DETERMINATION OF ELECTRONIC CHARGE FROM MEASUREMENTS OF SHOT-EFFECT IN APERIODIC CIRCUITS

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Abstract

An outstanding difficulty in the measurement of electronic charge from shot-effect in a tuned circuit has been regeneration into this circuit. Former methods of dealing with that feature have added tedious computations to Schottky's original calculation. The solution presented here yields the result in such a form that much of the labor is eliminated; the solution is of great mathematical simplicity, and measurement of high-frequency resistance and resonance frequency becomes unnecessary. The treatment may be generalized to include the non-resonant case if the assumption be made that the resonance curve of the amplifier is symmetrical.

A non-inductive resistance may be used as the medium of coupling, and experimental work with this system yields the mean result for electronic charge of 4.774×10^{10} e.s.u.

In work under conditions other than those of temperature limitation, the diminution of fluctuations seems to be due to high current densities rather than space charge in the region of the filament.

Introduction

If THE emission of electrons from the hot cathode of a vacuum tube is a random process, probability fluctuations of anode current are to be expected. In 1918, Schottky[1] derived an expression for the root mean square voltage one might expect to exist between the terminals when a temperature-limited thermionic current is caused to flow through a tuned circuit. This expression involves certain constants of the circuit, the current, and the magnitude of the electronic charge, ϵ; thus a measurement of the r.m.s. voltage, the current and the other factors involved, yields a measure, by purely electrical methods, of the value of ϵ.

Fürth,[2] Ornstein and Burger,[3] and Fry[4] have arrived at the same theoretical results by a variety of methods, while experimental work has been done by Hartmann,[5] Hull and Williams,[6] and Johnson.[7] Of these latter, the results of Hull and Williams are in agreement with the

[1] W. Schottky, Ann. d. Physik **57**, 541-67 (1918); **68**, 157-76 (1922).

[2] R. Fürth, Phys. Zeits. **23**, 354 (1922).

[3] L. S. Ornstein and H. C. Burger, Ann. d. Physik **70**, 622 (1923).

[4] T. C. Fry, J. of Franklin Inst. **00**, 199 (1925).

[5] C. A. Hartmann, Ann. d. Physik **65**, 51 (1921).

[6] A. W. Hull and N. H. Williams, Phys. Rev. **25**, 147 (1925).

[7] J. B. Johnson, Phys. Rev. **26**, 71 (1925).

theory,—the others, working at much lower frequencies, have failed to realize the expected magnitudes.

Schottky developed the thermionic current ι into a Fourier series,

$$\iota = \Sigma C_k \sin(\omega_k t + \phi_k)$$

and proceeded to determine the mean energy produced. If, however, use is made of an amplifier of the tuned-circuit type, each component of the series is amplified by a different amount, and this must be taken into account in the computations. The adaptations made by Hull and Williams to render the discussion applicable to the case where the voltage across the condenser of the tuned circuit is applied to an amplifier of this type, yields for the mean square output voltage of the amplifier

$$\overline{E^2} = A_0^2 L \iota_0 \epsilon F / 2C^2 R \cdot$$

A_0 is the voltage amplification at the resonance frequency; L is the self-inductance; C, the capacity, and R, the resistance of the tuned circuit, R to include losses in condenser and vacuum tube; ι_0 is the average thermionic current as read by a D.C. ammeter; ϵ is the electronic charge, and F is a factor which is obtained from the resonance curve, $f(x)$, of the amplifier, where $x = \omega/\omega_0$, ω_0 being 2π times the resonance frequency of both the amplifier and the tuned circuit. It is the mean square voltage output divided by the output that would be experienced if all components were subject to the same voltage amplification, A_0.

If the shot-voltage be disconnected and a sine wave of frequency $\omega_0/2\pi$ and r.m.s. voltage v_1 be applied to the amplifier, the output r.m.s. voltage will be $A_0 v_1$. If this produces the same mean square effect as the shot-voltage, we have $\overline{E^2} = A_0^2 v_1^2 = A_0^2 L \iota_0 \epsilon F / 2C^2 R$ or $v_1^2 = L \iota_0 \epsilon F / 2C^2 R$ whence $\epsilon = 2C^2 R v_1^2 / L \iota_0 F$.

