THE FLOW OF LIQUIDS THROUGH CAPILLARIES

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Abstract

A simple theory of the flow of a liquid from a reservoir, through a capillary, into a second reservoir is developed, and the conclusions from it are shown to accord with the observations of Bond and of Poiseuille, and with a qualitative study of the flow by means of colored streams. It is shown that, for viscosimeters of this type, the common interpretation of that term (with coefficient m) in the viscosimeter equation which is frequently called the kinetic energy correction is entirely incorrect. That term is an inertia correction; it does not arise from any loss of head attendant upon the imparting of kinetic energy to the liquid, but solely from a progressive change in the distribution of the flow in the exit reservoir. It is not the correction which was considered by Hagenbach. For ideal conditions, m is probably equal to unity. At very low velocities, the distribution of the flow in each reservoir is independent of the velocity, and consequently the inertia term vanishes; at this stage the Couette correction is $2e$, twice what it is when the distribution in the exit reservoir is changing. From Bond's data, it is found that $e=0.573r$. At a certain velocity, simply related to e, the distribution of the flow in the exit reservoir begins to change, the inertia term appears. Bond found that this occurs when the Reynolds number is 10. Under certain stated conditions, the initial distribution of flow can be retained to a much higher velocity; in these cases the inertia correction does not enter, and pt is independent of the velocity. There are indications that for very short tubes pt ceases to be linear in the velocity before the flow in the tube becomes turbulent. An explanation is offered. Changes in the terminal configurations affect both the Couette correction and the inertia correction, and the second may be markedly affected by changes in the size and form of the exit reservoir. As commonly used, the subdivided tube method for determining e is entirely unreliable.

THE problem to be considered is the nature of the flow of a liquid from a large reservoir, through a cylindrical capillary of circular cross-section, into a second large reservoir. It will be assumed that the free surface of the liquid in each reservoir is great as compared with the sectional area of the capillary, that the edge at each terminus of the capillary is sharp and smooth, that at each end of the capillary the terminal face is normal to the axis of the capillary, and that the radial extent of this face, in every direction, and all other distances from the terminus to the wall of the reservoir are severally so great that the distribution of the flow of the liquid is essentially the same as if they were infinite.

This problem is of much interest in itself, and it is also of prime importance in the theory of the capillary viscosimeter. The equation for the flow in such viscosimeters is usually written in one of the following equivalent forms.

$$\mu = \frac{\pi r^4 pt}{8V(l+e)} - \frac{m\rho V}{8\pi(l+e)t} \tag{1}$$

$$pt = \frac{8\mu V(l+e)}{\pi r^4} + \frac{m\rho V^2}{\pi^2 r^4 t} = \frac{8\mu V}{\pi r^4}(l+e) + \frac{m\rho Vv}{\pi r^2} \tag{2}$$

$$= \frac{8\mu V}{\pi r^4}\left[l+e+\frac{m\rho r^2 v}{8\mu}\right] \tag{3}$$

where μ is the coefficient of viscosity, p is the over-all driving pressure (the algebraic sum of the hydrostatic pressure, arising from the elevation of the free surface of the liquid in the entrance reservoir above that in the exit reservoir, and the excess of the pressure just below the free surface of the liquid in the entrance reservoir over that just below the free surface in the exit reservoir), t the time required for the volume V to flow through the capillary of length l and radius r, ρ the density of the liquid, e the Couette correction, m a numerical factor (usually taken as 1.12), and v the average velocity of the liquid at any cross-section of the capillary.

The term involving m is generally called either the kinetic energy correction or the Hagenbach correction; and it is interpreted as a subtractive correction to be applied to pt to cover a loss of head arising from the imparting of kinetic energy to the liquid. Such interpretation is entirely wrong, for it is evident that the conditions of the problem are such that all of the work done by the pressure is expended against viscous forces; there is no loss of head due to other actions. The liquid starts from practical rest at the surface of one reservoir, and reaches the surface of the other with a velocity which is essentially zero; if the areas of the two free surfaces are the same, then at the end of its journey the liquid has exactly the same kinetic energy as it had at the beginning. This has been pointed out by several writers, but its truth has not been generally recognized, probably because no satisfactory explanation of the origin of the term has been offered and no satisfactory picture of what actually occurs has been proposed.

