BATEMAN'S EXTENDED ELECTRODYNAMICS, AND THE MASS AND RADIATION REACTION OF AN ELECTRON

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Abstract

From Bateman's earlier stress-energy tensor, there are deduced by analogy with the momentum and energy equations based on the classical electromagnetic stress-energy tensor, an extra-classical body-force and a condition that may be understood to state the conservation of charge in combination with the Lorentz transformation. It is then shown how this procedure may be reversed so as to derive, from the new body-force and the condition just mentioned, Bateman's tensor with all of its components physically interpreted. Next the mass and radiation reaction of an electron in non-uniformly accelerated, non-periodic motion are calculated from the force (classical and extra-classical) exerted on it by its own field, on the assumptions that the electron is spherical in a rest-system and has a centrally symmetrical distribution of charge. Both come out zero. The result for the radiation reaction is new. Finally the same result for the mass is obtained from the total momentum of the field of an electron in uniform, or quasi-stationary, motion. The present method differs from one recently published by Bateman in not using the restricted relativity transformation for tensors, and in requiring of the distribution of charge only that it be centrally symmetrical.

Introduction

To the stress-energy tensor T^e of classical electrodynamics, comprising the stress-system X_x^e, X_y^e. . . Z_z^e, the energy-flux (Poynting's) vector $\mathbf{S}^e$ and related density of momentum $\mathbf{G}^e$ and the energy-density W^e as deducible from the equations of Maxwell and Lorentz, Bateman has successively added new tensors[1] designed to remedy defects of the classical theory. The fundamental one of these tensors, and the only one that will be considered here, is T^s, having as components

$$X_x^s = -2\psi\frac{\partial^2\psi}{\partial x^2} + \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2$$

$$X_y^s = Y_x^s = -2\psi\frac{\partial^2\psi}{\partial x\partial y}\cdot$$

$$W^s = 2\psi\frac{\partial^2\psi}{\partial t^2} + \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2$$

$$S_x^s = c^2 G_x^s = -2\psi\frac{\partial^2\psi}{\partial x\partial t}$$

[1] H. Bateman, Phys. Rev. **20,** 243 (1922); Bull. Nat. Res. Council, **24,** p. 99 (1922); Messenger of Mathematics **52,** 116 (1922); Ibid. **53,** 145 (1924); Phil. Mag. **49,** 1 (1925).

where ψ is a retarded scalar potential defined by

$$\psi = \int [\rho\sqrt{1-(v^2/c^2)}\,]_{t-r/c}(dV/r)$$

and consequently satisfying the differential equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=-\rho\sqrt{1-(v^2/c^2)}$$

Since the right side of this equation is an invariant with respect to the Lorentz transformation, as is also the differential operator on the left side, it follows that the potential ψ is such an invariant. The tensor T^s is then readily shown to be a symmetrical world-tensor[2] with the same transformational properties as T^e.

The components of the body-force associated with T^s are determined by relations of the type

$$F_x{}^s=\frac{\partial X_x{}^s}{\partial x}+\frac{\partial X_y{}^s}{\partial y}+\frac{\partial X_z{}^s}{\partial z}-\frac{\partial G_x{}^s}{\partial t} \tag{1}$$

which are identical in form with the relations between the classical body-force

$$\mathbf{F}^e=\rho(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H})$$

and the tensor T^e. The results may be combined in the vector equation

$$\mathbf{F}^s=2\psi\nabla\{\rho\sqrt{1-(v^2/c^2)}\,\}\ . \tag{2}$$

Corresponding to the classical energy-equation, we have also

$$\mathbf{v}\cdot\mathbf{F}^s+\mathrm{div}\mathbf{S}^s+\frac{\partial W^s}{\partial t}=0 \tag{3}$$

provided that

$$\psi\frac{d}{dt}\{\rho\sqrt{1-(v^2/c^2)}\,\}=0 \tag{4}$$

It is possible to proceed in another manner, following the usual treatment of classical electron theory. If we assume Eq. (2), then by an integration of $\mathbf{F}^s$ over all space and a reduction by means of Green's Theorem, the expressions previously labeled as $X_x{}^s$, $X_y{}^s$ . . . $Z_z{}^s$ and $\mathbf{G}^s$ are obtained and seen to have physical meanings consistent with the names assigned to them. Details will not be given; the process, though rather lengthy, is direct, and offers no essential difficulties. Eq. (1) follows as a corollary. In the same way, starting with an integration of the