Difficulties Attending Regeneration

This substitution method will yield correct results only if regeneration effects are absent. This condition was realized by Hull and Williams in their first work on these measurements, but with larger values of A_0 there are good grounds for believing that regeneration is unavoidable. That is, energy from the output of some or all of the tubes is fed back into the initial circuit where the shot-effect is generated. An ingenious method of procedure enables one to take account of feed-back. If the sine voltage v_1 be introduced into the tuned circuit as shown in Fig. 1 the resonance curve of the system, $f(x)$, may be obtained by the application of voltage of different frequencies. This will give a measure of the amplification with regeneration. If v_1 be applied to the system by the potential drop across a small impedance L_1 of negligible resistance, carrying a known high-frequency current I_1, the circuit is capable of development of shot-effect with the same efficiency as before. In this case one uses either the vacuum tube $V\,T$ to produce a thermionic current and shot-effect, or, cutting off the filament current, causes I_1 to flow through L_1 to produce the voltage v_1 for calibration or comparison. The tuned circuit is never disconnected nor detuned and regeneration causes no difficulty since it produces the same effect in both $f(x)$ and subsequent measurements.

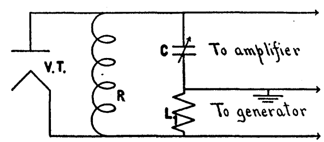

Fig. 1. Diagram showing method of introduction of calibrating voltage into shot-circuit.

Inconvenience is experienced however with this curve, $f(x)$. What one wishes to know is not how a voltage introduced into the circuit, but rather the voltage across the condenser, is amplified.

If v_{oc} is the condenser voltage at the frequency $\omega_0/2\pi$, and v_{kc} at $\omega_k/2\pi$, then with I_1 held constant

$$v_{0c}/v_{kc}=\sqrt{\omega_k^2+(L^2/R^2)(\omega_k{}^2-\omega_0{}^2)^2}/\omega_k$$

If $\phi(x)$ be the curve obtained by plotting relative power output against x, and $f(x)$ the curve that would be obtained were the condenser voltage maintained constant, we have

$$\phi(x)=E_k^2/E_0^2=A_0^2 vkc^2 f(x)/A_0^2 v_{0c}^2$$

and

$$f(x)=\phi(x)v_{0c}^2/v_{kc}^2$$

That is, each ordinate of the experimentally determined curve $\phi(x)$ must be multiplied by the ratio $v_{oc}{}^2/v_{kc}{}^2$ to obtain the curve desired for the determination of F.

This inconvenience becomes on further analysis a remarkable simplification of the work of Schottky. The following development due to the authors is here presented for the first time.

Simplification of the Tuned-Circuit Problem

The power factor of the circuit varies with the thermionic current, but it may be maintained constant by the addition of suitable resistances. Let this be done.

If now we take the expression given by Hull and Williams for the mean square output voltage due to the k-component of the thermionic current, we have

$$\overline{E_k^2}=\tfrac{1}{2}L^2\omega_0^2A_0^2C_k^2\,\frac{x^2f(x)}{(1-x^2)^2+r^2x^2}$$

$$=\tfrac{1}{2}L^2\omega_0\,A_0^2C_k^2\,\frac{x^2}{(1-x^2)^2+r^2x^2}\times\frac{v_{0c}^2}{v_{kc}^2}\phi(x)$$

where $r=R/L\omega_0$. But

$$\frac{v_{0c}^2}{v_{kc}^2}=\frac{\omega_k^2+(L^2/R^2)(\omega_k^2-\omega_0^2)^2}{\omega_k^2}$$

$$=\frac{(1-x^2)^2+r^2x^2}{r^2x^2}\,.$$

So that

$$\overline{E_k^2}=\frac{L^2A_0^2}{2R^2C^2}C_k^2\,\phi(x)$$

Substituting for C_k^2 its value $\overline{C_k^2}=(4\iota_0\epsilon\omega_0/2\pi)dx$ we obtain for the total mean square output voltage

$$\overline{E^2}=\Sigma\overline{E_k^2}=\frac{L^2A_0^2}{2R^2C^2}\times\frac{4\iota_0\epsilon\omega_0}{2\pi}\Sigma\phi(x)dx$$

$$=\frac{2\iota_0\epsilon L^2A_0^2}{R^2C^2}\int_0^\infty\psi(\nu)d\nu$$

where ν (the frequency) $=\omega/2\pi$; $\psi(\nu)=\phi(x)$; and $dx=d(\omega/\omega_0)=(1/\omega_0)$ $d(2\pi\nu)=(2\pi/\omega_0)d\nu$. $\int\psi(\nu)d\nu$ becomes simply the area under the curve $\phi(x)$ plotted against frequency. If we call this area A, the above equation yields