For any given velocity of efflux, the total work expended against viscous forces may be conceived as made up of three terms: One represents the work which would be performed in the capillary if throughout its length the flow were exactly the same as that at the central section of a very long capillary of the same sectional dimensions; another represents the amount by which the actual viscous work performed between the surface of the entrance reservoir and the central section of the tube exceeds half the work done in the capillary, computed as in the preceding case. This may be called the entrance work, or entrance correction. The third represents the corresponding term for the exit reservoir, and may be called the exit work, or exit correction. Each of these corrections may be expressed in terms of the length of the capillary within which the same amount of work will be done in the same time. Let these lengths be denoted by δ_i and δ_o, respectively. Then

$$pt = \frac{8\mu V}{\pi r^4}\left[l + \delta_i + \delta_o\right] \tag{4}$$

Eq. (4) has exactly the same form as Eq. (3).

The amounts of work corresponding to δ_i and δ_o depend solely upon the relative velocities of adjacent portions of the liquid, and the same is true of the work done in the capillary. If the flow is steady, the entire volume of the liquid may, in imagination, be divided into a large number of elementary tubes of flow; and, so long as the distribution of these tubes remains unaltered, the relative velocities of the adjacent portions of the liquid throughout the system will depend solely upon v, and will vary proportionally to it. Hence, so long as the distribution of the tubes remain unaltered, the values of δ_i and δ_o will remain unaltered, and pt will remain constant no matter how great v may become. But if either δ_i or δ_o vary, then pt will vary. It is such variation that gives rise to the m term in Eqs. (1), (2), and (3). The changes which occur in the distribution of the tubes of flow in the exit reservoir can be readily followed if a colored liquid is allowed to discharge into clear water. When the flow is exceedingly slow, the colored liquid oozes out of the capillary, and flows away in all directions, forming a slowly growing, nearly hemispherical cap seated against the end of the tube. As the velocity is slowly increased, this condition persists for a time; but presently, while the velocity is still very low, the cap is seen to move bodily from the end of the tube, developing a stem. As the velocity continues to increase, the stem lengthens, the cap increases in size and takes on a mushroom form, the rim of the cap acquires the appearance of a vortex ring, the whirls become more and more pronounced, until further regular development is interfered with, either by the surface of the liquid or by the walls of the reservoir. By using clear water in the entrance reservoir, and introducing colored liquid by means of a capillary pipette, the tubes of flow in that reservoir also were studied. There was no indication that the distribution of these tubes changed as the velocity was increased from a very low value to one corresponding to a long jet in the exit reservoir. The change appeared to be exclusively in the exit reservoir.

The m term does not arise from any loss of head resulting from an imparting of kinetic energy to the liquid, but solely from the inertia of the liquid, from its tendency to preserve the direction of its velocity. The term is much more properly described as an inertia correction, than as a kinetic energy correction. Certainly it should not be called the Hagenbach[1] correction, because the case he was considering was quite different. In that, the liquid discharged into a gas, and the pressure which entered into the equation similar to Eq. (1) was that between the entrance reservoir and the outlet of the tube. In that case, the kinetic energy of the issuing stream was actually thrown away; that portion of the work done by the pressure was not expended in viscous work done between the boundaries at which the pressures were measured. But in the case here considered, there is no such discarding of kinetic energy, but merely a temporary storage of energy in that form. The correction considered by Hagenbach does not apply to the present case.

Most of the attempts to determine the quantity e in Eqs. (1), (2), and (3) are of little value, largely on account of assumptions based upon the false interpretation of the m term. In fact, only Bond's[2] data appear to be at all satisfactory for the purpose, though he did not so use them. Using very carefully made, thick walled, cylindrical tubes having circular lumens and square-cut ends with sharp edges, Bond determined the value of $\delta_i+\delta_o$ for tubes of different radii and for a wide range of velocity and of kinematic viscosity (μ/ρ). He found that below a certain velocity, corresponding to Reynolds number $R(=2\rho vr/\mu)=10$, $\delta_i+\delta_o$ is independent of v, and equals $1.146r$. He does not state explicitly how the sum varies at higher velocities, but by scaling the figure reproduced in his paper, it is found that from $R=10$ to, about, $R=700$, the sum increases linearly with v, $d(\delta_i+\delta_o)/dv$ being $0.98\rho r^2/8\mu$, with an uncertainty of perhaps 2 percent. The numerical factor, 0.98, corresponds exactly to m in the viscosimeter equation. For $R>700$, the slope decreases, at first rapidly and then more slowly, finally approaching asymptotically the value corresponding to $m=0.735$.

