to the incandescent filament are among these. The integral equation which applies to the part of an electrically heated, thermionically emitting filament over which the cooling due to the thermal conductivity of the filament is small may be solved directly upon this machine. Furthermore, by a method of successive approximations, the integro-differential equation which takes into account the thermionic current as well as cooling by thermal conduction to the filament supports mav be solved by means of this new device. In every case the actual resistivity, thermal conductivity, radiation emissivity, and thermionic emission, as functions of the temperature, are considered; graphical, rather than analytical relations being used throughout.

The object of this paper is to show the method of attack upon this filament problem, to present some of the typical results obtained, comparing them with measured quantities, and to bring together in a single check various empirically determined properties of tungsten and thoriated-tungsten filaments.

Equations Which Express Physical Conditions

No cooling due to thermal conduction. Consider the central portion of a long filament heated by a direct current, and let the filament be emitting thermionically at such a rate that the total emission over the portion considered is great enough to affect the filament current, all emitted electrons being drawn from the wire to a neighboring plate by an electric field. The emission is sometimes comparable with the filament current in the case of thoriated-tungsten or oxide-coated filaments, or with filaments in alkali vapor. Let it be assumed that the cooling due to thermal conductivity is negligible over this part of the filament, as it will be if the filament is very long. The following identity applies to any element in this portion of the filament;

Electrical Input = Radiation output + Cooling due to thermionic emission. Or,

$$i^2 R_T = f(T) + \phi I \tag{1}$$

where i is the filament current, which is a function of the distance along the filament (considered throughout this paper as measured from the negative end of the portion of filament); R_T is the resistance of the filament per unit length, as a function of the absolute temperature, T; $f(T)$ is the radiation per unit length of filament at the temperature T, and ϕI is the cooling per unit length of filament due to the thermionic emission, I, at the temperature T, ϕ being the work function for the filament material in equivalent volts.

If i_0 represents the filament current at the negative end of the portion of filament considered, then i is given by

$$i = i_0 - \int_0^x I dx \tag{2}$$

and thus Eq. (1) becomes

$$i_0 - [(f(T) + \phi I)/R_T]^{1/2} = \int_0^x I dx \tag{3}$$

which may be written in the form

$$\psi(T) = \int_0^x F(T)dx \tag{4}$$

Emission neglected. If the filament is such that the emission current is negligible compared with the filament current, then the filament current, i, remains constant along the wire, as is the case usually with pure tungsten filaments. In this case the cooling due to thermionic emission is usually entirely negligible. Taking the cooling due to thermal conductivity into account,

Electrical input = Radiation output + Conduction output, or

$$i^2R_T = f(T) + \left[-K_T\frac{d^2T}{dx^2} - \frac{dK_T}{dT}\left(\frac{dT}{dx}\right)^2 \right] \tag{5}$$

where K_T is the thermal conductivity of the filament per unit length, at the temperature T, x is the distance from a support, and the other symbols have the same meaning as before.

In one case (Fig. 7), the second conduction term in the square brackets of Eq. (5) was neglected. There is no loss of generality of the method in doing this, as a new temperature scale, T', may be adopted with which the conduction output will be given by one term only, $[-Kd^2T'/dx^2]$, K in this case being constant. This will be demonstrated below. After making this substitution we may write

$$d^2T'/dx^2 = (1/K)[f(T) - i^2R_T] = \lambda(T) = \lambda'(T'). \tag{6}$$

Emission and thermal conductivity both accounted for. If the emission is sufficient, and enough of the emitted electrons are removed by an electric field, the filament current is appreciably affected. Assuming that all emitted electrons are removed, the equation which represents conditions along the filament is,

$$i^2R_T = f(T) - K(d^2T'/dx^2) + \phi I \tag{7}$$

the temperature scale which makes the thermal conductivity constant being adopted. x here represents the distance from the support at the negative end of the filament.

Eq. (7) may be written

$$\left[i_0 - \int_0^x Idx\right]^2 R_T = f(T) - K(d^2T'/dx^2) + \phi I \tag{8}$$

or

$$(f(T) - Kd^2T'/dx^2 + \phi I)/R_T]^{1/2} = \int_0^x Idx \tag{9}$$

Temperature scale to give constant thermal conductivity. In terms of degrees absolute, the conduction output per unit length of filament is given by the bracketed expression in Eq. (5). But in terms of the new temperature scale T', this is assumed to be $[-Kd^2T'/dx^2]$ where K is an arbitrary constant. Thus

$$K_T(d^2T/dx^2) + (dK_T/dT)(dT/dx)^2 = K(d^2T'/dx^2) \qquad (10)$$

This has the solution

$$T' = (1/K)\int_0^T K_T\, dT + K_0 \qquad (11)$$

which gives the relation between T' and T, K_0 being a second arbitrary constant. Any value of K and K_0 may be assumed; the final result in terms of T will be the same in any case.[1]

