FLOW OF IONS THROUGH A SMALL ORIFICE IN A CHARGED PLATE

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ABSTRACT

Measurements of ion and electron current densities and temperatures in a low pressure arc may be made with a collector which is screened by a close parallel electrode pierced with uniform circular holes. *The electrostatic potential distribution in the neighborhood of a pierced electrode*, pierced with either slit or hole, has been worked out. *The volt-ampere characteristic for random electrons* is exponential for the higher retarding voltages and the exponent gives a temperature slightly too high. When the anode drop is known the current density of electrons can be calculated. *The volt-ampere characteristic for positive ions* is linear for small accelerating voltages. For small retarding voltages the characteristic is not a pure exponential. The characteristic gives anode drop, ion current density, "transverse" ion temperature and "longitudinal" ion temperature. The limitations and possible errors are discussed.

IN ORDER to separate the ionic and electronic parts of the electric current to an electrode immersed in an electric discharge in a gas, A. F. Dittmer suggested allowing these carriers to pass through small holes in a plane electrode and collecting them on a second electrode parallel to and behind the first, keeping the gas pressure low enough so that no collisions occurred between the sheath edge, A, and the second electrode, C, cf. Fig. 4. By adjusting the potentials on these electrodes it can be so arranged that either ions alone or electrons alone reach electrode C. The necessary variation in electrode potentials has a negligible effect on the discharge because of the presence of the sheath before the pierced electrode B. The thickness of this sheath must be greater than the hole diameter to prevent the discharge from penetrating through the hole. Other conditions, also, must be fulfilled, as will appear when the different cases are analyzed.

The carriers, which it is desired to keep from reaching C, can be prevented from doing so either by the field between B and C, or by the field between sheath edge and B. Thus there are two fundamental potential arrangements; in the first the carriers to be collected on C are initially retarded then accelerated, in the second they are initially accelerated then retarded.

The first arrangement is unsuited to measurements on positive ions (hereafter called simply ions) because it requires that B be made positive with respect to the anode. This situation cannot usually be realized for it is the most positive electrode of any considerable size which acts as anode in the discharge. The second arrangement, on the other hand, is unsuited to measurements on electrons. They should be collected in an accelerating field because of electron reflection. On account of this specific application of each arrangement, we shall hereafter speak of positive and negative voltages, rather than accelerating and retarding voltages.

The present paper treats first, the measurement of the current density in a high velocity beam of electrons; second, the volt-ampere characteristic of electrons having a Maxwellian distribution of velocities in the discharge; and third, the volt-ampere characteristic of ions having Maxwellian and other velocity distributions in the discharge. Preliminary to this, the equations for the electric potential field in the neighborhood of both a slit and a circular hole in an infinite charged plane are derived, and the transverse velocity and displacement acquired by a fast electron (ion) in traversing such a field are calculated.

I. The Electrostatic Field about a Slit

The potential distribution about a slit in an infinite plane charged electrode[1] is found by applying a Schwartz transformation[2] to the polygon $ABC \cdots H$ of Fig. 1. Proceeding to the limit by allowing

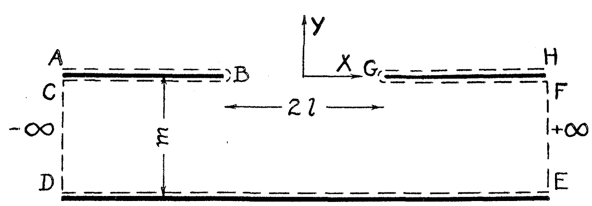

Fig. 1.

m to approach ∞ as $\partial V/\partial y$ at $y = +\infty$ remains finite and equal to $-E$, the potential V near a slit is found to be given by the parametric equations:

$$x = -\frac{U}{E} - \frac{El^2}{4} \frac{U}{U^2+V^2} \tag{1}$$

$$y = -\frac{V}{E} + \frac{El^2}{4} \frac{V}{U^2+V^2}, \quad V \geqq 0 \tag{2}$$

[1] The hydrodynamic case corresponding to this problem and also to the problem of the hole in a plate has been treated by Lamb, Hydrodynamics, 3rd edition, p. 77 and p. 137 respectively, but it has been thought worth while to sketch derivations of the potential using more usual coordinate systems.

[2] Jeans, Electricity and Magnetism, 2nd edition, p. 271.

and the boundary conditions of the field reduce to those shown in Fig. 2. To realize the general case where $\partial V/\partial y$ is different from zero on both sides of the electrode, it is only necessary to superpose a uniform electric field parallel to the y-axis modifying U and V in Eqs. (1) and (2) in such a way as to retain their conjugate properties.

U, the parameter, and function conjugate to V, has a very simple physical meaning. Keeping U constant and taking V as parameter in Eqs. (1) and (2) (or these equations modified as contemplated above) gives the lines of electric force just as V = constant gives the equipotentials, and the value of U associated with a line of force is proportional to the total charge (per unit length) between the foot of that line of force and the foot of the line of force for which $U=0$. Along any line x = constant, in Fig. 2, the change in U between any

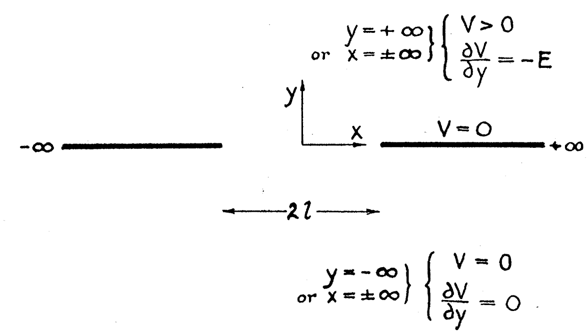

Fig. 2.

two fixed points, as given by Eqs. (1) and (2), is the same as the change in U, given by these equations modified to include the superposed uniform field. Therefore, Eqs. (1) and (2), rather than the more complicated modifications, can be used for the general case whenever, as in the next section, changes in U along a line parallel to the y-axis are involved.

The equipotentials given by Eqs. (1) and (2) are sufficiently plane when y is somewhat greater than zero for these equations to be good approximations even when the parallel electrode, C, is near. For instance, Eq. (2) shows that the equipotential $V=-6lE$ lies at $y=6l$ at some distance from the slit (x large, U large,) and lies at $y=6l(1-1/144)$ at $x=0$. In terms of potential, the potential of a conducting plane at $y=6l$ is too low at $x=0$ by $El/24$. When y is somewhat less than zero, the electric field is so weak and the potential is so nearly zero that the plane equipotential, A, may also be used there without altering the field for our purpose.

II. Transverse Velocity Acquired by a Charged Particle Passing Through the Slit

A particle of charge ϵ and mass m moving with high velocity parallel to the y-axis is subject to a transverse acceleration due to the non-uniformity of the field.