Poiseuille[3] also worked most carefully with thick walled tubes which had square-cut ends, and of which at least the exit terminals had sharp edges. He divided his observations into two classes. Into one he placed those for which pt seemed to be independent of the velocity; into the other, he placed those for which pt varied markedly with the velocity. For the latter, the variation is linear in v, except for the presence of

[1] Hagenbach, E., Pogg. Ann. **109**, 385-426 (1860).

[2] Bond, W. N., Proc. Phys. Soc. London, **33**, 225 (1921); **34**, 139–144 (1922).

[3] Poiseuille, Mém. des Savants Étrangers, l'Inst. de France, **9**, 433–543 (1846).

certain abnormal observations. The values of m deduced from Poiseuille's observations are collected in Table I. The C-tubes are plainly abnormal, and the observations with tubes D^{iii}, D^{iv}, E^{ii}, and F^{i} are very erratic. For the others the average is $m = 1.04$. The differences between the values for the several tubes, as well as the difference of the mean from Bond's, appear to be real; they probably arise from slight differences in the configurations of the terminals of the capillaries and of the reservoirs. It seems probable that for the ideal case of truly circular, thick walled, cylindrical tubes with square-cut ends and sharp edges, $m = 1$. The high values of m (averaging about 1.12) usually found for capillary viscosimeters, and the high value for the C-tubes are probably to be explained by irregularities in the edges, or by some other departure from the ideal conditions.

Some of the observations which Poiseuille placed in the first class actually belong in the second, but as $l+e$ was very large as compared with $\rho r^2 v/8\mu$ (cf. Eq. (3)) the percent change in pt was small, and was overlooked by him. It is customary to assume that the same is true of all the observations of this class, but the recorded data do not warrant such a conclusion. The observations for some of these tubes are shown in Fig. 1; in each case the dotted line indicates the slope of the graph which would have been obtained had they varied in the manner generally assumed. It is obvious that the values do not lend themselves to this assumption. True, in each case, even the extreme variation required by the dotted line is small, but the erratic variations are much smaller. For these tubes, we are forced to conclude that the observations do not follow the dotted lines, but that, to a high order of precision, pt is independent of the velocity to the very highest velocity represented by the observations. In each case, the velocity corresponding to $R = 10$ is marked; it will be noticed that the constancy of pt extends to considerably higher velocities. This appears to contradict Bond's conclusion, but it should be remembered that these tubes were very long ($l > 400r$), while those used by Bond were much shorter ($l < 150r$).

Combining the two sets of observations, it is seen that the nature of the flow may be represented as in Fig. 1a; ABD represents Bond's observations, ABC and BD represent Poiseuille's. Above B, which corresponds to $R = 10$, there are two distinct regimes, BD representing the preferential one. But the regime BC can be established, and when established, may be stable. The interpretation of these observations is not difficult, and leads to a clear idea of the origin of the change in regime.

As inertia effects vary according to a higher power of the velocity than do the corresponding viscous ones, it is evident that, as the velocity is reduced, the former will presently vanish as compared with the latter. At such low velocities, the flow takes place as though the liquid were devoid of inertia; and, in any region, the elementary tubes of flow are distributed in such a way that the rate at which work is done against the viscous forces is as small as possible, consistent with the velocities of the liquid at the boundaries of that region.[4] Hence, at these velocities both δ_i and δ_o are as small as possible, consistent with the value of v; and

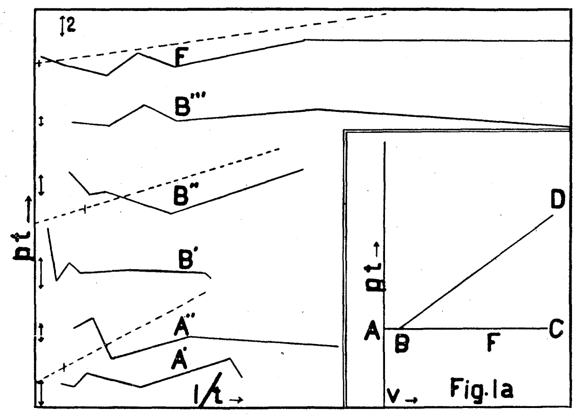

Fig. 1. Poiseuille's data for tubes A^i, A^{ii}, B^i, B^{ii}, B^{iii}, and probably F indicate that pt is independent of the velocity. Continuous lines connect the individual observations. The three dotted lines indicate the slopes of the graphs which would have been obtained for the A, B, and F tubes, respectively, if pt had varied in the manner commonly assumed; the short vertical lines indicate the positions at which $R=10$. The length of each arrow-tipped vertical line corresponds to a change of 0.1 % in pt for the adjacent graph, except in the case of F for which the change is 2.0 %.