Solution of Equations

In solving the differential and integral equations, the values of resistivity, thermal conductivity, and radiation intensity of tungsten, as functions of the temperature, were taken from a paper by Forsythe and Worthing,[2] while the thermionic emission from tungsten and thoriated tungsten, as a function of the temperature, and the work function for thorium were taken from a paper by Dushman.[3] For lack of better knowledge, the thermal conductivity has been assumed constant up to 1000°K, and equal to its value at that temperature, the variation above this temperature being taken into account.

Eq. (5) may be solved directly upon the integrating machine, to give the thermionic emission per unit length, against distance from any given point on the filament. This solution, as has been noted, neglects all effects due to thermal conductivity. It is particularly fortunate that the emission can be directly determined, as it is the emission that is wanted, more often than not, in such a problem, and if the temperature distribution is desired it may readily and accurately be obtained from the emission curve.

Fig. 1 shows a typical result obtained by this method. This is drawn for a thoriated tungsten filament three mils in diameter carrying 0.976

[1] A description of the integrating machine and its use in evaluating such integrals as that of Eq. (11) will be found in the Jour. Frank. Inst. Jan. 1927. The adaptation of this machine to the solution of equations such as (5), and (8), is not included in that paper, but the authors of the present paper hope to publish an article on that subject in the near future.

[2] Forsythe and Worthing, Astrophys. J. **61,** 147 (1925).

[3] Dushman, Gen. Elec. Rev. March 1923.

amperes at the point from which the distance is measured (which is the negative end of the portion of filament considered), and it is assumed

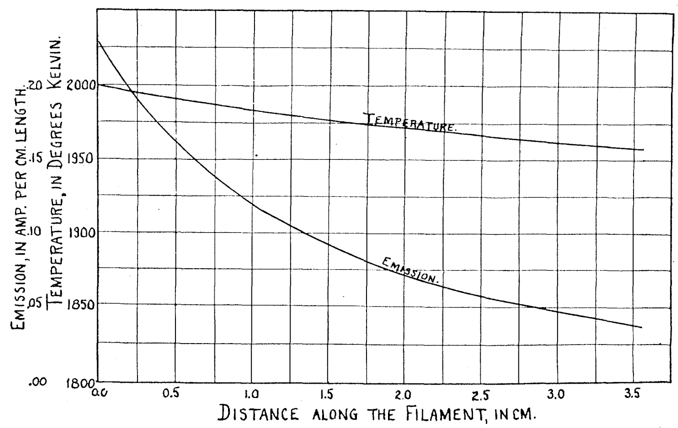

Fig. 1. Variation of temperature and thermionic emission along a thoriated tungsten filament, diam. 3 mils, current 0.976 amps.

that all emitted electrons are carried away to an adjacent plate by an electric field.

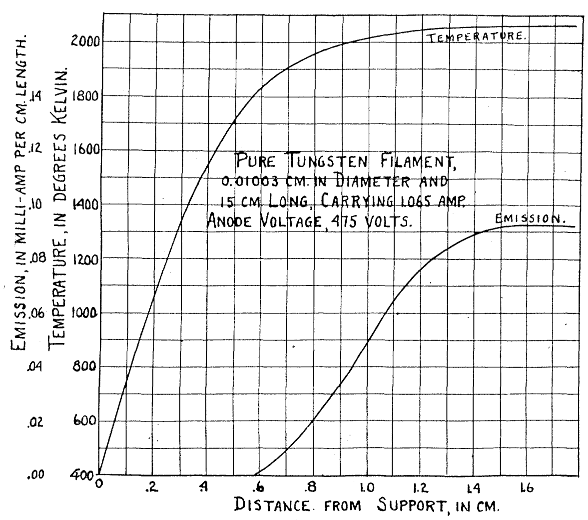

Fig. 2. Solution of Eq. (6) neglecting the cooling term.

Fig. 2 represents the solution of Eq. (6), using the absolute temperature scale, with a variable thermal conductivity, but neglecting the second cooling term. The boundary conditions were: at $x=0$, $T=$

400°K, and when $i^2R_T = f(T)$, $dT/dx = 0$. The latter assumption is equivalent to saying that the filament was so long that the temperature at the center became that of a similar filament infinitely long. This is practically the case. As the filament was of pure tungsten, the thermionic emission had virtually no effect upon the temperature distribution.