This acceleration is

$$\frac{d^2x}{dt^2}=-\frac{\epsilon}{m}\left(\frac{\partial V}{\partial x}\right)_{x=x'}$$

where V is the potential in the field and x' is the distance of the particle path from the center line of the slit. Now the discussion above shows that such non-uniformity is quite local, so that the particle acquires the same transverse velocity in traveling from $y=-\infty$ to some point near electrode B as in going from electrode A, the sheath edge of Fig. 4, to the same point. Hence the transverse velocity acquired will be

$$\dot{x}=-\frac{\epsilon}{m}\int_{-\infty}^{t_1}\frac{\partial V}{\partial x}dt=-\frac{\epsilon}{m}\int_{-\infty}^{y}\frac{1}{\dot{y}}\frac{\partial V}{\partial x}dy \tag{3}$$

If $\dot{y}$ is large it changes but slightly in the neighborhood of the slit, so $\dot{y}$ will be supposed constant and

$$\dot{x}=-\frac{\epsilon}{m\dot{y}}\int_{-\infty}^{y}\frac{\partial V}{\partial x}dy=\frac{\epsilon}{m\dot{y}}\int_{-\infty}^{y}\frac{\partial U}{\partial y}dy=\frac{\epsilon}{m\dot{y}}\left|\,U\right|_{-\infty}^{y} \tag{4}$$

where U is the function conjugate to V.

Eqs. (1) and (2) are applicable, as has been remarked above. From them

$$U=-Ex',\text{ when } y=+\infty,\ x=x'$$
$$U=0,\text{ when } y=-\infty$$

Accordingly the total transverse velocity acquired in passing from electrode A to electrode C ($y=-\infty$ to $y=+\infty$) is

$$\dot{x}_f=-\epsilon Ex'/m\dot{y} \tag{5A}$$

Now E is the electric field in the simple case corresponding to Eqs. (1) and (2). It is equal to the difference of the actual fields on the two sides of B so that

$$\dot{x}_f=\frac{\epsilon(E_1+E_2)x'}{m\dot{y}} \tag{5B}$$

where E_1 and E_2 are the two fields, each being counted positive when directed towards B. Expressing velocities as potentials and using the mean field strength $\bar{E}=\frac{1}{2}(E_1+E_2)$

$$V_x V_y = \bar{E}^2 x'^2 \tag{6}$$

One conclusion from the preceding is that the particle has acquired one-half its transverse velocity when it reaches the plane of B. This is obviously true for a symmetrical arrangement of voltages; any unsymmetrical arrangement can be realized by superposition of a uniform field, hence it holds also for any other arrangement. In general, however, this conclusion is not valid if the slit has finite thickness.

On the other hand, it is true that $\dot{x}_f$ is given by Eq. (5B) no matter what the thickness of electrode B and what the shape of the sides of the slit so long as the slit's cross-section is constant. This follows immediately from the fact that U at A and C does not depend on the shape of the slit. Of course there is the difficulty of assigning an origin of coordinates when the slit is unsymmetrical.

A further integration of Eq. (4) would give the lateral displacement of the particle. Inasmuch as the calculation for the case of the hole is simpler and the displacement in the present case is of no interest here, that result is not calculated.

III. The Electrostatic Field about a Hole[1]

The problem is to find the potential distribution due to an infinite conducting plane with a circular hole of radius a charged to potential

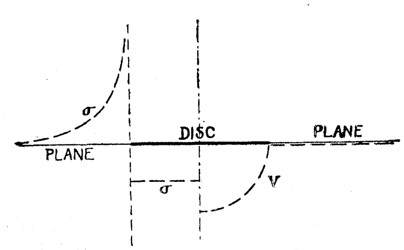

Fig. 3.

$+V$ and placed symmetrically between two parallel infinite grounded plates at a great distance away.

The limiting value of surface charge density on the central plate far from the hole is taken to be σ_0; then the uniform density on each grounded plate is $-\frac{1}{2}\sigma_0$. A uniform density $-\sigma_0$ is then superimposed over the central plane and hole in addition to the charge already existing, and to compensate for this $-\frac{1}{2}\sigma_0$ is removed from each distant grounded plate. The central conducting plane is thereby brought to zero potential, and the problem is changed to: An infinite grounded conducting plane has a circular hole in which is placed a disk charged with a uniform density of negative electricity. Find the potential distribution.

The general nature of the distributions of potential V and charge σ are indicated in Fig. 3. It can be proved by direct substitution that a solution of Laplace's equation in cylindrical coordinates is

$$V(r,z)=\int_0^\infty e^{-uz}udu\int_0^\infty f(s)J_0(ur)J_0(us)sds$$

where $f(s)$ is arbitrary. V vanishes for $z=+\infty$, and for $z=0$ is

$$V(r,0)=\int_0^\infty udu\int_0^\infty f(s)J_0(ur)J_0(us)sds=f(r)$$

The last equation is a consequence of the *Fourier-Bessel integral formula* analogous to the Fourier integral in the case of circular functions. Thus $f(r)$ is simply the potential distribution in the plane $z=0$. Taking the hole to be of unit radius, for convenience, $f(s)=0$ for $s>1$, and

$$V(r,z)=\int_0^\infty e^{-uz}udu\int_0^1 f(s)J_0(ur)J_0(us)sds \tag{7}$$

Applying Gauss' theorem at the disk,

$$\left(\frac{\partial V}{\partial z}\right)_{z=0}=2\pi\sigma \tag{8}$$

to Eq. (7) gives an integral equation to determine $f(s)$. The shape of the curve for V in Fig. 3 suggested the solution

$$f(s)=A\sqrt{1-s^2} \tag{9}$$

This is tried in the equation and found to satisfy it,[3] giving for A the value $-4\sigma_0$. Hence the potential distribution in the plane $z=0$ is given by

$$f(r)=V(r,0)=-4\sigma_0\sqrt{1-r^2},\ \ 0\leqq r\leqq 1$$

[3] Invaluable reduction formulae are found in Watson, "A Treatise on the Theory of Bessle Functions," pp. 373, 54, and 405, Nos. 2 and 7.

The potential distribution for the modified problem is found by putting this in Eq. (7) and reducing slightly

$$V(r,z) = -4\sigma_0 \int_0^\infty \frac{e^{-uz}}{u}\left(\frac{\sin u}{u} - \cos u\right) J_0(ur)\,du, \quad z \geqq 0$$

The potential distribution for the original problem involving the hole of radius a is then easily seen to be

$$V(r,z) = V_0 - \sigma_0\left[2\pi z + 2a \int_0^\infty \frac{e^{-uz/a}}{u}\left(\frac{\sin u}{u} - \cos u\right) J_0(ur/a)\,du\right], z \geqq 0 \quad (10)$$

where V_0 is the potential of the plate with the hole.

The equipotential surfaces given by this equation are more nearly plane than those for the slit. On the axis of the hole Eq. (10) reduces to

$$V(0,z) = V_0 - \sigma_0\left[2\pi z + 4a\left(1 - \frac{z}{a}\cot^{-1}\frac{z}{a}\right)\right]$$

so that the equipotential

$$V = V_) - 2\pi\sigma_0 z_1 ,$$

which lies at $z = z_1$ at a distance from the hole, lies at

$$z = z_1\left[1 - \frac{2}{\pi}\frac{a}{z_1}\left(1 - \frac{z_1}{a}\cot^{-1}\frac{z_1}{a}\right)\right]$$

$$= z_1\left(1 - \frac{2}{3\pi}\frac{a^3}{z_1^3}\right)$$

at the axis. Thus the surface at $z_1 = 3a$ only decreases its distance from the plane of B by 0.8 percent.