Fig. 1a. Variation of pt with the velocity. From A to B there is no variation; beyond B there are two possible regimes, BD is the preferential one, but BC can be realized and is stable for long tubes and quiet conditions. If p_c is the difference in pressure between the surface of the liquid in the entrance reservoir and the exit terminus of the capillary, and p is the difference in pressure between the surface of the liquid in one reservoir and that in the other, then, in the region AB, $p_c<p$; in BD, $p_c=p$; in BC, $p_c>p$.

a constancy of their sum requires the individual constancy of each. Furthermore, if the two reservoirs are identical in form and size, the distribution of the tubes will, at these velocities, be the same in each, and δ_i will equal δ_o. As Bond found that so long as R was less than 10, $\delta_i+\delta_o$ remained constantly equal to $1.146r$, the distribution of the tubes must have remained unaltered—must have been that characteristic of an inertialess liquid—and $\delta_i=\delta_o=0.573r$, which may be denoted by ϵ.

At excessively low velocities, the pressure in the stream at its exit from the capillary exceeds that in the quiescent liquid at the same level

[4] Lamb, H., Hydrodynamics (1st ed.), p. 537, Section 297.

in the exit reservoir by an amount p_o, which is equal to the viscous work done in that reservoir per unit volume of efflux. As the velocity increases, the kinetic energy delivered by the stream to the exit reservoir presently becomes appreciable as compared with the viscous work done in the reservoir, and p_o decreases in such a manner that $p_o+\rho v^2$ continuously equals that viscous work per unit volume of efflux. That is, $p_o+\rho v^2 = 8\mu v\delta_o/r^2 = 8\mu v\epsilon/r^2$ if $R<10$. When v becomes equal to $8\mu\epsilon/\rho r^2$, $p_o=0$; and if v continues to increase, p_o becomes negative unless δ_o increases; that is, unless the distribution of the tubes of flow changes. When p_o is negative, a portion of the energy of pressure is transformed into kinetic energy which is retransformed into pressure energy as the issuing stream does work against the back pressure arising from the negative value of p_o. Such pointless transformation and retransformation seems to be avoided whenever possible. When, with slowly increasing v, p_o becomes 0, the distribution of the tubes of flow begins to change. Until this velocity is reached, the tubes of flow are distributed over a hemisphere resting against the end of the capillary; now this hemispherical cap moves out, developing a stem, and, if the reservoir is large so that the advance of the cap and the attendant changes are unimpeded, the change proceeds at such a rate as to keep p_o continuously equal to zero. That is, δ_o remains always equal to $\rho v r^2/8\mu$, which is exactly the value of the m term in Eq. (3) when $m=1$.

If the preceding description is correct, the point B (Fig. 1a) should correspond to the velocity $v=8\mu\epsilon/\rho r^2$. Putting into this expression the value of $\epsilon(=0.573r)$ deduced from Bond's observations, it is found that the resulting Reynolds number is 9.2, while Bond's value for this point is 10. The two values agree as closely as one should expect. It should be remembered that Bond's determination of the position of B is entirely independent of his determination of 2ϵ.

The preceding assumes that the changes in the velocity are so slow that the distribution of pressure is always that which corresponds to an unaccelerated flow; otherwise, $p_o+\rho v^2$ will not be equal to $8\mu v\delta_o/r^2$. If the acceleration is positive, p_o is greater than it would otherwise be. In particular, if there is a considerable positive acceleration as the velocity passes through the value corresponding to B, the kinetic energy of the incoming liquid is not sufficient to enable it to get out of the way of the following liquid, even when it keeps to the most favorable distribution of the tubes of flow—that corresponding to very low velocity. Hence, p_o continues positive, and the original distribution persists. Thus, this distribution of flow may be established at a velocity corresponding to some point F beyond B. Having been so established, the distribution of the flow in the reservoir can change to that corresponding to the line BD (Fig. 1a) only by decreasing the velocity of efflux by a finite amount. Such decrease involves the slowing up of the entire column of liquid in the capillary. This can not be done instantly, but only progressively, by the propagation of a series of compressional waves through the column. As these waves are damped by the viscosity of the liquid, a given amplitude at one end of the capillary will produce no appreciable effect at the other if the length of the capillary exceeds a certain amount. Hence a flow corresponding to BC may be stable in relatively long tubes, and under very quiet conditions.