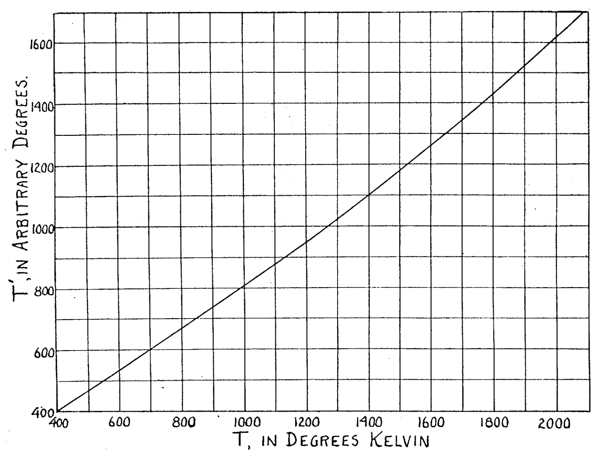

Fig. 3. Relation of the temperature scale T' for which the thermo-conductivity of tungsten is constant to the absolute temperature scale, T.

Fig. 3 shows the relation of the temperature scale T', for which the thermal conductivity of tungsten is constant, and equal to 1.23 watts per cc per degree C, to the absolute temperature scale T, and represents a solution of Eq. (11). Fig. 3 was obtained by integrating, with the integrating machine, the thermal conductivity, K_T, plotted against the absolute temperature, K being chosen to give a convenient scale for T', and K_0 being such as to make the two scales coincide at 400°, that is,

$$T' = (1/K)\int_{400}^{T} K_T\, dT + 400$$

Fig. 4 represents the solution of Eq. (8), using the temperature scale T', for the filament of Fig. 2, under similar conditions. Thus Fig. 4 should give the correct temperature distribution, rather than Fig. 2, which is only an approximation. However, a comparison of Figs. 2 and 4 will show that the effect of neglecting the second cooling term is, in this case at least, very small indeed.

The data for the filament to which Figs. 2 and 4, as well as Fig. 5, apply were taken from a paper by Dushman, Rowe, Ewald, and Kidner[4] (Tube 107–1). The calculated values check quite well with

[4] Dushman, Rowe, Ewald and Kidner, Phys. Rev. March, 1925.

the measured quantities, as will be seen from the data in Table I. Fig. 5 is just the same as Fig. 4, except that it is drawn for a smaller filament current, and was determined as a further check of the method, and also of the data in the paper noted above.

TABLE I

Data for the filaments referred to in Figs. 2, 4 and 5.
The quantity f is the ratio of emission which would exist if the entire filament were at maximum temperature to the actual total emission.

	Fig. 2		Fig. 4		Fig. 5	
	Calc.	Meas.	Calc.	Meas.	Calc.	Meas.
Filament voltage (volts):	11.99	11.77	12.02	11.77	9.32	9.12
Maximum temperature (°K):	2070	2065	2070	2065	1900	1897
Total emission (m.a.):	1.21	1.15	1.21	1.15	.110	.1065
f:	1.147	1.143	1.147	1.143	1.18	1.167

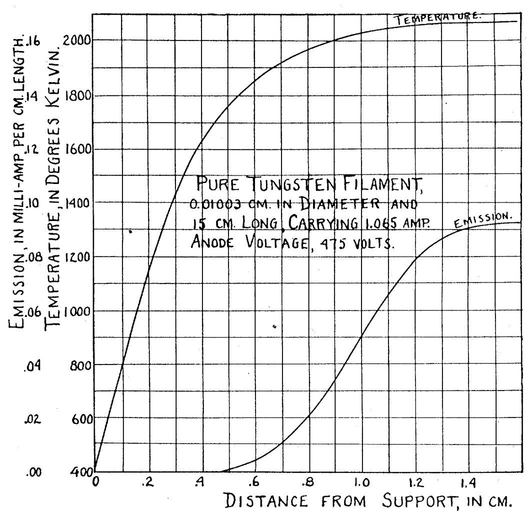

Fig. 4. Solution of Eq. (8) using the temperature scale, T'.

Fig. 6, which is drawn for a thoriated tungsten filament with zero potential gradient at the cathode (that is, with an electric field just great enough to remove all emitted electrons), represents the solution of Eq. (8), with the boundary conditions; when $x=0$, $T=400°K$, when $x=$length of filament, $T=400°K$, and the assumption that over the central part of the filament there was no cooling due to thermal conductivity. This cooling was found to be negligible in the actual case. It will be noted that the emission falls off very rapidly as the distance from the negative end of the filament increases, because of the drop in filament temperature produced by the decrease of filament current.