[2] M. v. Laue, Die Relativitätstheorie, 4. Aufl., 1. Bd., p. 102.

activity of the body-force, $\mathbf{F}^s \cdot \mathbf{v}$, we can derive the rest of the tensor T^s and Eq. (3), if

$$\frac{d}{dt}\{\rho\sqrt{1-(v^2/c^2)}\}=0\ .$$

The last assumption, which is not altogether the same as Eq. (4), may be granted, since it amounts to the conservation of charge as combined with the Lorentz transformation.

Mass and Radiation Reaction

The reaction on an electron of its own field will now be considered. The electron will be taken as spherical when at rest, with a volume charge distributed symmetrically about the center, but otherwise arbitrarily. The motion of the electron will be unspecified except for the supposition that there is always a reference system of the type dealt with in the restricted theory of relativity in which the electron is momentarily at rest, with acceleration and derivative of acceleration not varying from point to point, and that, relative to this system, the electron is spherical and has the stationary distribution of charge.

On these assumptions, the classical reaction, so far as it depends on the acceleration ($\mathbf{f}$) and derivative of the acceleration ($\dot{\mathbf{f}}$), is[3]

$$-(\mathbf{f}/6\pi c^2)\int \rho dV \int \rho' dV'(1/(R^2+R'^2-2RR'\cos\theta)^{1/2})+e^2\dot{\mathbf{f}}/6\pi c^3$$

where ρ and ρ' are the densities of charge at volume elements dV and dV' which lie, respectively, at distances R and R' from the center, along radii vectors enclosing the angle θ.

To get the extra-classical reaction, we start with Eq. (2). Here, to allow for the rest-system used, we put $v=0$. The part of ψ due to element of charge de at a field-point relative to which the position of de is given by the vector $\mathbf{r}$ is

$$de[\sqrt{1-(v^2/c^2)}]/4\pi[r(1+(\mathbf{r}\cdot\mathbf{v}/cr))]$$

where the bracketed quantities must be assigned the values pertaining to the previous time $t-[r]/c$. This expression, reduced by means of known series expansions[4] so as to involve only magnitudes evaluated at the time t becomes, as far as terms in $\dot{\mathbf{f}}$,

$$\frac{de}{4\pi}\left(\frac{1}{r}+\frac{1}{2}\,\frac{\mathbf{f}\cdot\mathbf{r}}{rc^2}-\frac{1}{3c^3}\dot{\mathbf{f}}\cdot\mathbf{r}\right)$$

[3] L. Page, Introduction to Electrodynamics, pp. 51 52.
[4] Ref. 3, pp. 39, 40.

Next, changing de to $\rho' dV'$, multiplying by $2\nabla\rho dV$, and integrating twice through the electron, we have, omitting the electrostatic term that contributes nothing,

$$\frac{\mathbf{f}}{4\pi c^2 f}\int dV \cos\alpha(\mathbf{f}\cdot\nabla\rho)\int\frac{\rho' dV'(R'\cos\theta - R)}{\sqrt{R^2+R'^2-2RR'\cos\theta}}+\frac{e\dot{\mathbf{f}}}{6\pi c^3}\int\cos^2\beta\frac{d\rho}{dR}dV$$

$$-\frac{\mathbf{f}}{4\pi c^2}\int\rho_{R=a}\cos^2\alpha dS\int\left[\frac{\rho' dV'(R'\cos\theta-R)}{\sqrt{R^2+R'^2-2RR'\cos\theta}}\right]_{R=a}$$

$$-\frac{e\dot{\mathbf{f}}}{6\pi c^3}\int a\rho_{R=a}\cos^2\beta dS$$

where α, β are the angles between $\mathbf{R}$, and $\mathbf{f}$ and $\dot{\mathbf{f}}$, respectively. The surface integrals arise from the infinite discontinuity of $\nabla\rho$ at the surface, $R=a$.

By the use of Gauss' Theorem, the surface integrals may each be proved equal to the negative of the sum of the corresponding classical and extra-classical volume reactions, so that for any centrally symmetrical distribution of charge (which may be very different from the distribution designed by Bateman to secure equilibrium within the stationary electron[5]) the $\mathbf{f}$ and $\dot{\mathbf{f}}$ reactions vanish.