$$\overline{E^2}=2\iota_0\epsilon L^2A_0^2A/R^2C^2$$

If as before we introduce a voltage v_1 which causes the same output as the shot-effect, we have $i=v_1/R$ where i is the current in the tuned circuit and v_1 has the frequency $\nu_0=\omega_0/2\pi$, and the condenser voltage v_c is i/ω_0C or $v_1/RC\omega_0$. Also $\overline{E^2}=A_0^2v_c^2=A_0^2v_1^2/R^2C^2\omega_0^2=2\iota_0\epsilon L^2A_0^2A/R^2C^2$ so that $\epsilon=v_1^2/2L^2\omega_0^2\iota_0A$.

It may be noted that by this means we may avoid the complicated evaluations of the original shot-integral. Also, since v_1 is introduced by a current I_1 flowing through an inductance L_1 of negligible resistance, we have

$$\epsilon=L_1^2I_1^2\omega_0^2/2L^2\iota_0A\omega_0^2=L_1^2I_1^2/2L^2\iota_0A$$

This measurement is independent of all frequency measurements except those of frequency differences for the area A. This results in the possibility of much greater accuracy than does the method of substitution.

Analysis of the General Case

Hitherto all the work that has been published on shot-effect has considered tuned circuits only, and indeed the preceding development is of the same class. The following shows however that a general solution may be made which is applicable to all circuits suitable for the production of shot-effect. This may be stated in general as an inductive resistance in parallel with a capacitance. A series capacity obviously makes impossible the application of an accelerating potential to the anode.

The above discussions apply to circuits tuned to the resonance frequency, ν_0, of the amplifier, and of such small resistance that R^2 may be neglected in comparison to $L^2\omega_0^2$. The following discussion is more rigorous and completely general for the circuit in that it involves no assumption regarding the relative magnitude of R, L, and C, and makes no mention of a natural period of the circuit. It applies, indeed, equally well to circuits which are aperiodic. In the following, ω_0 will refer to 2π times the resonance frequency of the amplifier, and has no connection with the natural period (if one exists) of the circuit where the shot-effect is generated. It is here assumed that the power factor of the circuit is invariant, and this condition, as has been pointed out above, may be realized.

We have as above, $\iota_k = C_k \sin(\omega_k t + \phi_k)$ whence $\overline{\iota_k^2} = \frac{1}{2}C_k^2$. If the shot-current be caused to flow through an impedance Z, the mean square voltage produced across it by the k-component is given by

$$\overline{v_k^2} = \tfrac{1}{2}Z_k^2C_k^2$$

and

$$\overline{\mathrm{E}_k^2} = \tfrac{1}{2}Z_k^2C_k^2A_0^2f(x)$$

where A_0 and $f(x)$ have the significance assigned above.

Thus

$$\overline{E^2} = \tfrac{1}{2}A_0^2\Sigma Z_k^2C_k^2f(x).$$

If the impedance is of the form shown in Fig. 1, and $\phi(x)$ is the curve obtained as power output when the input current I_1 through the inductance L_1 is maintained constant, we have, where i is the current generated in the circuit, $i = L_1I_1\omega/\xi$ where ξ is the series impedance of the circuit, $\xi = \sqrt{R^2 + (L\omega - 1/\omega C)^2}$. So that $i = L_1I_1\omega/\sqrt{R^2+(L\omega-1/\omega C)^2}$ and $v_c = L_1I_1\omega/\omega C\sqrt{R^2+(L\omega-1/\omega C)^2}$ where v_c is the condenser voltage. Thus

$$\frac{v_{0c}^2}{v_{kc}^2} = \frac{L_1^2I_1^2}{C^2[R^2+(L\omega_0-1/\omega_0C)^2]} \times \frac{C^2[R^2+(L\omega_k-1/\omega_kC)^2]}{L_1^2I_1^2}$$

where $\nu_0=\omega_0/2\pi$, is the resonance frequency of the amplifier and v_{oc} the condenser voltage at ν_0.