For tubes of circular section, it can be shown[5] that the amount a wave is damped while traveling the length of the tube is determined by the value of the dimensionless quantity $\mu l^2/\rho\lambda s r^2$, where s is the velocity of sound in the liquid, and λ is the wave-length. Unless λ is simply related to l, the effect of successive waves will not be cumulative. In the case of Poiseuille's observations, as well as under the usual conditions obtaining in viscosity measurements, the mechanical disturbances are not simply periodic, but are very heterogeneous. Under such conditions, the component which is fundamental to the tube is of greatest importance and is automatically picked out by it. Hence the quantity which determines the damping of the important constituent of the disturbance is $D = \mu l/\rho s r^2$.

If, for a given series of tubes, the corresponding fundamental constituents of the disturbance were all of the same intensity and persisted for the same number of oscillations, a definite, limiting value of D would mark the division between those tubes for which flow of the type BC was stable, and those for which it was not. But if certain of these fundamental constituents were more intense, or more persistent than others, they would be more effective, and for the corresponding tubes a larger value of D would be required to insure stability of the BC type of flow. For weak, or brief constituents, the reverse would be true. Furthermore, the intensity and other characteristics of accidental disturbances are subject to large fluctuations; these will cause marked variations in the limiting value of D. Hence, in general, this limiting value will not be very definite, and it is probable that even under the most favorable conditions certain of the data will seem to be exceptional. Nevertheless, it seemed desirable to see whether Poiseuille's data would give any indication of a definite limiting value of D. For certain series of his tubes, both types of flow of water at 10°C were observed; the values of l/r^2 for the adjacent members for which the types of flow differed are given

[5] Rayleigh, Theory of Sound, Vol. 2, p. 331.

in Table I. It will be noticed that, excepting the F-tubes, the smallest value for which the flow was definitely of the BC type is 73,000 cm^{-1}, and the greatest for which it was definitely of the BD type is 51,000 cm^{-1}, or possibly 73,000 cm^{-1}. These limits are surprisingly close. Taking 60,000 cm^{-1} as the limiting value, the limiting value of D is 0.005. The data for the F-tubes are distributed erratically, but the BC type of flow seems to persist to $R=300$ if $l/r^2=36{,}000$ cm^{-1} (F), and possibly even if l/r^2 is as small as 19,000 $cm^{-1}(F^i)$. For larger values of R, the observations with tube F appear to correspond to points lying between BC and BD, but much nearer the former. This may result from the regime changing from BC to BD during the course of a single observation; such a change might well result from a violent jar. The observations with F^i lie fairly close to BD when $R>300$; this suggests that during the observations at low velocity the significant mechanical disturbance was weaker than it was when the high velocity observations were taken.

What of the distribution of the tubes of flow in the entrance reservoir? Does this remain unchanged, as our observations with colored liquid seemed to indicate? As any change from that corresponding to very low velocity would result in an increase in δ_i, Bond's observation that pt remains independent of v so long as R does not exceed 10 shows that for this range of v there is no change. Poiseuille's observations indicate that there is no change in pt while v varies from a value near $R=10$ to $R=104$ (tube A^{ii}), to $R=117$ (tube B^{iii}), and possibly to $R=300$ (tube F); consequently in this range there is no change in δ_i. Within this higher range, the flow with Bond's tubes and also with many of Poiseuille's was of the type corresponding to BD; as there seems to be no reason for expecting that the type of flow in the entrance reservoir will change at exactly the same time as that in the exit reservoir, the absence any break in the BD graph within this range indicates that, within this range, δ_i is changeless even for the BD type of flow. Furthermore, Bond's curve shows that BD retains its linear character until $R=700$, and, with one exception, Poiseuille's observations give no indication that the linear character of BD changes before the condition of turbulence is closely approached. Whence, it appears that the distribution of the tubes of flow in the entrance reservoir remains unchanged throughout the range of velocity for which the flow in the capillary is steady, provided that the capillary fulfills the ideal conditions stated at the beginning of this article.