IV. Transverse Velocity Acquired by a Charged Particle Passing through the Hole

The expression for the transverse velocity can be written immediately from inspection of Eq. (3)

$$\dot{r} = -\frac{\epsilon}{m\dot{z}}\int_{-\infty}^{\infty}\frac{\partial V}{\partial r}dz = -\frac{2\epsilon}{m\dot{z}}\int_0^\infty \frac{\partial V}{\partial r}dz$$

Evaluating this[4] and introducing $\overline{E}$ gives the same results as in section II, namely

$$\dot{r}_f = -\epsilon E r'/m\dot{z} \text{ or } V_r V_z = \overline{E}^2 r'^2 \quad (11)$$

[4] Watson, p. 405, No. 3, used twice, once integrated with respect to b.

The similarity between this case and the slit case extends further. Here as there the particle acquires one-half its transverse velocity in reaching the plane of B in the ideal case of zero thickness of B. Here also the total transverse velocity acquired is independent of the electrode thickness and the hole shape as long as the hole has circular symmetry. This follows from the existence of a "current function" which has been defined by Stokes for three dimensional problems concerning configurations having axial symmetry. This function is analogous to the function U conjugate to the potential V in two dimensional problems.

V. Measurement of the Current Density in a Beam of High Velocity Electrons

Under certain conditions the electrons which leave the incandescent tungsten cathode in a low pressure arc travel with nearly equal velocities in straight lines radially from the filament. One way of collecting these without at the same time collecting the random low velocity electrons and without introducing reflection errors is by using the present double electrode scheme. The potential requirements here are that a low negative voltage V_B be put on B, and a low positive voltage V_C be put on C. These potentials are relative to the sheath edge which serves as electrode A, V_B must be sufficient to repel all low velocity electrons, V_C must be sufficient to repel all positive ions, and the electrons must be moving parallel to the hole axis.

All the fast electrons which are headed for the hole pass through it and reach C. In addition some electrons which are directed toward B just outside the hole are displaced inward so that they penetrate the hole. The decrease in radial distance which enables an electron to graze the edge of the hole can be calculated readily on the assumption that B is infinitely thin and with the approximation that the transverse field is the same along the straight line path $r=a$ as along the actual path. Thus the displacement at the hole is given by

$$\Delta r' = -\frac{\epsilon}{m\dot{z}^2}\int_{\infty}^{0}\int_{\infty}^{z}\left(\frac{\partial V}{\partial r}\right)_{r=a} dz dz$$

evaluating this[5] it is found that

$$\frac{\Delta r'}{a} = \frac{\sqrt{2}}{3\pi}\,\frac{a\epsilon\overline{E}}{m\dot{z}} = \frac{\sqrt{2}}{6\pi}\,\frac{a\overline{E}}{V_D - V_B} = .075\frac{a\overline{E}}{V_D - V_B} \tag{12}$$

[5] Watson, pp. 54 and 403, No. 2.

where V_D is the potential, intrinsically negative for electrons, necessary to reduce the drift velocity $\dot{z}$ to zero. The difference $V_D - V_B$ is used instead of V_D because it is the value of $\dot{z}$ at the hole which is significant. The effective area of the hole is then fractionally greater than the true area by

$$\Delta A/A = -2\,\Delta r'/a = 0.15\,a\bar{E}/(V_B - V_D) \tag{13}$$

It is to be noted that whereas the field, E_2, between B and C is given by $(V_C - V_B)/\delta$, where δ is the electrode separation, the field, E_1 on the sheath side of B is more nearly $-4V_B/3\sigma$, σ being the sheath thickness, on account of the space charge there. Accordingly

$$\bar{E} = \tfrac{1}{2}\left(-\frac{4}{3}\,\frac{V_B}{\sigma} + \frac{V_C - V_B}{\delta}\right) \tag{14}$$

As an example, consider the following case: hole radius, $a = 0.0125$ cm; sheath thickness $\sigma = 0.0375$ cm; electrode separation $\delta = 0.075$ cm; $V_B = -10$ v.; $V_C = +10$ v. Then $E_1 = 356$, $E_2 = 267$ and $a\bar{E} = 3.9$ v.

Calculating $\Delta A/A$ in percent for various values of V_D the following is found:

V_D :	-100	-50	-30	-20
$\Delta A/A$:	0.0065	0.015	0.029	0.059

The true electron current density in the beam is found by dividing the current to C by $A(1 + \Delta A/A)$.

If the hole has appreciable thickness the side of B with the stronger longitudinal field has the stronger transverse field, and, therefore, contributes the major part of the transverse velocity. In the present example, the sheath-side field is the stronger on account of the space charge there. Accordingly $\Delta A/A$ as calculated is too small.

Electrode B collects all the ions which strike it but reflects some of the electrons. The ion component can be determined by the method of section VIII and in other ways. Thus it becomes possible by comparing the electron current densities to B and C to determine the reflection coefficient of B for electrons of different voltages in the presence of gas and ion bombardment.

VI. Measurements on the Random Electrons in an Arc

The random electrons, having, presumably, a Maxwellian distribution of velocities, can be investigated in the absence of drift electrons with essentially the same potential arrangement as in the previous case. When the electric fields on the two sides of B are equal, the surface of minimum potential (m.p.s.), which caps the hole, becomes plane and the potential at any point on it is given by

$$V = V_B + (2\bar{E}/\pi)\sqrt{a^2 - r^2} \tag{15}$$

A general theorem pertaining to Maxwellian distributions, which is discussed in a forthcoming paper by Langmuir and Mott-Smith, Jr., applies here in all cases where $V < 0$ over the minimum potential surface. In this case, the entire hole constitutes a collector which repels electrons, and there will be no "interior orbits" for the electrons between sheath edge and B. Accordingly the electrons in the plane m.p.s. have a Maxwellian distribution of velocities and a density which varies according to Boltzmann's equation. The current, therefore, which flows through the hole is

$$i_C = I_- \int_0^a e^{-\epsilon V/kT} 2\pi r dr$$

Using (15) to integrate and then (14) together with the substitutions

$$\eta_1 = -\epsilon V_B/kT, \; \eta_2 = -\epsilon(V_C - V_B)/kT$$

$$\nu = \frac{a}{\pi}\left(-\frac{4\eta_1}{3\sigma} + \frac{\eta_2}{\delta}\right), \; \Phi(\nu) = \frac{2}{\nu^2}[1 + e^\nu(\nu - 1)]$$

we obtain

$$i_C = \pi a^2 I_- e^{\eta_1}\Phi(\nu) \tag{16}$$

Since electrons are involved ϵ is negative. Besides, it has been assumed that V_B is negative and V_C positive. Hence η_2 and ν are always positive, and η_1 negative.

From the form of Eq. (16) it is seen that $\Phi(\nu)$ is the ratio of the currents per unit area to the hole and to electrode B (assuming no reflection), and since $\ln\Phi$ is an almost linear function of ν whose derivative changes only from $2/3$ at $\nu = 0$ to 1 at $\nu = \infty$ it follows that the slope of the $\ln i_c$ against V_B (η_2 constant) characteristic will differ from its usual value in other cases, namely, ϵ/kT, by a nearly constant amount. It will correspond to a higher temperature.