If, as above, we set $x=\omega_k/\omega_0$,

$$\frac{v_{0c}^2}{v_{kc}^2}=\frac{1}{R^2C^2\omega_0^2+(1-LC\omega_0^2)^2}\times\frac{R^2C^2\omega_0^2x^2+(1-LC\omega_0^2x^2)^2}{x^2}$$

Now from our method of its derivation

$$\phi(x)=\frac{E_k^2}{E_0^2}=\frac{v_{kc}^2A_0^2f(x)}{v_{0c}^2A_0^2}$$

So that $f(x)=\phi(x)v_{oc}^2/v_{kc}^2$; $\overline{E^2}=\frac{1}{2}A_0^2\Sigma Z_k^2C_k^2f(x)$ where $Z_k^2=(R^2+L^2\omega_k^2)[R^2C^2\omega_k^2+(1-LC\omega_k^2)^2]$ and $\overline{C_k^2}=4\iota_0\epsilon\omega_0dx/2\pi$.
So that

$$\overline{E^2}=\tfrac{1}{2}A_0^2 4\iota_0\epsilon(\omega_0/2\pi)\int_0^\infty\frac{R^2+L^2\omega_k^2}{R^2C^2\omega_k^2+(1-LC\omega_k^2)^2}f(x)dx.$$

$$=2A_0^2\iota_0\epsilon(\omega_0/2\pi)\int_0^\infty\frac{R^2+L^2\omega_0^2x^2}{R^2C^2\omega_0^2x^2+(1-LC\omega_0^2x^2)^2}\times\frac{v_{0c}^2}{v_{kc}^2}\phi(x)dx$$

$$=\frac{2A_0^2\iota_0\epsilon}{R^2C^2\omega_0^2+(1-LC\omega_0^2)^2}\times\frac{\omega_0}{2\pi}\int_0^\infty\frac{R^2+L^2\omega_0^2x^2}{x^2}\phi(x)dx\cdots$$

$$=\frac{2A_0^2\iota_0\epsilon}{R^2C^2\omega_0^2+(1-LC\omega_0^2)^2}\times\frac{\omega_0}{2\pi}\left[\int_{1-\kappa}^{1+\kappa}\frac{R^2}{x^2}\phi(x)dx+\int_{1-\kappa}^{1+\kappa}L^2\omega_0^2\phi(x)dx\right]$$

where $1-\kappa$ to $1+\kappa$, $\kappa\ll1$, are the limits over which the integration need be extended to include all values of x for which $\phi(x)\neq0$. In

$$\int_{1-\kappa}^{1+\kappa}\phi(x)dx/x^2$$

set $z=1/x$, $dz=-dx/x^2$. Then if $\phi(x)$ is symmetric with respect to $x=1$, so that $\phi(1/(1+\delta))=\phi(1-\delta)=\phi(1+\delta)$,

$$\int_{1-\kappa}^{1+\kappa}\phi(x)dx/x^2=\int_{1-\kappa}^{1+\kappa}\phi(1/z)dz=\int_{1-\kappa}^{1+\kappa}\phi(z)dz=\int_{1-\kappa}^{1+\kappa}\phi(x)dx$$

since a definite integral is independent of the variable of integration.

$$\overline{E^2}=\frac{2A_0^2\iota_0\epsilon}{R^2C^2\omega_0^2+(1-LC\omega_0^2)^2}\times\frac{\omega_0}{2\pi}\left[\int_{1-\kappa}^{1+\kappa}R^2\phi(x)dx+\int_{1-\kappa}^{1+\kappa}L^2\omega_0^2\phi(x)dx\right]$$

$$=\frac{2A_0^2\iota_0\epsilon(R^2+L^2\omega_0^2)}{R^2C^2\omega_0^2+(1-LC\omega_0^2)^2}\left[\frac{\omega_0}{2\pi}\int_{1-\kappa}^{1+\kappa}\phi(x)dx\right]$$

$$=\frac{2A_0^2\iota_0\epsilon(R^2+L^2\omega_0^2)A}{R^2C^2\omega_0^2+(1-LC\omega_0^2)^2}$$

$$=2A_0^2Z_0^2\iota_0\epsilon A$$

where Z_0 is the impedance of the shot-circuit at the resonance frequency of the amplifier, and A has the same significance as formerly. This discussion is independent of the relative magnitudes of L, R and C.

If ξ be the series impedance of the circuit at the frequency ν_0, we have, where a voltage v_1 is introduced into the circuit by the current I_1 flowing through the inductive reactance $L_1\omega_0$, $i=v_1/\xi=L_1I_1\omega_0/\xi$ where i is the current in the circuit. The condenser voltage

$$v_c=i/\omega_0C=L_1I_1\omega_0/\omega_0C\xi=L_1I_1/\xi C$$

If the voltage v_1 introduced into the circuit causes the same mean square output voltage $\overline{E^2}$, as is occasioned by the shot-effect,

$$A_0^2v_c^2=\overline{E^2}=2Z_0^2A_0^2\iota_0\epsilon A$$

or substituting for v_c its value given above,

$$A_0^2L_1^2I_1^2/\xi^2C^2=2A_0^2Z_0^2\iota_0\epsilon A$$

Whence $\epsilon=L_1^2I_1^2/2Z_0^2\xi^2C_0^2A$. For this type of system $\xi^2=R^2+(L\omega_0-1/\omega_0C)^2$.

