THE RATIO OF HEAT LOSSES BY CONDUCTION AND BY EVAPORATION FROM ANY WATER SURFACE

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Abstract

It is shown that the process of evaporation and diffusion of water vapor from any water surface into the body of air above it is exactly similar to that of the conduction or "diffusion" of specific heat energy from the water surface into the same body of air. Because of this similarity it is possible to represent the ratio R of the heat loss by conduction to that by evaporation by the formula

$$R = .46 \left(\frac{T_w - T_a}{P_w - P_a} \right) \frac{P}{760}$$

where T_a and P_a are the original temperature and vapor pressure of the air passing over the lake, and T_w and P_w are the corresponding quantities for the layer of air in contact with the water surface. The substitution of R times the evaporation loss for the value of the conduction heat loss in the Cummings equation for evaporation makes it an exact equation for the determination of evaporation from any water surface in terms of the net radiant energy absorbed by the water and the heat stored in the water.

IN his study of evaporation from lakes, Cummings[1] has assumed, as a first approximation, that the determining factor is the incoming solar radiation, i. e. that the evaporation from any lake is roughly equal to the radiant energy falling on the lake divided by the latent heat of vaporization, corrected, of course, for heat stored in the lake due to any change in its temperature. This would be strictly true if we could neglect other heat losses such as conduction and convection, for since evaporation increases rapidly with the temperature, the lake would warm up to such a temperature that the heat lost by evaporation would be just equal to that gained by solar radiation. Wind and humidity would then not affect the total evaporation, but only the temperature that must be reached before equilibrium between evaporation and solar radiation is attained. The present paper is a theoretical attempt to evaluate losses by conduction and convection in terms of easily measurable quantities, and hence to determine whether they are small enough to be neglected; and if not, how they may be corrected.

In order to simplify calculations, we shall assume in the first part of the paper that diffusion coefficients, conductivities, and densities are independent of the temperature. We can write at once for the

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[1] Cummings, Phys. Rev. **25,** 721 (1925); Journal of Electricity **46,** 491.

mass m, of water vapor diffusing per unit time across a space of length l, and area a,

$$m = D_1 a (\rho_1 - \rho_2)/l \tag{1}$$

ρ_1 and ρ_2 are the densities of water vapor on the two faces of the space and D_1 is the diffusion coefficient of water vapor through air.

If we multiply both sides of this equation by L, the latent heat of vaporization, we get

$$Lm = D_1 a (L\rho_1 - L\rho_2)/l \tag{2}$$

and if let θ_1 and θ_2 represent the energy per unit volume in the form of latent heat, the energy Q_L carried across becomes

$$Q_L = D_1 a (\theta_1 - \theta_2)/l \tag{3}$$

Likewise we can write the formula for the heat energy Q_S carried, by conduction, across the same space as above:

$$Q_s = Ca(T_1 - T_2)/l \tag{4}$$

where C is the conductivity of the air in the space and T_1 and T_2 are the temperatures of the two faces.

If we substitute $C = D_2 C_p d$, where D_2 is the "diffusion coefficient" or diffusivity for the heat energy, C_p is the specific heat at constant pressure, and d is the density of the air, (4) becomes

$$Q_s = D_2 a (T_1 C_p d - T_2 C_p d)/l \tag{5}$$

Or if we let φ_1 and φ_2 be the densities of specific heat energy at the two faces,

$$Q_s = D_2 a (\varphi_1 - \varphi_2)/l \tag{6}$$

It is seen at once that (3) and (6) are of exactly the same form, the only difference being in the values of D_1 and D_2, which, in fact, differ only by a few percent, (a relationship predicted by the kinetic theory). This leads one to expect that heat losses by evaporation and diffusion, and by conduction, will follow the same laws and will be affected in the same way by convection.

Consider the case of a lake over which a wind is blowing (Fig. 1). Let y be the distance from the windward side of the lake measured parallel to the direction of the wind. Assume that the velocity of the wind at a distance x above the surface of the lake is $f(x)$ and also that this velocity is large enough so that the velocity of horizontal diffusion is negligible in comparison with it. Then for any element of volume $dxdydz$ the amount of latent heat entering the volume per unit time

from the lower side is $-D_1(\partial\theta/\partial x)dy\,dz$, while on the upper side the amount leaving is

$$-\left[D_1\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial x}\left(D_1\frac{\partial\theta}{\partial x}\right)dx\right]dydz\ .$$