When V_C is made rather large so that ν is large, the equation of the characteristic is

$$i_C = \frac{2\pi^2 a I_-}{-4\eta_1/3\sigma + \eta_2/\delta} e^{a\eta_2/\pi\delta} \, e^{\eta_1(1 - 4a/3\pi\delta)} \tag{17}$$ [6]

[6] See the discussion following Eq. (23) for a necessary modification to this equation.

which shows that for η_2 constant, ln i_c is nearly a linear function of η_1 whose slope is

$$\frac{d\ln i_C}{d\eta_1}=1-\frac{4a}{3\pi}\frac{d}{d\eta_1}(\eta_1/\sigma), \text{ nearly.}$$

Since σ is proportional to $\eta_1^{3/4}$ this becomes

$$d\ln i_C/d\eta_1=1-a/3\pi\sigma, \text{ nearly.} \tag{18}$$

Consequently the temperature determined from this slope is slightly greater than the true electron temperature in the ratio $1 : 1-a/3\pi\sigma$. Practically, η_1 and η_2 cannot be made very small because of incalculable changes in the sheath field, but it is instructive to consider the case. Eq. (16) becomes

$$i_C=\pi a^2 I_- e^{\eta_1}\left[1+\frac{2a}{3\pi}\left(-\frac{4\eta_1}{3\sigma}+\frac{\eta_2}{\delta}\right)\right] \tag{19}$$[6]

and

$$d\ln i_C/d\eta_1=1-2a/9\pi\sigma, \text{ nearly,} \tag{20}$$[6]

giving again a temperature slightly greater than the true one, now in the ratio of $1 : 1-2a/9\pi\sigma$. Accordingly the slope of the characteristic consists of a constant corresponding to the true temperature less a correction term which is small (not over 3.5 percent since σ must not be less than $3a$) and inversely proportional to the three-fourths power of V_B.

If now the sheath edge potential and the positive ion current density are known (from measurements on the positive ion current which will be discussed below in section VIII, for instance) so that V_B and V_C are known, and σ can be calculated, everything but I_- in Eq. (17) is known and the electron current density can be found.

Three factors affecting this method and introducing certain corrections must be treated in greater detail. They are, first, the effect of the random ion velocities on the electric field and sheath thickness; second the condition that the fields on the two sides of B should be equal, and third, the condition that the potential everywhere over the hole shall be less than zero.

The presence of initial velocities among the ions changes the space charge equation[7] to

$$I_+=5.44\times 10^{-8}(-V_B)^{3/2}(1+\mu)/\sqrt{M}\sigma^2 \tag{21}$$

[7] Langmuir, Phys. Rev. **21**, 419 (1923), Eq. (17).

where M is the molecular weight of the ion, and

$$\mu = 0.0247\sqrt{-T_+/V_B} \tag{22}$$

T_+ being the positive ion "temperature." From this it is found that

$$E_1 = -\frac{4}{3}\frac{V_B}{\sigma}\frac{1+\mu}{1+\frac{2}{3}\mu} \tag{23}$$

Thus Eqs. (17), (18), (19), and (20) should be modified by replacing $1/\sigma$ by

$$\frac{1}{\sigma}\frac{1+\mu}{1+\frac{2}{3}\mu}$$

For σ, Eq. (21) gives

$$\sigma = 7.37\times 10^{-4}(-V_B)^{3/4}\sqrt{1+\mu}/M^{1/4}I_+^{1/2} \tag{24}$$

The condition that $E_1 = E_2$ is seen to be

$$-\frac{4}{3}\frac{V_B}{\sigma}\frac{1+\mu}{1+\frac{2}{3}\mu} = \frac{V_C - V_B}{\delta} \tag{25}$$

and using (24) it becomes

$$\frac{V_C - V_B}{\delta} = \frac{4}{3}\frac{M^{1/4}I_+^{1/2}(-V_B)^{1/4}}{7.37\times 10^{-4}}\frac{\sqrt{1+\mu}}{1+\frac{2}{3}\mu} \tag{26}$$

The theory has been worked out on the basis of keeping $V_C - V_B$ constant, so that if a run includes an N-fold variation of V_B and (26) holds for some mean value of V_B then (26) will be out each way by the factor $N^{1/8}$ at the extremes. This results in a bulging of the m.p.s. so that its center lies at a distance z_0 from the plane of B given by the equations

$$\gamma = \frac{2}{\pi}\left(\frac{z_0/a}{1+z_0^2/a^2} + \tan^{-1}(z_0/a)\right), \quad E_2/E_1 = (1+\gamma)/(1-\gamma) \tag{27}$$

or approximately by

$$E_2/E_1 = \frac{1+1.274z_0/a - 1.70z_0^3/a^3}{1-1.274z_0/a} \tag{28}$$

which makes E_2/E_1 5 percent too large for $z_0 = .4a$ and 8 percent too large for $z_0 = -.4a$. The potential at the center of the m.p.s. is

$$V = V_B + 2a\overline{E}/\pi(1+z_0^2/a^2) \tag{29}$$

A permissible error of 1 percent in V allows an error of at least 1 percent in the second term of the right member. This in turn allows z_0/a to be as great as 0.1 and E_2/E_1 to be 1.29 by (28). But this is $N^{1/8}$, so that V_B may have a $(1.29)^8 = 7.7$-fold variation when properly chosen without exceeding the 1 percent limit set above.

Finally, the condition that the center of the m.p.s. shall be negative is given by Eq. (15).

$$\frac{\eta_2}{-\eta_1} = \frac{V_C - V_B}{-V_B} \leqq \frac{1 - 4a/3\pi\sigma}{a/\pi\delta} = \frac{\pi\delta}{a}, \text{ nearly},$$

the effect of initial ion velocities being neglected. Inspection shows that when condition (25) is satisfied this condition is satisfied also since $\sigma/a \geqq 3$.

VII. Measurements on Positive Ions : Calculation of the Volt-Ampere Characteristic of Ions Collected on C

It was noted in the preliminary remarks that to make measurements on the ions V_B must be sufficiently negative to keep any electrons

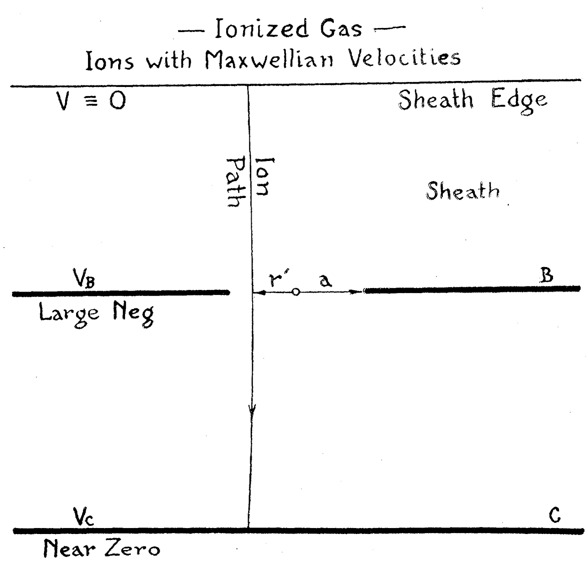

Fig. 4.