If this be applied to a periodic circuit of small resistance having a natural period $2\pi/\omega_0$, we have $Z_0=L/RC$, $\xi=R$, so that $\epsilon=L_1^2I_1^2/2(L^2/R^2C^2)R^2C^2\iota_0A=L_1^2I_1^2/2L^2\iota_0A$ which is precisely the result obtained for this type of circuit in the previous discussion.

Analysis for a Pure Resistance

The experimental part of this investigation deals with the case in which the shot-circuit is a non-inductive resistance R. Since a substitution method was used in these measurements, a modification of the general method of the preceding paragraph becomes necessary. No effects of energy fed back from the output into this resistance were to be observed, so a method of substitution should lead to correct experimental results. The method of computation follows closely those previously outlined.

If, as above, the thermionic current traverses the resistance R, we have for the k-component of the Fourier series into which the whole current may be developed

$$\iota_k=C_k\sin(\omega_kt+\phi_k)$$

and $v_k=R\iota_k$ where v_k is the voltage introduced across the resistance by ι_k. Thus $v_k=RC_k\sin(\omega_kt+\phi_k)$ and $\overline{v_k^2}=\frac{1}{2}R^2C_k^2$. If $\overline{E_k^2}=v_k^2A_0^2f(x)$ where $\overline{E_k^2}$, A_0 and $f(x)$ have the significance previously assigned, then the total mean square voltage output

$$\overline{E^2}=\overline{\Sigma E_k^2}=A_0\Sigma v_k^2f(x)$$
$$=\tfrac{1}{2}R^2A_0^2\Sigma C_k^2f(x)$$

If, as before, we substitute for C_k^2 its value

$$\overline{C_k^2} = 4\iota_0\epsilon\omega_0 dx/2\pi$$

we have

$$\overline{E^2} = \tfrac{1}{2}R^2A_0^2\,4\iota_0\epsilon(\omega_0/2\pi)\int_0^\infty f(x)dx$$

where A is the area under the curve which represents relative voltage-square amplification plotted against frequency.

If now the system producing the shot-effect be disconnected and a voltage v_1 at the resonance frequency, ν_0, of the amplifier be applied to the input, it becomes after amplification A_0v_1. v_1 may be adjusted so that the output has the same r.m.s. value as that occasioned by the shot-effect.

We have then $\overline{E^2} = A_0^2v_1^2 = 2R^2A_0^2\iota_0\epsilon A$ whence $\epsilon = v_1^2/2R^2\iota_0 A$.

The Resistance R

The selection of a resistance suitable for the measurement of these effects is a matter of some interest. High resistances of vanishingly small self-inductance are readily obtainable. These usually consist of a paper base with a coating of India ink, Aquadag or other similar substance, but unfortunately they exhibit the well-known property of fuzziness. That is, when carrying a direct current from a battery the resistance is not constant, but exhibits a random fluctuation.

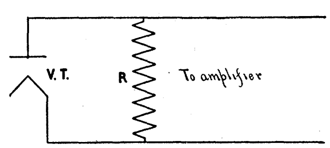

Fig. 2. Diagram of aperiodic circuit used in obtaining measurements submitted.

The resistance used in this work was a line of graphite on the etched surface of a glass tube ½ inch in diameter and 3 inches long. Investigation showed that this when properly prepared had a constant resistance, and was in every way suitable.

Capacity and Leakage Effects

One could not assume, however, that the magnitude of R was the impedance in the path of the thermionic current. If we examine the circuit, we note that we have in parallel with R the leakage across the socket and tube V.T. and across the amplifier input as well as attendant capacity effects. Thus it seemed necessary to measure the impedance of the circuit as it was actually used. The capacity effects alone are dependent on the frequency, and if these are small and the resonance curve not too broad, it will make little difference at what point in the region the impedance is measured. If it be measured at the lowest frequency of the resonance curve, it will be too great, at the highest, too small. Hence it was measured at such a frequency that an ordinate through this point passes through the center of area of the region under the resonance curve. If we consider that the impedance of the circuit is the one obtained in this way, and that it is constant throughout the pass-band of the amplifier, the error introduced is vanishingly small.