The difference, if we assume D_1 constant, is then $D_1(\partial^2\theta/\partial x^2)dxdydz$. When a steady state is reached this net gain must be carried away by the air passing through the volume in unit time. The air in passing
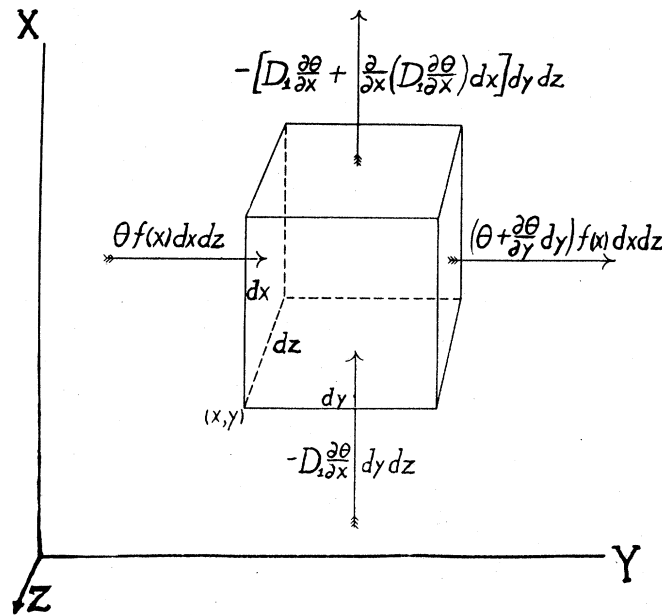

Fig. 1.

through the volume $dxdydz$ has its density of latent heat changed from θ to $\theta+(\partial\theta/\partial y)\,dy$; and since the volume passing per unit time is $f(x)dxdz$ the total amount carried away is $(\partial\theta/\partial y)\,f(x)dxdydz$.

Equating these two quantities we get

$$\frac{\partial\theta}{\partial y}f(x)\ =\ D_1\frac{\partial^2\theta}{\partial x^2}\tag{7}$$

and by similar reasoning,

$$\frac{\partial\varphi}{\partial y}f(x)\ =\ D_2\frac{\partial^2\varphi}{\partial x^2}\tag{8}$$

as the general equations determining θ and φ at all points above such a lake. The boundary conditions are $\varphi=\varphi_1$, $\theta=\theta_1$ at $x=0$, and $\varphi=\varphi_2$,

$\theta = \theta_2$ at $y = 0$. φ_2 and θ_2 represent the energy density, in the form of specific and latent heat respectively, of the incoming air, and φ_1 and θ_1 represent the corresponding densities for the layer of air in contact with the water.

Since it is impossible to obtain a general solution of these equations for any function $f(x)$ we will consider certain special values of $f(x)$.

Case I. $f(x) = K$ from $x = 0$ to $x = a$ and $f(x) = 0$ from $x = a$ to $x = \infty$. If a and K are small and the lake large, then the layer of air from $x = 0$ to $x = a$ will have its temperature changed to that of the water and will become saturated with water vapor, i. e., its specific heat density will change from φ_2 to φ_1 and its latent heat density from θ_2 to θ_1. Evidently the ratio R of heat lost by conduction to that lost by evaporation is

$$R = (\varphi_1 - \varphi_2)/(\theta_1 - \theta_2) \tag{9}$$

since this is the ratio of the two kinds of energy taken on by each cubic centimeter of the air passing over the lake.

Case II. $f(x) = 0$, from $x = 0$ to $x = a$; $f(x) = K$, from $x = a$ to $x = \infty$. If the area of the lake is small and the velocity K large, then the water vapor and heat diffusing through the stationary layer from $x = 0$ to $x = a$, will be carried away immediately and we can assume $\varphi = \varphi_2$, $\theta = \theta_2$ at the layer $x = a$. Under these conditions the rates at which heat and water vapor leave the water surface are determined solely by diffusion and the ratio between them is simply equation (6) divided by (3).

$$R = D_2(\varphi_1 - \varphi_2)/D_1(\theta_1 - \theta_2) \tag{10}$$

Case III. $f(x) = Kx^n$ from $x = 0$ to $x = \infty$. Professor Epstein has obtained equation (11) as a solution for this value of $f(x)$.

$$\theta = \theta_1 + (\theta_2 - \theta_1) \frac{\int_0^x \left[\frac{K}{(n+2)^2 D_1 y} \right]^{\frac{1}{n+2}} e^{-\sigma^{n+2}} d\sigma}{\int_0^\infty e^{-\sigma^{n+2}} d\sigma} \tag{11}$$