from reaching C. The situation as regards the ions is illustrated in Fig. 4 and may be described as follows: An ion of a certain u, v, w velocity class (u perpendicular to the electrodes, v and w parallel to them) leaves the ionized gas, is accelerated by B, is given an added transverse velocity by the hole, and is then retarded by C so that it may not reach C. The v velocity axis is chosen in the sense and direction of the radius vector, r, from the center of the hole to the point at which the ion path cuts the plane of the hole. The transverse velocity $\dot{r}$ acquired by the ion increases its "transverse energy" by $\frac{1}{2}(\dot{r}^2+2\dot{r}v)$ and decreases its "longitudinal energy" by the same amount, Hence only when

$$\tfrac{1}{2}u^2-\tfrac{1}{2}\dot{r}^2-\dot{r}v\geqq\epsilon V_C/m \qquad (30)$$

is the ion able to reach C, V_C being the retarding voltage on C. From Eq. (11)

$$\dot{r}_f=nr', \quad n=-\epsilon E/m\dot{z}, \text{ a constant}$$

so that the relation (30) can be written

$$\tfrac{1}{2}n^2r'^2+nr'v-(\tfrac{1}{2}u^2-\epsilon V_C/m)\leqq 0 \qquad (31)$$

The two roots r_1, r_2 ($r_1<r_2$) of the equality partially fix the respective radial limits ρ_1 and ρ_2 between which an ion of a certain velocity class in a certain retarding field (V_C) must pass in order to be collected. But besides the mathematical relations

$$\rho_1=r_1, \quad \rho_2=r_2 \qquad (32\text{A})$$

there are the additional dominant physical limitations on ρ_1 and ρ_2

$$0\leqq\rho_1\leqq\rho_2\leqq a \qquad (32\text{B})$$

Now the effective area of the hole for (u, v, w) ions is

$$\pi(\rho_2^2-\rho_1^2)$$

Accordingly the current of (u, v, w) ions is

$$i_{u,v,w}=2h'h''m^2I_+(\rho_2^2-\rho_1^2)u\exp[-h'mu^2-h''m(v^2+w^2)]du\,dv\,dw \qquad (33)$$

the u velocity distribution being characterized by a different temperature from the v and w velocity distribution because in the body of the discharge there is a small electric field perpendicular to the electrodes which makes the longitudinal components of velocity greater. The relation between the h's and temperature are given below Eq. (36B).

Denoting the sum of $(\rho_2^2-\rho_1^2)$ for a positive and an equal negative value of v by A, and using the variable

$$\bar{u}^2=u^2-2\epsilon V_C/m$$

($\bar{u}$ being the final u velocity before impact on C of an ion passing through the middle of the hole), (33) becomes

$$i_{u,v}=2\sqrt{\pi h'' m}\,h' m I_{+} e^{-2h'\epsilon V_C/m} A\bar{u}\exp(-h'm\bar{u}^2-h''mv^2)d\bar{u}dv \tag{34}$$

after integration for all values of w. The limitations on ρ_1 and ρ_2 imposed by (31), 32A), (32B) necessitate breaking up the $\bar{u}$, v plane into several sections for the integration of (34) as shown in Fig. 5.

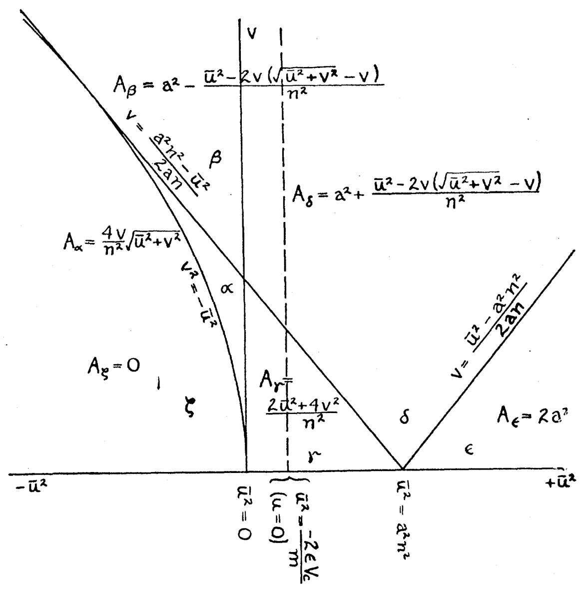

Fig. 5.

The values of the A's there given are the values appropriate to a hole all points of which are at the same potential. This is not true of the actual situation, the potential in the hole being variable and given by Eq. (15), Within the mathematical limitations of the present theory, it is sufficient, except in Eq. (39B), to write

$$A_\alpha=4\Theta\frac{v}{n^2}\sqrt{\bar{u}^2+v^2},\qquad A_\gamma=2\Theta\frac{\bar{u}^2+2v^2}{n^2}$$

where
$$\Theta=1+2a\bar{E}/\pi V_B=1-\frac{2}{\pi}\sqrt{V_t/-V_B}\,. \tag{35}$$

The integration limits are 0 and $+\infty$ for v and $-2\epsilon V_C/m$ and $+\infty$ for $\bar{u}^2$. For accelerating potentials, $V_C<0$, that is, the lower $\bar{u}^2$ limit lies to the right of the v-axis as indicated by the dotted line, while for retarding potentials, $V_C > 0$, that is, the lower $\bar{u}^2$ limit lies to the left of the v-axis. Practically, as long as $2\epsilon V_C/m$ is considerably less than a^2n^2 and $h'm\ a^2n^2 >> 1$, the integration may be simplified by taking $A_\beta = A_\alpha$ and $A_\delta = A_\epsilon = A_\gamma$.

The integration then gives

$$i = -\Theta\frac{i_C}{V_t}(V_C + V' + V'') \qquad \text{for } -V_t << V_C \leqq 0 \quad (36A)^8$$

$$i = -\Theta\frac{i_C}{V_t(V' - V'')}\left(V'^2\exp\frac{V_C}{V'} - V''^2\exp\frac{V_C}{V''}\right) \text{ for } V_t >> V_C \geqq 0 \quad (36B)^8$$

Here $i_c = \pi a^2 I_+$, is the constant ion current when all ions are collected; $V_t = m(na)^2/2\epsilon$ is the voltage equivalent to the transverse velocity acquired by an ion grazing the edge of the hole; $V' = -1/2h'\epsilon = -kT'/\epsilon$ is the reciprocal of the slope of the semi-log plot (base e) corresponding to the temperature T' of the longitudinal velocity distribution; and $V'' = -1/2h''\epsilon = -kT''/\epsilon$ is the same for the transverse velocity distribution.

The behavior of i for values of V_C less (more accelerating) than those covered in (36A) is very complicated except when $V'' = 0$ and therefore the general case will not be dealt with.

When transverse and longitudinal temperatures are equal (36B) reduces to

$$i = -\Theta\frac{i_C}{V_t}(2V' - V_C)\exp\frac{V_C}{V'},$$

$$\text{for } V' = V'',\ V_t >> -V',\ V_t >> V_C \geqq 0 \qquad (37)^8$$

the equation of a curve on a semi-log plot which has half the slope at $V_C = 0$ that it has at $V_C = \infty$ and only approximates to a straight line of slope $1/V'$ for large values of V_C.

When the transverse temperature is zero, $V'' = 0$ and the Eq. (36*B*) gives a straight semi-log plot of slope $1/V'$. In this case it can also be shown that

$$i = i_C \qquad \text{for } V_C \leqq -V_t \qquad (38A)$$

$$i = -\Theta\frac{i_C}{V_t}\left\{V_C + V'\left[1 - \exp\frac{V_C + V_t}{V'}\right]\right\} \text{ for } V_t >> -V',\ V_t < V_C < 0 \quad (38B)$$

[8] See section IX, §§B and C.