Thus in the expression for $\overline{E}^2$ we shall write Z,—where Z is the impedance obtained under the conditions mentioned above,—in place of R in our former consideration. The value of $\overline{E}^2$ then becomes

$$\overline{E^2}=2Z^2A_0^2\iota_0\epsilon A$$

or solving for ϵ as before,

$$\epsilon=v_1^2/2Z^2\iota_0A$$

Measurement of v_1

The measurement of v_1 was carried out by precisely the method used by Hull and Williams,[6] and it will be discussed here very briefly. The method utilizes the inductive drop in a short length of a long straight brass rod surrounded by a concentric cylindrical return conductor. Hull and Williams have shown that if the cylindrical return be connected to the central rod by a block of brass or other material of small resistance, carefully sweated in so that it makes contact with the cylinder at its inner edge, then the field is uniform right up to the inner surface of the disc, and hence this may be considered the zero point and all measurements made therefrom. The inductance per centimeter of such an arrangement is

$$L_{cm}=2\log_e(R/r)\ \text{e.m.u.}$$

where R is the radius of the cylindrical return and r the radius of the central rod.

In the apparatus used $R/r=6$, so that $L_{\overline{cm}}=2\ \log_e 6\times10^{-9}$ henries, and the inductance per length used, -1, which has been referred to formerly as L^1, $L_1=2\ 1\ \log_e 6\times10^{-9}$ henries. Thus if a current I_1 from a generator traverses this inductance, v_1, the potential across it will be $v_1=L_1I_1\omega$ and in particular when the generator frequency is the same as the resonance frequency of the amplifier,

$$v_1=L_1I_1\omega_0=2\,1\log_e 6\times10^{-9}I_1\omega_0$$

The design of the apparatus permits the use of 3, 6, 9, 12, 15, 18, 21, or 24 centimeters of length since it is tapped at these points.

The Amplifier

The amplifier is a five stage cascade tuned-impedance-coupled system. The impedances consist of 400 turn coils wound on hard rubber cores 1 inch in diameter and $\frac{1}{2}$ inch in length, the coils being in parallel with suitable capacities. The tubes used in the amplifier are screen grid tubes similar to those used by Hull and Williams, and supplied through the courtesy of Dr. Hull of the Research Laboratories of the General Electric Company. Each stage is contained in a metal case which shields from outside influence, and all high voltage lines are carefully filtered. The amplification used was not measured but was estimated from previous experience to be a voltage amplification of about 10^6.

Detection

The method of detection is perhaps deserving of special mention since it is not precisely that used by Hull and Williams in obtaining the results they have published, but a method later devised by them and used in some more recent work.

A coil, whose ends are connected across the heater element of a 900 ohm vacuum thermocouple, is placed in the field of the coil in the plate circuit of the last amplifier tube. The output is read on a galvanometer connected to the couple side of the thermocouple. These leads are connected inside the case of the amplifier by a large condenser. Investigation showed that the apparatus obeys the square detection law within the limit of error of the measurements; that is, the galvanometer reading is proportional to the power input.

Experimental Procedure

The method of finding the particular v_1 which corresponds to a certain ι_0 in our formula, is as follows:

The thermionic current ι_0 is caused to traverse the impendance Z. The voltage across Z is applied to the input of the amplifier and the deflection of the galvanometer in the output noted. This system is now disconnected and the voltage v_1 across the self-inductance L_1 is used as input. If v_1 be now adjusted so that the output galvanometer experiences the same deflection as before, we have, where v_1 has the frequency ν_0

$$A_0^2 v_1^2 = \overline{E^2} = 2Z^2 A_0^2 \iota_0 \epsilon A$$

or $\epsilon = v_1^2/2Z^2\iota_0 A$. We have shown before that $v_1 = 2\ \mathrm{l}\ \log_e 6\times10^{-9}\ I_1\omega_0$. The determination of l consists in the selection of a suitable tap on L_1 so that to produce the desired voltage, the current I_1 should be of a magnitude convenient for measurement.

The Generator System

The current is generated in a movable coil placed in inductive relationship to another coil carrying the generator output. Capacity coupling is avoided by the use of a static screen. The mutual inductance and hence I_1 is altered by rotating the coil. The measurement of the current is effected by the use of a vacuum thermocouple together with a calibration curve.

A large cage of wire mesh about the amplifier and a smaller one about the generator prevent interaction. The source of energy for each system is within the cage that contains it.