But what of the exceptions? Bond's curve shows that for $R>700$ the slope of BD progressively changes, becoming smaller; and Poiseuille's observations at high velocities with tube A^{vii} lie below the BD graph corresponding to low velocities. Tube A^{vii} was very short ($l=0.1$ cm,

$l/r = 14$), and so were several of Bond's tubes; it seems probable that the non-linearity of BD is intimately connected with the shortness of the tube, but until additional data are available it is not possible to speak with certainty. It may possibly arise somewhat as follows. One might well anticipate that the velocity of the entering liquid will not be fully adjusted throughout the cross-section until after the liquid has advanced a certain distance σ_i into the capillary, and that σ_i will increase with Rr. Likewise it would not be surprising to discover that the distribution of the velocity begins to change before the exit is reached, and that the distance σ_o from the exit to the point where this change begins, increases with Rr. In these terminal regions of adjustment, the kinetic energy per unit volume of the liquid is less than it is in the central region of complete adjustment; hence, after these regions have met one another, after $\sigma_i+\sigma_o$ has become equal to the length of the capillary, the kinetic energy per unit of volume delivered to the exit region will be less than ρv^2 and consequently δ_o will be less than its normal value $\rho vr^2/8\mu$. Hence, the rate of increase of pt with v will begin to decrease, probably perceptibly, when $\sigma_i+\sigma_o$ has become equal to l. It may be for this reason that the graph for short tubes ceases to be linear at some value of R which is smaller than that corresponding to turbulent flow in long tubes. But if such is the case, then at low velocities, $R<10$, $\sigma_i+\sigma_o$ must be exceedingly small, as Bond found that the difference between the value of $\delta_i+\delta_o$ for a perforated plate ($l=0.0075$ cm, $l/r=0.102$) and for long tubes lay within the limits of his experimental error, being only $0.014r$. Furthermore, the increase of $\sigma_i+\sigma_o$ with v can be accompanied by only a very small increase in the rate of dissipation, otherwise pt could not have appeared to remain independent of v to such high velocities as was observed for some of Poiseuille's tubes.

Although the data we have considered give no indication of any change in the distribution of the tubes of flow in the entrance reservoir, the fact that the kinetic energy per unit volume of liquid in the capillary is ρv^2—exactly twice what it would be were the velocity the same over the entire transverse section of the tube—suggests that with increasing velocity a change in distribution will probably occur. Reynolds[6] observed that in no case did the first suggestion of turbulence occur nearer the entrance than 60 radii. Whatever its significance, it is interesting to note that when the velocity is such that, in transferring a unit volume of liquid from the surface of the reservoir to a point 60 radii beyond the terminus, the viscous work done is equal to $\frac{1}{2}\rho v^2$, the Reynolds' number

[6] Reynolds, O., Scientific Papers Vol. 2, p. 77; Phil. Trans. **174**, 935–982 (1883).

is 1938; Reynolds[7] observed that turbulence set in when R lay between 1900 and 2000.

Departures from the ideal conditions mentioned at the beginning of this paper will, in general, be accompanied by changes in the amount of viscous work performed at the termini, and by changes in those quantities which depend upon this work. Hence, in practice, we should expect the values of δ_i, δ_o, ϵ, m and the value of R at which δ_o begins to vary with v, to vary from case to case. There seems to be no satisfactory data which will enable one to form an idea of the magnitude of such variation in δ_i, ϵ, and the critical value of R, but there is a considerable amount of data bearing upon the rate of variation of δ_o with the velocity. These indicate that for actual viscosimeters approximating to the ideal conditions, the variations of δ_o with the velocity are such that m averages about 1.12, and varies from very near 1.00 nearly to 2. Probably these variations are caused mainly by variations in the configurations of the terminals of the capillaries. But it is obvious that changes in m will also occur if the reservoir is so constructed as to impede the development of the jet. If, in its advance, the head of the jet meets an obstruction, the uniform development of the jet will be impeded, the work done in the reservoir will be increased, the pressure in the jet will rise, and the observation will lie above the line BD (Fig. 1a). Whether beyond this point the line will have the same slope as BD depends upon the nature of the obstruction. In some cases it is to be expected that the slope will be greater. On the other hand, if the reservoir is small, it is conceivable that the distribution of the tubes of flow may presently become such that no further change with the velocity is possible. Then, pt will cease to increase with v, and the graph will become parallel to AC, but will lie above it.