When both transverse and longitudinal temperatures are zero but the ions have a longitudinal drift velocity corresponding to V_D volts the equations for the volt-ampere characteristic become

$$i=i_C \qquad \text{for } V_C \leqq V_D - V_t \tag{39A}$$

$$i=-\Theta'\frac{i_C}{V_t}(V_C-V_D) \qquad \text{for } V_D - V_t \leqq V_C \leqq V_D \tag{39B}$$ [8]

$$i=0 \qquad \text{for } V_C \geqq V_D \tag{39C}$$

Here θ' is similar to θ except first, that the value of $\bar{E}$ to be used in it is that corresponding to $V_C=V_D$ instead of $V_C=0$, and second that when V_C leaves the neighborhood of V_D and approaches V_D-V_t, θ' increases and approaches 1.

VIII. Measurements on Positive Ions: Interpretation of Experimental Curves

A. The potential of the discharge with respect to the anode. The voltage readings, E_C, do not give the position of $V_C=0$ for they measure potential from the anode which is a few volts negative with respect to the discharge itself. It follows that the reading on the E_C scale of the line $V_C=0$ is the potential of the discharge with respect to the anode. Now, Eq. (36A) shows that if a linear plot be made of the volt-ampere characteristic, a portion of it up to $V_C=0$ will be straight. The curves of Fig. 6[9] obtained by Mr. C. G. Found using a mercury vapor arc in a tube built by Mr. A. F. Dittmer show this well. Unfortunately the deviation from a straight line is so slow at small positive values of V_C that the actual position of $V_C=0$ cannot be found by simple inspection of the curve.

A method for finding the point $V_C=0$ is to plot a certain root of the deviations from the straight line, against E_C. The straight line drawn through the resulting set of points cuts the E_C-axis at $V_C=0$. The root to be used varies from the square root for $V''/V'=0$ to the 2.7th root for $.2 \leqq V''/V' \leqq 1$ and this results in an uncertainty in the value of the discharge potential.

As an example, consider the curves of Fig. 6. Both the square root and 2.7th root of the deviations are plotted as crosses, the intersections of the straight lines through these with the axis are marked,

[9] These curves were made before any of the present theory had been developed and no particular precautions in regard to the shape or uniformity of the holes in electrode B, to the measurement of the distance between B and C, or to the accurate measurement of the small ion currents at the higher retarding voltages were made. Experiments in which these precautions are being observed are under way.

and it is noted that the uncertainty is about 0.1 volt. For curve A, E_C, at $V_C=0$, lies between 3.6 and 3.7, while for curve B, E_C at $V_C=0$, lies between 3.7 and 3.8 volts.

B. *The fundamental transverse voltage*, V_t. The slope of the linear portions of the characteristics in Fig. 6 is $-\theta i_c/V_t$. Since i_c is known

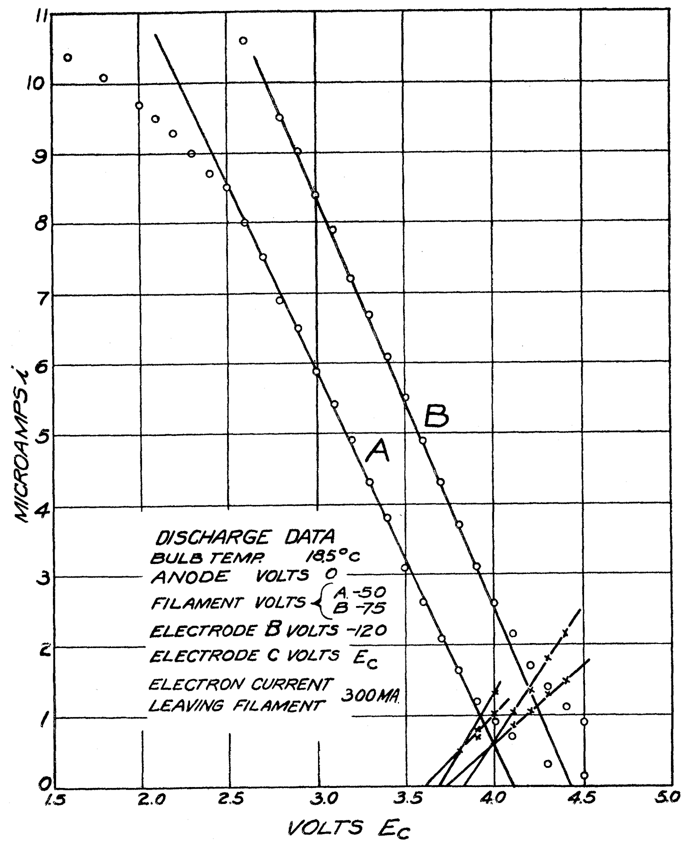

Fig. 6.

in each case, V_t can be calculated by using Eq. (35) and is found to be 1.9 and 2.2 volts respectively for curves A and B.

An independent calculation of V_t can be made using Eq. (11), $\bar{E}$ being calculated by Eq. (14) with $V_C=0$. The distance between B and C was not, unfortunately, accurately known, but lay between 0.1 and 0.2 cm and was estimated at 0.15 cm for the field strength calculation. On this basis it was found that:

For curve A; V_t (calc) $=2.4$ v., V_t (from curve) $=1.9$ v.
For curve B; V_t (calc) $=2.65$ v., V_t (from curve) $=2.2$ v.

a satisfactory agreement in view of the uncertainty of hole and size and electrode distance.

C. The voltage V' corresponding to the "longitudinal temperature," $T' = -11{,}600\ V'$. Eq. (36B) shows that when a semi-log plot of the volt-ampere characteristic becomes straight at positive values of V_C, then the reciprocal of the slope is V'. The experimental difficulty is encountered that the determination of this quantity depends on the smaller and, therefore, usually the less accurate current measure-

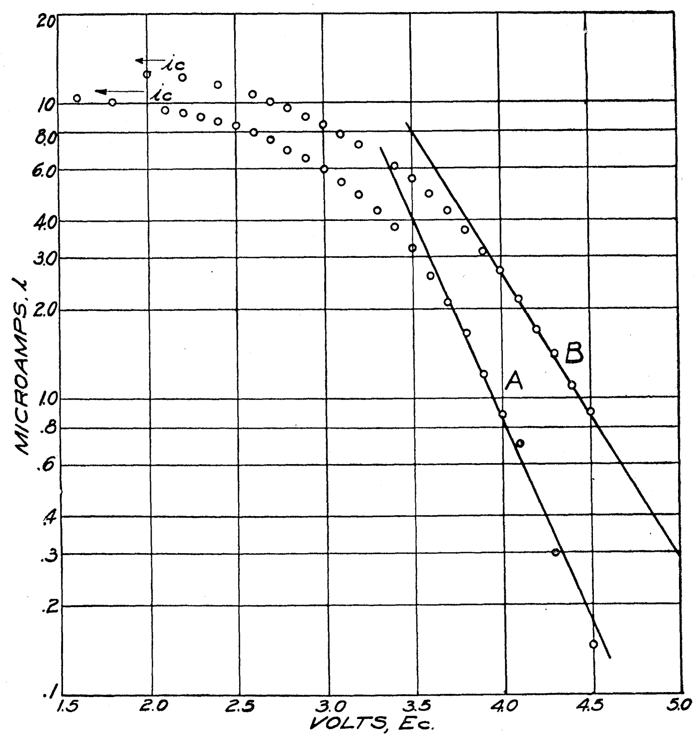

Fig. 7.

ments. This trouble appears in Fig. 7, the semi-log plot of the observations already shown in Fig. 6. Using the straight lines drawn here as the best approximation:

For curve A; $V' = -0.32$ v., $T' = 3360°K$

For curve B; $V' = -0.45$ v., $T' = 5200°K$.