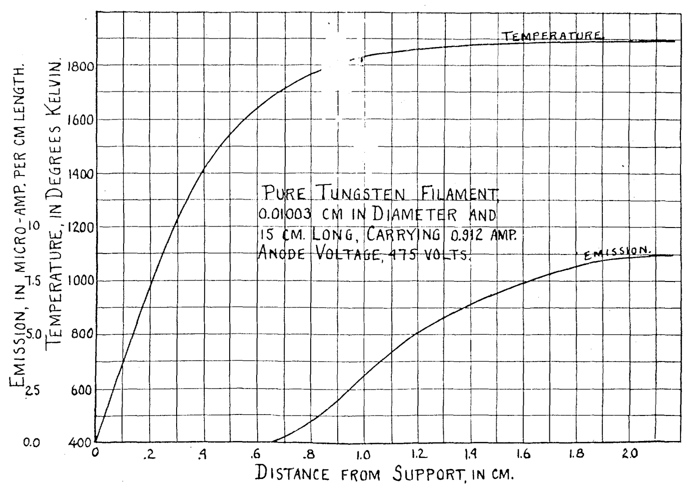

Fig. 5. Solution of Eq. (8) using the temperature scale, T'. The filament current is less than for Fig. 4.

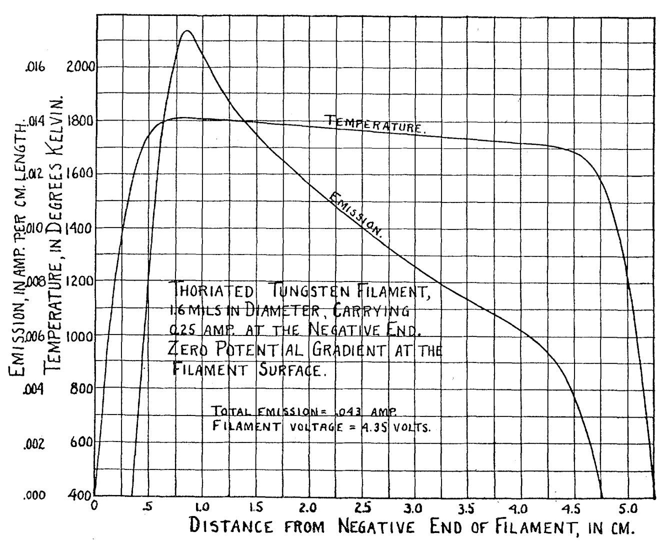

Fig. 6. Solution of Eq. (11) assuming the temperature to be 400°K at the ends of the filament.

Fig. 7 shows that the cooling due to thermionic emission from the thoriated tungsten filament is considerable, the error in maximum temperature which would result if this cooling term were neglected being about 125°K. Fig. 7 also gives some idea of the effect of other terms in Eq. (8), and shows the several steps in obtaining the temperature distribution of Fig. 6 from $x=0$ to the point where dT/dx is 0. First the filament current was assumed constant, and curve (4) was determined. The second cooling term was neglected here, but the effect

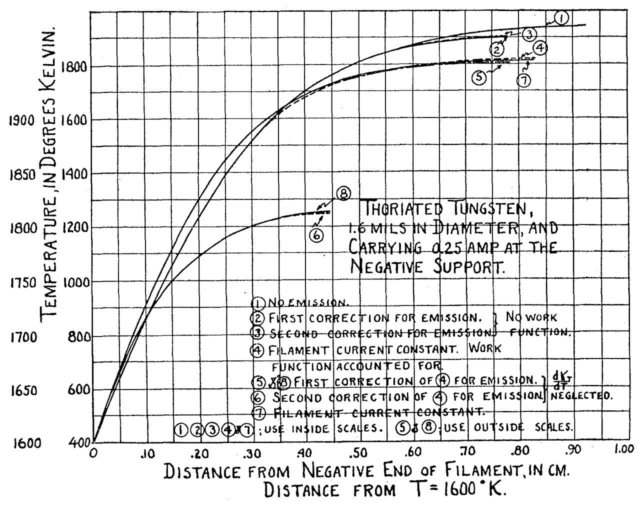

Fig. 7.

is very small, as a comparison of curves (7) and (4) will show. Then the value of $\int_0^x Idx$, as a function of x, was determined from curve (4), and placed in Eq. (8) which was solved, as before, for curve (5). Curve (6) was obtained from (5) in a similar manner, and as these last two were practically coincident, the maximum difference being about 2°K, the successive approximations were carried no further. The central portion of the emission curve of Fig.6 represents the solution of Eq. (3), and the temperature distribution near the positive end of the filament was determined just as for the negative end.

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