Experimental Work

The first work with this aperiodic circuit was carried out at a frequency of about 50 kilocycles. The effect was easily perceptible, but certain other considerations led to the use of higher frequencies. When the amplifier was actuated by the shot-voltage, the output galvanometer exhibited random fluctuations in deflections. These were believed to be other than variations in battery voltage, since this random effect disappeared when the generator system was used as input. The magnitude of these fluctuations was decreased by the introduction of a thermocouple of larger thermal capacity, but they still caused such inconvenience that, while the mean of a set of readings yielded results of the proper order of magnitude, individual readings would differ by as much as twenty percent. This phenomenon is perhaps a consequence of the flicker effect discussed by Schottky in his recent paper.[8] A change to higher resonance frequency led to the disappearance of this annoyance. The first work which is to be presented here was carried out at a frequency of 146 kilocycles.

The resistance used had a value of 48300 ohms, and a measurement of the impedance by the method outlined in a preceding paragraph yielded the value 34500 ohms.

The Area A

The area A was to be determined as the area under the curve representing relative voltage square amplification plotted against frequency. Since the apparatus obeys the square detection law, the reading of the output galvanometer at any fixed frequency is proportional to the power

[8] W. Schottky, Phys. Rev. **28**, 74-103 (1926).

input. If a voltage v_1 of constant magnitude and varying frequency be applied to the input, the galvanometer deflections, divided by the deflection for ν_0, will give the ordinates of the required curve.

In actual practice it is the input current I_1, and not v_1 which is maintained constant. If $\phi(\nu)$ be the curve obtained in this way, and $f(\nu)$ the curve required, we have

$$\phi(\nu_k) = A_0^2 v_k^2 f(\nu_k) / A_0^2 v_0^2$$

where v_k is the value of v_1 at the frequency ν_k. Thus

$$f(\nu_k) = (v_0^2/v_k^2)\phi(\nu_k) = (L_1^2 I_1^2 \omega_0^2 / L_1^2 I_1^2 \omega_k^2)\phi(\nu_k) = \phi(\nu_k)\omega_0^2/\omega_k^2.$$

That is, each ordinate of the experimentally determined curve $\phi(\nu_k)$ must be multiplied by the ratio ω_0^2/ω_k^2.

This correction produces an increase in the ordinates on one side of the resonance frequency, and a decrease on the other. The change in area is quite insignificant, but was taken into account in obtaining the value of A.

TABLE I

Summary of measurements leading to the determination of electron charge. v_1 is given in microvolts ϵ in coulombs $\times 10^{19}$

ι_0	I_1	l	v_1	Z	A	ϵ
		Frequency = 146 kilocycles				
.206	1.070	9	31.7	34500	12880	1.587
.310	1.320	9	39.1	34500	12880	1.605
.367	1.432	9	42.4	34500	12880	1.595
.404	1.500	9	44.4	34500	12880	1.590
.447	1.587	9	46.7	34500	12880	1.590
.508	1.260	12	49.7	34500	12880	1.586
.614	1.387	12	54.7	34500	12880	1.590
.715	1.497	12	59.1	34500	12880	1.591
.811	1.274	15	62.8	34500	12880	1.587
					Ave	1.5912
		Frequency = 114.25 kilocycles				
.206	1.019	9	23.6	36870	6231	1.595
.310	1.248	9	28.9	36870	6231	1.590
.404	1.425	9	33.0	36870	6231	1.590
.417	1.500	9	34.7	36870	6231	1.593
.508	1.198	12	37.0	36870	6231	1.589
					Ave	1.5914

EXPERIMENTAL DIFFICULTIES AND LIMIT OF ERROR

A consideration of the foregoing results leads to a matter of some interest, viz. how far experimental difficulties may be overcome and the resulting limit of error that may be placed on results obtained by this method.

Variation in battery voltage is a source of some annoyance, since small variations, with an amplification of the order of magnitude used, produce very appreciable changes in results. If sufficient care be taken to avoid this difficulty, individual readings vary between narrow limits. It seems that if calibrations could be made and maintained with such accuracy that the systematic error were known to be negligible, then considerable precision might be anticipated.

The region of uncertainty in the preceding measurements is about 1 percent.

It will be noted that the mean values obtained agree, within the limit of experimental error, with the value given by Millikan,—1.5913×10^{-19} coulombs.

Space Charge

A study of the effect produced by the probability fluctuations of thermionic emission in a vacuum tube may be carried out under conditions other than those of temperature limitation. The foregoing work is of interest in that it leads to a determination of the electronic charge ϵ; investigation under other conditions may lead to some knowledge of the effect produced by amplifier tubes in actual use.