D. *The voltage V'' corresponding to the "transverse temperature,"* $T'' = -11{,}600\ V''$. Eq. (36A) shows that the intercept of the E_C-axis in Fig. 6, which is terminated by $V_C = 0$ at one end, and at the other by the intersection of the straight line plot continued is the quantity

$-(V'+V'')$. This relation may be used to calculate V''. Assuming that the value of E_C corresponding to $V_C=0$ is, for curve A, 3.6 v. and, for curve B, 3.7 v., and noting that curve A intersects the E_C axis at 4.1 v. and curve B at 4.4 v., we have

For curve A; $V''=-0.18$ V., $T''=2100°$K
For curve B; $V''=-0.25$ V., $T''=2900°$K.

Even if the other values of E_C, namely 3.7 and 3.8 v. had been used, the ratio V''/V' would still have exceeded 0.2, showing that 2.7 was the correct root of the deviations and that the proper values for E_C were used above.

A further approximation to V'' can be made using (36B) by solving it for the V'' which appears in the denominator, using an approximate value of V'' in the term involving the exponential and using a value of i at which the second exponential term is relatively small. Thus for curve B at $E_C=4.4$ v., $i=1.1\mu$a., it is found that $V''=-0.24$. Accordingly $T''=2800°$K and $T''/T'=V''/V'=.53$.

E. *The decrease in i when* $V_C \leqq -V_t$. Eqs. (38) show that when $V''=0$, i is constant up to $-V_t$. But the existence of initial transverse velocities results in some ions being cut off prematurely so that a decrease in i at $-V_t$ indicates a transverse temperature different from zero. In the present case part of the decrease must be ascribed to non-uniform hole size which affects the characteristic shape most vitally in this region. Accordingly this portion of the present characteristics is unsuited to interpretation. Even in the ideal case, however, the theoretical equation for the current is very complicated in this region and also subject to errors listed in section IX, §§B and C, so that any quantitative interpretation appears impossible.

IX. Measurements on Positive Ions: Errors and Experimental Precautions

A. *The condition* $V_t >> -V'$ *of Eqs.* (36) *and* (37). The necessity for simplifying the integration of the ion distribution equation led to this condition. By taking $A_\delta = A_\epsilon = 0$ an upper limit to the error is found to be about 4 percent for $V_t=-5V'$, but the actual situation is much more favorable than this. Probably the best criterion of the fulfillment of this condition is the straightness of the linear plot, Fig. 6.

B. *Effect of ion paths not parallel to hole axis.* The ions which go through the hole follow paths which deviate appreciably from parallelism with the hole axis. Insofar as this arises from the "transverse temperature," it is small, about $\sqrt{V''/V_B}$ or 0.045 radians in the case of curve B, and its effect is negligible.

The non-parallelism which arises from the acquired transverse velocity itself is only slightly larger, but its effect is not negligible. It results in a change in the effective area of the hole, which can be calculated from Eq. (13). $V_D = -V'$ may be neglected so that $\Delta A/A = .15\ a\bar{E}/V_B = -.022$ for curve B. That is, the effective area of the hole is 2 percent less than its true area. But cases in which this error is appreciable may arise. An examination of the integration leading to Eqs. (36A) and (36B) reveals that the hole size is not involved. This quantity enters only when the final substitutions listed under (36B) are introduced. Accordingly the difficulty can be readily analyzed by more exact definition. Retaining the definition of i_c as the total ion current through the hole it becomes necessary to append the factor $1+\Delta A/A$ to the mathematical expression for i_c which follows Eq. (36B). This introduces a correction coefficient of $1-\Delta A/A$ which is to be applied to the right members of Eqs. (36A), (36B), (37), (38B), and (39B). The values of V_t (from curve) of section 8B are thus 2 per cent low. *None of the other calculations based on the experimental curves is, however, affected by the non-parallelism of the ion path and the axis of the hole.*

C. *Thickness of the hole.* In sections II and IV it was concluded that neither the depth of the orifice nor its shape, provided it was of the proper symmetry, has any effect on the transverse velocity acquired by an ion whose path is perpendicular to B. The path is not perpendicular to B primarily on account of acquired transverse velocity, so if the hole is cylindrical and has appreciable depth the radial distance at which the ion enters it is less than the radial distance at which the ion leaves it. Such an ion acquires a maximum transverse velocity when it grazes the edge of the hole as it leaves. The radius appropriate for calculating this velocity is the mean of the radius of entrance and the radius of the hole itself. This mean radius is seen to be $a-\frac{1}{4}t\sqrt{-V_t/V_B}$ [or $a(1+t\bar{E}/4V_B)$ by Eq. (11)]. Here again the hole size is involved, and the reasoning of §B applies. But since here the maximum acquired transverse velocity is not directly observed, the mathematical definition of V_t as it appears below Eq. (36B) will be retained. Accordingly no change has to be made in the equations because of this change in effective radius. There is, however, an accompanying decrease in the effective hole area which allows ions to pass. The complete fractional increase (algebraic) in area is by Eqs. (13) and (11)

$$\Delta A/A - t/a\sqrt{V_t/-V_B} \text{ or } -\sqrt{V_t/-V_B}(0.15+t/a) \qquad (40)$$

where t is the depth of the cylindrical part of the hole. In the new experiments which are contemplated it is planned to make the holes conical so as to reduce t to zero.

The initial transverse velocities of the ions also cause the ion paths to deviate from the perpendicular, the effect here being to reduce the effective area of the hole for ions having $v>0$ and leave it unchanged for ions having $v<0$. As this only involves one half the ions, and as $V_t >> -V''$, any error from this cause is quite small compared with those already dealt with.

D. *Equivalence of mathematical and physical equipotentials.* In order that the mathematical solution of the field about a hole (slit) should correspond to the physical conditions it has been pointed out in section 3 (section 1) that the distances, sheath edge to electrode B and electrode B to electrode C, should be at least 3 radii (3 slit widths).

E. *Return of ions through the hole.* Any ions which are not collected by C and re-enter the sheath through the hole in B increase the space chatge there and decrease the primary ion current. As the space charge conrribution of the returned ions relative to the space charge contribution of the primary ions at any point is equal to the density of returned ions relative to that of primary ions at that point, an estimate of this error is easily made. Neglecting the initial transverse velocities of the ions for the moment, that is, assuming that all ion paths in the sheath are perpendicular to B, the transverse velocity acquired by an ion in passing through the hole at a distance r' from the center corresponds to an angular deflection of the path at B of $\sqrt{V_r/-V_B}$ radians, V_r being the acquired transverse voltage. The distance traveled before the ion reaches B again is 2δ (δ being the electrode separation) and consequently the lateral displacement of the ion when it again reaches the plane of B is $\Delta_1 r = 2 \cdot 2\delta\sqrt{V_r/-V_B}$, assuming that it follows a parabolic path.