Several have attempted to derive the value of the quantity e in Eq. (1) from two sets of observations; one made with a long capillary, and the other with the same capillary cut into a number of sections which were then connected one to another by short lengths of tubing of considerably greater internal diameter. The computation was based upon the assumption that each of these connecting tubes acted as an exit and as an entrance reservoir, each of these functions being in every way the same as that performed by the corresponding, large, terminal reservoir. This required that the viscous work done in each connecting tube should be equal to the sum of those done in the same time in the two terminal reservoirs. But in the small, intermediate reservoirs, the distribution of the tubes of flow entering the capillary was surely not exactly the same

[7] Reynolds, O., Scientific Papers, Vol. 2, p. 536; Phil. Trans. **186**, 123–164 (1895).

as in the large entrance reservoir; and the distribution of the tubes emerging from the capillary was certainly quite significantly different from that in the large exit reservoir. Hence, correct values of e could not be so obtained.

As already stated, the only satisfactory data from which e can be obtained are those of Bond. These lead to the value $0.573r$, which we have denoted by ϵ. This is for the BD regime (Fig. 1a); for the region AB and for the BC regime, in which the inertia term vanishes, the Couette correction ($1.146r$) is twice as great as it is for the BD regime. This is because the inertia term arises from the variation in δ_o; and the minimum value of δ_o is ϵ.

TABLE I

Certain data for Poiseuille's tubes.
(See pages 837 and 841.)

Tube	m	Tube	m	$l/1000r^2$			
				BC		BD	
A^{iii}	1.03_8	$*D^{iii}$	0.79	A^{ii}	102	A^{iii}	50.8
A^{iv}	1.06_1	$*D^{iv}$	1.52	B^{iii}	73	B^{iv}	28
A^{v}	1.07_2	$*E^{ii}$	0.24(?)	C^{iii}	134	$*C^{iv}$(?)	55.5
A^{vi}	1.04_5	F^{i}	0.51	$*D^{iii}$	208	$*D^{iv}$(?)	73
A^{vii}	1.06_1	F^{ii}	1.04_6	$*F$	36	$*F^{i}$(?)	19
$*B^{iv}$	1.07_4	F^{iii}	0.97_5	$*F^{i}$(?)	19	F^{ii}	9.4
B^{v}	1.03_6	F^{iv}	1.09_6				
$*C^{iv}$	1.16_9	F^{v}	0.91_4				
C^{v}	1.81						
	**Mean	1.03_8					

* Observations are very erratic. For B^{iv} m lies between 1.01 and 1.14; some points of C^{iv} indicate m is as low as 0.79; some points of D^{iv} give $m = 0{\cdot}51$
** Omitting the C, D, and E tubes and F^{i}.

Summary. For viscosimeters of the type considered: 1. The m-term in the viscosimeter equation arises from progressive changes in the distribution of the tubes of flow. It is an inertia correction; it has nothing to do with any loss of head attending the acceleration of the liquid, and is not the Hagenbach correction. It is not dependent upon the existence of turbulence or of eddies, although the latter are frequently present, and the former may exist.

2. Bond's data show that when Reynolds' number $R < 10$, the inertia correction vanishes and the Couette correction is $1.146r$.

3. For $R > 10$, approximately, two regimes are possible if the tube is long and the mechanical disturbances are slight. In one, the conditions continue the same as when $R < 10$. In the other, the Couette correction is half as great as it is when $R < 10$, and m is probably equal to unity.

4. From theoretical considerations, it is shown that the value of R at which the inertia correction firstappears is $16\epsilon/r$, where ϵ is the Couette correction for the BD-regime.

5. As the velocity is increased, it is to be expected that the distribution of the flow in the entrance reservoir will ultimately change, but there is no evidence that such a change occurs before the flow in the tube becomes turbulent.

6. For very short tubes, m seems to decrease at high velocities. This may indicate that near the ends of the capillary the distribution of the velocity over the cross-section of the tube is not quite the same as it is at the center of a long tube; and that the length over which this abnormal distribution extends increases with the velocity.

7. The value of the Couette correction, of m, and of the velocity at which the inertia term first appears, all depend upon the configuration of the terminals and of the reservoirs, and may be expected to vary from case to case. The variation in m is of special importance in viscosimetry.

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