The vanishing of the $\mathbf{f}$ reaction is interpreted to mean that the mass is zero, and this is in agreement with the result in the final section where the case of uniform (or quasi-stationary) motion is treated. By introducing (in the last three of his papers[1]) new tensors to remove certain surface discontinuities in the components of T^e+T^s for the stationary electron, Bateman succeeds in rendering the mass different from zero—the actual value obtained, regardless of the way in which density varies with distance from the center, is three times the mass of a Lorentz electron with a uniform surface distribution. But the $\dot{\mathbf{f}}$ reaction remains zero. This result seems to indicate that radiation is impossible. When, however, the

[5] This distribution is obtained by putting $F^e+F^s=0$ inside the stationary electron. The advantage it enjoys over other distributions is that, when additional tensors are introduced, it leads to the exact relativistic relation between energy of the stationary field and rest-mass; without the additional tensors, this energy is zero, as well as the rest-mass.

Bateman, however, repeatedly states that the resultant body-force should be postulated to vanish everywhere. While this is in the spirit of relativity mechanics, it may be pointed out that for the immediate purposes of his analysis such a generality is not essential. In one of his papers (Messenger of Mathematics **52**, 122 (1922)) Bateman appears to make elaborate use of it, but his results there can readily be derived without it.

energy radiated by a point charge is considered, it develops[1] that the energy radiation vanishes, on the average, only for periodic motion.[6]

Momentum of the Field for Uniform Motion

That the mass of an electron is zero may also be inferred from the vanishing of the total momentum of the field of an electron in uniform (or quasi-stationary) motion. Bateman[7] obtains this result by using a transformation formula applying to tensors in the restricted theory of relativity. The momentum in question is thus shown to be a multiple of the total energy of the field of a stationary electron with his equilibrium distribution of charge, and this he has found to be zero.

While the argument can be modified so as not to involve any particular distribution of charge, another deduction requiring only central symmetry and using simpler relativity transformations may easily be made as follows: As can be shown[8] by changing to the reference system in which the uniformly moving electron is at rest, the electromagnetic momentum of its field is

$$\frac{\mathbf{v}}{c^2\sqrt{1-(v^2/c^2)}}\int(\nabla\Phi)^2\sin^2\theta dV \tag{5}$$

where the integral is extended over all space. Φ is the electrostatic potential due to the electron; dV is in polar coordinates so that θ has its usual significance, the polar axis being the direction of motion (velocity = $\mathbf{v}$) in the original reference system. It remains to calculate the extra-classical momentum, whose effective density is

$$\mathbf{G}^s = -\frac{2\mathbf{v}}{c^2 v}\psi\frac{\partial^2\psi}{\partial x\partial t}$$

if the electron moves along the x-axis. The integration will again be carried out in the rest-system. As may be inferred from the definition of ψ and its invariance with respect to the Lorentz transformation, or may otherwise be proved directly, on changing the reference system ψ becomes Φ. Applying the Lorentz transformation to the differential co-

[6] In his later papers, Bateman regards compensating negative energy radiation as a blemish in his theory. It is hard to gather from the context whether the tensor introduced in Messenger of Mathematics, **52,** 125 (1922) is meant to remove this blemish; if so, it does not serve its purpose, for calculation shows that it leaves the radiation of energy *in statu quo.*

[7] Messenger of Mathematics **53,** 147 (1924).

[8] L. Silberstein, Theory of Relativity, p. 214.

efficient and to the element of volume, we have for the extra-classical momentum

$$\frac{2\mathbf{v}}{c^2\sqrt{1-(v^2/c^2)}}\int \Phi \frac{\partial^2\Phi}{\partial x^2} dV$$

or, by symmetry,

$$\frac{2\mathbf{v}}{3c^2\sqrt{1-(v^2/c^2)}}\int \Phi\nabla^2\Phi dV \tag{6}$$

The trigonometric factor in the integrand of (5) may be replaced by 2/3. The sum of (5) and (6) is reducible to a surface integral by Green's Theorem, and this surface integral vanishes since Φ and $\nabla\Phi$ are finite and continuous at the surface of the electron and vanish at infinity.

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