The density of returning ions relative to outgoing ions is then $r'^2/(r'+\Delta_1 r)^2$ at B, and this can be evaluated using the above equation together with Eq. (11) and calculating $\bar{E}$ as in section VI, §B. Since, however, the returning ions contribute most effectively to the space charge when they are near the sheath edge the calculation of their radial distances must be carried one step further, that is, into the sheath once more, noting, of course, that in repassing the hole the ions acquire additional transverse velocity. To calculate the relative space charge contribution of these ions their relative density one quarter of the distance from sheath edge to electrode B was selected as a fair average value. As these ions have insufficient longitudinal energy to penetrate the sheath edge they are reflected from it and make a second space charge contribution as they return to B which must also be added. The computation shows that the relative space charge contribution is a function only of the ratio of electrode separation, δ, to sheath thickness, σ. It has a maximum value amounting to 0.7 per cent when $\delta/\sigma=0.7$ and drops to 0.35 per cent when δ/σ equals either 0.2 or 2.5.

Turning now to the effect of the presence of random initial transverse velocities it is seen that this has the general tendency to decrease the returned ion density, and therefore to decrease the error. Under all circumstances, then, the return of ions through the hole may be neglected.

F. *Limiting of ion current to C by space charge.* With an electrode separation greater than the sheath thickness it is evident that if B were perfectly transparent to ions, then theoretically no ions could reach C on account of space charge. Experimentally, however, there is only a narrow beam of ions flowing through a space devoid of charge, and this beam tends to diffuse on account of initial and acquired transverse velocities. It is evident that the thinner the beam is in the sheath, the greater the separation δ of the electrodes relative to the sheath thickness, σ, may be. The condition that space charge limitation shall not set in may, therefore, be roughly expressed as $\delta/\sigma \leqq \kappa\sigma/a$ or $\delta a/\sigma^2 \leqq \kappa$, where κ is a constant. In the present case $\delta/\sigma=2$, $\sigma/a=6$, hence $\delta a/\sigma^2=1/3$. Taking this as the maximum advisable value the condition above becomes

$$a\delta/\sigma^2 \leqq 1/3 \tag{41}$$

G. *Reduction of potential at the center of the hole by space charge.* If no non-uniformity is introduced into the space charge distribution in front of B by the hole, this charge gives no transverse field and cannot, therefore, change the potential at the hole center except insofar as the field E_1 is affected, a factor already dealt with in Eq. (14). But the ions which have passed through B have a non-uniform distribution and may affect the potential. Assuming, (1) a uniform field between B and C to calculate the ion velocities, (2) that the ions form a cylindrical beam behind B, and (3) that the density of ions returning from C is negligible, it is found that the change in potential sought amounts to only a few millivolts in the case of curve B. Accordingly it is most unlikely that this factor would ever become appreciable.

H. *Effect of ion repulsion.* On account of the concentration of charge in the beam, any ion which is not on the beam's center line is subject to a transverse repulsive force which tends to diffuse the beam still more and by increasing transverse velocities, reduce longitudinal ones. No really satisfactory estimate of the magnitude of this effect has been made on account of numerous complicating factors. But in order to understand the situation better, a calculation based on certain simplifying assumptions has been made. It is easily seen that the only class of ions which is of interest is that composed of ions which just reach the plate at some fixed collector voltage. Assuming that the beam suffers no diffusion such as dealt with in §E, those ions of this class will be most affected which travel along the outside of the beam. We consider these ions only. The predominating part of the repulsion velocity acquired by such an ion will be acquired while the ion is moving either most slowly or in the vicinity of the densest space charge. Both conditions occur at the same place, namely, in the vicinity of the collector. On this basis the following assumptions were made: (1) the ion beam is bounded by the right circular cylinder which has the hole for a base, as already mentioned above, (2) an ion acquires a negligible repulsion velocity until it is within a distance of the collector of the same order as the radius of the beam (hole), (3) on account of the electrical image of the beam in the collector, the charge effective in repulsion is that in a cylinder cut from the beam which is bounded by the planes $x=x_i/4$, $x=7x_i/4$ and of diameter $3x_i/2$, x being distance from the collector, and x_i being the x of the ion, (4) the repulsive force exerted by this charge is that of a sphere of uniform charge density, of the same total charge as the cylinder, of the same diameter as the cylinder, which is tangent to the cylindrical beam boundary at the ion, i.e., x_i, (5) the charge density in the beam is given by the space charge equation treating the collector as a plane cathode and using the current density to electrode B.

Then for mercury ions it is found that the voltage corresponding to the repulsion velocity is

$$V_R''=8.43\times10^3 I_+^{2/3}a^{4/3} \tag{42}$$

For curve B, Fig. 7, $I_+=8\times10^{-4}$ amps/cm², $a=0.0125$ cm. Hence $V_R''=0.21$ v.

Contrary to assumption (3) the repulsion velocity acquired by the ion in travelling from $x_i=3a$ to $x_i=a$ is not negligible, though less than this. Assuming the charge between $x=a$ and $x=3a$ (as given by the space charge equation) to be redistributed uniformly throughout a sphere of radius a which is always tangent to the beam boundary at the ion as the ion travels from $3a$ to a, it is found that

$$V_R'=3.88\times10^3 I^{2/3}a^{4/3} \tag{43}$$

whence $V_R'=0.097$ v., and the total repulsion voltage is

$$V_R=(\sqrt{V_{R'}}+\sqrt{V_R''})^2=0.59.$$

This is very large compared, for instance, with V'' which is 0.24 for this curve. A re-examination of the assumptions shows that (1) is certainly not fulfilled and (5) is open to doubt. It is not unreasonable, however, to suppose that the actual space charge, including ions returning to B, if confined as contemplated in assumption (1), would be near enough on the average to that given by assumption (5). Accordingly, attention must be confined to (1). The considerations of §E show that three quarters of the way from B to C only a fraction, λ, say, of the ions lie within the cylindrical boundary of assumption (1). Accordingly, the actual force exerted on the ion by the charge inside the cylinder is certainly less than λ times the calculated force. λ is given by the approximate formula $\lambda=.29\sigma/\delta$, in the range $1.4<\delta/\sigma<4.0$. But the repulsive force of ions outside the cylinder balances a portion of this force, say $\sqrt{\lambda}$. Thus the actual force is $\lambda^{3/2}$ of the calculated force, and hence the repulsion voltage is λ^3 or $0.024\,(\sigma/\delta)^3$ of that calculated. In the present case V_R becomes $0.59\times 0.003=0.002$. This, if correct, is small enough compared with V'' to be neglected, but the uncertainty involved demands that an experiment with ions or electrons of known temperature be done. With the above corrections the repulsion voltage in terms of the molecular weight, M, of the ion becomes

$$V_R=99M^{1/3}I_+^{2/3}a^{4/3}(\sigma/\delta)^3 \tag{44}$$

Summarizing the factors which vitally affect the interpretation of an experimental curve in the form of conditions to be satisfied, we have, from the paragraphs noted,

A. $$V_t>>-V' \quad \text{(Validity of integration)} \tag{45}$$

F. $$\frac{\delta}{a}\leqq\frac{1.82\times 10^{-8}V_B^{3/2}}{Ma^2I_+} \quad \text{(Space change limitation)} \tag{46}$$

H. $$\left.\begin{matrix}-V''\\-V'\end{matrix}\right\}>>99M^{1/3}I_+^{2/3}a^{4/3}(\sigma/\delta)^3 \quad \text{(Ion repulsion)} \tag{44}$$

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