But the rate of diffusion from unit area of the water surface, i. e., diffusion across the plane $x = 0$, is

$$Q_L = -D_1 \left(\frac{\partial \theta}{\partial x} \right)_{x=0} = \frac{D_1(\theta_1 - \theta_2) \left[\frac{K}{(n+2)^2 D_1 y} \right]^{\frac{1}{n+2}}}{N} \tag{12}$$

where

$$N = \int_0^\infty e^{-\sigma^{n+2}} d\sigma$$

Similarly

$$Q_s = -D_2\left(\frac{\partial \varphi}{\partial x}\right)_{x=0} = \frac{D_2(\varphi_1 - \varphi_2)\left(\dfrac{K}{(n+2)^2 D_2 y}\right)^{\frac{1}{n+2}}}{N} \qquad (13)$$

which gives at once

$$R = \frac{Q_s}{Q_L} = \left(\frac{D_2}{D_1}\right)^{\frac{n+1}{n+2}} \left(\frac{\varphi_1 - \varphi_2}{\theta_1 - \theta_2}\right) \qquad (14)$$

This ratio is independent of y and therefore holds for every square centimeter of the lake and also for the lake as a whole.

Discussion

Case I represents a condition where the whole quantity of air under consideration is completely changed in temperature and moisture content to that of the layer of air in contact with the water. For this reason diffusion is not the limiting factor and hence the diffusion coefficients do not enter into the formula for the ratio of the heats lost by the two processes. It is obvious, however, that if conditions are such that diffusion does enter, it will be in such a way as to increase the loss by the process having the larger diffusion coefficient relative to the loss by the other method. Hence under any conditions of wind we may say, since $D_1 > D_2$ that

$$R \leq (\varphi_1 - \varphi_2)/(\theta_1 - \theta_2) \qquad (15)$$

Case II, on the other hand, represents conditions where diffusion is the completely determining factor, the heat and water vapor being immediately carried away after diffusing through the stationary layer. If, however, these are not immediately carried away, they tend to build up the temperature and vapor density on the upper side of this layer, and thus decrease the gradient in the stationary layer. This in turn tends to slow down further diffusion. But the process with the larger diffusion coefficient will build up faster, and hence will be retarded more than the one with the lower diffusion coefficient. Hence we can write

$$R \geq D_2(\varphi_1 - \varphi_2)/D_1(\theta_1 - \theta_2) \qquad (16)$$

The results of Case III support these conclusions since for all values of $n \geq 0$.

$$\left(\frac{D_2}{D_1}\right)^{\frac{1}{4}} \left(\frac{\varphi_1 - \varphi_2}{\theta_1 - \theta_2}\right) \geq R \geq \frac{D_2}{D_1}\left(\frac{\varphi_1 - \varphi_2}{\theta_1 - \theta_2}\right) \tag{17}$$

Negative values of n are without physical meaning since they would involve an infinite velocity at $x=0$. The upper limit of R in (17) corresponds to the case $f(x) = K$, $i.$ $e.$, a wind of uniform velocity. The values between this limit and the limit set by Case I correspond to conditions where $f(x)$ is a decreasing function of x, and hence do not correspond to actual conditions over a lake.

The above discussion shows that R lies under all conditions between the values given by Case I and Case II. For these two very simple cases, however, it is no longer necessary to assume D, C, and ρ independent of the temperature.

Thus in Case I, if we consider the changes in an amount of gas that would occupy 1 cc at 0°C, the amount of heat necessary to warm it from the temperature of the air, T_a, to the temperature of the water, T_w, is

$$Q_s = d_0 C_p \frac{P}{760}(T_w - T_a) \tag{18}$$

where d_0 is the density of air at 0°C, 76 cm, C_p is the specific heat at constant pressure, and P is the pressure of the air in mm. Likewise the amount of heat taken on in the form of latent heat when this amount of air is brought to saturation at the temperature T_w is

$$Q_L = L\left(\frac{\rho_w T_w - \rho_a T_a}{273}\right) \tag{19}$$

where as before ρ_a and ρ_w are the densities of water vapor in the incoming air and in the layer of air in contact with the water respectively. Therefore

$$R = \frac{Q_s}{Q_L} = \frac{d_0 C_p \dfrac{P}{760}(T_w - T_a)}{L\left(\dfrac{\rho_w T_w - \rho_a T_a}{273}\right)} \tag{20}$$

Similar consideration, which take into account the known temperature coefficients of heat conduction and diffusion, give for Case II

$$R = \frac{C_0(T/273)^{\gamma_1}(T_w - T_a)}{LD_{1,0}(T/273)^{\gamma_2}(\rho_w T_w - \rho_a T_a)760/P} \tag{21}$$

where C_0 and $D_{1,0}$ are the value of the conductivity and diffusion coefficient at 0°C. Kinetic theory gives $\gamma_1 = \gamma_2 = \frac{1}{2}$, while experimental data indicate a somewhat larger value, but still make $\gamma_1 = \gamma_2$.