The tube used in the preceding work was a Radiotron U.X. 120. The grid and plate together formed the anode and were maintained at a voltage which exceeded that of the cathode by some hundred to a hundred and twenty volts. This straight filament tube was employed to eliminate suspicion of residual space charge. Fear was experienced that, with a filament of the Λ type such as is used in the Radiotron U.V. 201A, even with as high anode potentials as one might apply, some space charge might still exist in the region where the two branches of the filament lie near together. The following investigation will show that space charge may be eliminated even with a filament of the M type, such as is used in the U.X. 112 tube.

In customary use the potential of the grid is maintained at a much lower value than in the preceding work and the magnitude of the probability fluctuations varies proportionately.

For the following curves, the anode potential and the space current were maintained constant, the filament current and grid potential being the variables.

These curves which, within the limits of experimental error are straight lines, show that if account be taken of the change in tube impedance, very little variation occurs in the distribution of electrons striking the anode when the space charge, but not the plate current, is increased. It would seem that the diminution of shot-voltage experienced by others has been due in a great measure to higher current densities in the tube rather than to an accumulation of electrons in the region of the filament.

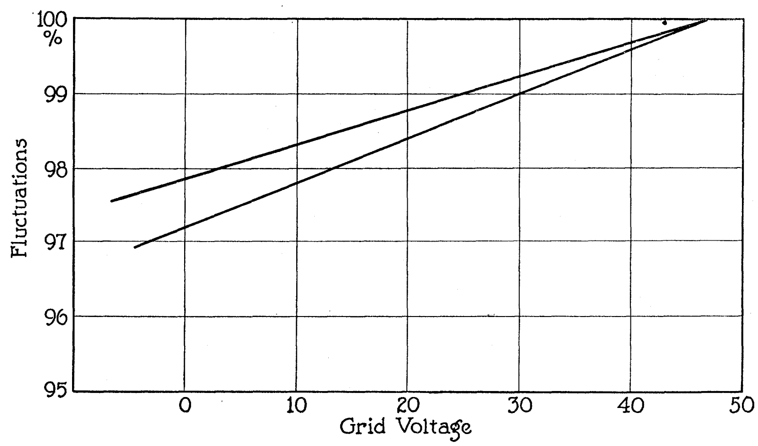

Fig. 3. Curves showing decrease of probability fluctuations with an increase of space charge. Ordinates represent percent fluctuations compared with those produced under conditions of temperature limitation,—the abscissae show corresponding grid voltages in volts. Due account is taken of the change in tube impedance. Upper curve taken for plate current of 1 m.a., lower, 2 m.a.

Discussion of Results

The theoretical work of this paper has a significance of fundamental importance in connection with the accuracy one may attain in these measurements. The analysis for the tuned circuit makes the measurements independent of a determination of the resonance frequency, and the results of this are very far-reaching. A wave-meter measuring ranges of frequencies is always subject to possible changes in the position of the mechanism, and resulting changes in calibration. Any variation which is likely to occur will affect the actual frequency reading far more than a reading of frequency differences. The method permits the use of a totally uncalibrated crystal oscillator. The crystal will maintain a definite frequency, and even if this be unknown, frequency differences may be read by a method of beat notes and sound standards. These latter may be known and maintained with great accuracy.

The analysis eliminates all necessity of a knowledge of the resistance of the circuit. Even an exact measure, in ohms, of resistance differences is unnecessary. If an impulse be applied to the circuit, and the condenser voltage to a detector tube, and if further conditions be adjusted so that the cessation of the thermionic current, and the simultaneous introduction of a resistance produces no change in the detector action, then the necessary condition is fulfilled.

The labor of computation is greatly reduced, since that required to obtain the area A is negligible compared with the manipulation entailed in the evaluation of F.

The use of an aperiodic circuit possesses some advantages. It is remarkably less subject to regeneration effects than is a tuned circuit. The accuracy of results with a periodic system depends in a large measure on a precise tuning of the circuit. This is entirely eliminated, increasing both ease of manipulation and freedom from error. Especially at low frequencies, where capacity effects become negligible, may a higher plate impedance be attained with a resistance than is possible for a tuned circuit.

Regarding the values given for electronic charge, the first set, while taken with considerable care, does not represent so great precision as the second. The authors hope to be able to publish in the near future a series of results whose region of uncertainty will be even more narrowly limited.

University of Michigan,
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