If we substitute in these formulas the following experimental data: $d_0 = .001293$, $D_0 = .206$, $C_0 = .0000566$, $C_p = .241$, $L = 585.$, $\rho_a T_a/273 = P_a/9.41 \times 10^5$, $\rho_w T_w/273 = P_w/9.41 \times 10^5$, where P_a and P_w are the corresponding partial pressures of water vapor expressed in mm of mercury, we get

Case I.

$$R = .501 \left(\frac{T_w - T_a}{P_w - P_a} \right) \frac{P}{760} \tag{22}$$

Case II.

$$R = .442 \left(\frac{T_w - T_a}{P_w - P_a} \right) \frac{P}{760} \tag{23}$$

Case III, however, shows that under ordinary conditions where $f(x)$ is an increasing function of x, the true R is nearer Case *II* than Case *I*. Hence we may take as the most probable equation

$$R = .46 \left(\frac{T_w - T_a}{P_w - P_a} \right) \frac{P}{760} \tag{24}[2]$$

This equation is, of course, valid only for value of T_w low enough that the volume of the air is not appreciably increased by the water vapor evaporating into it and also that the water vapor on diffusing into the cooler air above does not condense and fall back to the water surface. This last condition is only violated under certain extreme conditions since the air is being warmed by conduction nearly as fast as its dew point is being raised by the diffusion of water vapor into it.

As an example let $T_w = 20$°C, $T_a = 15$°C, Relative humidity $= .5$, $P = 760$ mm. This gives $R = .21$ which indicates that while the ratio is small and can be corrected for by the aid of this formula, yet it cannot be neglected. It may be noted that if we let $R = -1$, this becomes, as it should, the standard formula for the determination of P_a by means of the wet and dry bulb thermometer.

Conclusion

If we put in the form of an equation Mr. Cummings'[1] statement of the relation between evaporation and radiant energy, we have

$$I = S + LE + K \tag{25}$$

[2] Since the above was written, a somewhat similar formula has been derived empirically from wet and dry bulb data. D. K. Dean, Power **62**, p. 754 (1925).

where I is "the radiant energy . . . . integrated over any time interval" or perhaps a little more explicitly, the solar and sky radiation corrected for reflection, minus the back radiation, S is the "heat represented by the change in temperature of the water," LE is "the heat represented by the evaporation (E) during the same time interval," L, of course, being the latent heat of evaporation, and K is a "relatively small correction" to cover other losses. These other losses, however, are due to conduction and convection which we have put equal to R times the losses by evaporation, i. e., $R(LE)$, where R can be determined by Eq. (24). Substituting this value of K in (25) we have

$$I = S + LE(1+R) \tag{26}$$

which is an exact equation for any body of water that is thermally insulated on the sides and bottom (a condition holding for most lakes) since it takes into account all processes by which heat can be lost or gained.

Consequently if we know S, E and R, which can be determined easily in any small insulated test jar, we can, following the method proposed by Cummings[1] use this slightly modified formula (26) to determine I or, vice versa, if we know I and can measure S and R, we have a means for calculating the evaporation E from a large body of water where it cannot be measured directly.

Such a body of water, however, is very nearly a black body over a very wide range of wave-lengths, since throughout the infrared even a thin layer of water absorbs almost completely, while in the visible this is true if the water is deep enough, as in a lake or if the inside of the vessel is properly blackened, as can be easily done for any test jar. In addition the reflection at the surface due to the difference in index of refraction is small throughout the visible and infrared to 20μ,[3] which is well beyond the maximum of the back radiation. On account of this fact I as determined by Eq. (26) becomes at once, with only a very small correction for reflection, the value of the net radiation exchanged with a black body at the temperature of the water. Thus such a jar of water can be used to determine the net heat radiated by a black body at night or received by it during the day time.

In a succeeding article Mr. Cummings will present the results of experiments designed to check the modified form of his equation, Eq. (26), under a series of widely different condition. Since this Eq. (26) depends on the determination of R as given by Eq. (24), these experiments will

[3] Rubens and Ladenburg. Verh. d. Phys. Ges. **11**, 16 (1909).

also serve as a test of Eq. (24) as well. His preliminary experiments indicate that these equations will be verified completely.

In conclusion the writer wishes to thank Professor Epstein for his derivation of Eq. (11).

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