VISIBILITY PHENOMENA WITH INTERFERENCE BY MULTIPLE REFLECTIONS

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Abstract

Visibility of interference fringes due to multiple reflections of approximately monochromatic light.—Approximate expressions are developed for the intensity and visibility of interference fringes for both transmitted and reflected light when any number of multiple reflections exist and the source of light is one of small width. Where the number of reflections is large, the visibility is $V=2b^2C/P$ and $V=(1-b^2)C/P$ respectively, C and P depending on the distribution of intensity in the source and having the significance which was given them by Michelson and Rayleigh b is the coefficient of reflection at the first reflecting surface. The effects of the coefficients of reflection and absorption are examined and it is indicated that the method of visibility with interferometers which depend on multiple reflection may be used to advantage in the study of the fine structure of such spectra as those of hydrogen and helium, not only in order to obtain the components but also to determine the character of each component. It is also indicated that the development may be extended to x-ray spectra in which series of multiple reflections from numerous planes exist.

THE growing necessity of obtaining high resolution of the components of a source of radiation makes it desirable that the method of visibility of interference fringes developed by A. A. Michelson should be investigated for the case where the interference is obtained under the conditions of multiple reflections. Resolution of a very high order may be obtained and components very close in wave-length are revealed by this method, but it has the further distinct advantage that it gives the distribution of intensity in the radiation of various wave-lengths emitted by components of the source. Moreover recent developments in the quantum theory make it desirable to obtain this information of the fine structure, particularly of hydrogen and helium spectra.

Interference of homogeneous radiation of wave-number m as ordinarily obtained is represented by

$$I=2A^2(1+\cos 4\pi Dm). \tag{1}$$

For a source of small "width" this becomes

$$I=2\int_{-x}^{x} F(x)\left\{1+\cos 4\pi D(\overline{m}+x)\right\}dx. \tag{2}$$

If visibility is defined by the expression

$$V=\frac{I_{max}-I_{min}}{I_{max}+I_{min}} \tag{3}$$

and if the distribution of intensity in the source follows the probability law,

$$V=C/P=\frac{\int_{-\infty}^{+\infty}\exp\left[-h^2x^2\right]\cos 4\pi Dx\,dx}{\int_{-\infty}^{+\infty}\exp\left[-h^2x^2\right]dx}=\exp\left[-(2D)^2\pi^2/h^2\right] \tag{4}$$

Where there are several components in the source the visibility is expressed by

$$U^2=\frac{\sum r^2+2\sum rr'\cos 4\pi D(d'-d)}{(\sum r)^2}V^2 \tag{5}$$

If the width of the source be considered as due to a Doppler effect in which Maxwell's law of distribution of velocities exists it was shown by the late Lord Rayleigh that

$$V=\exp\left[-\pi(\pi\Delta v/\lambda c)^2\right] \tag{6}$$

and (4) has been expressed in terms of a defined width $\bar{a}$ by Michelson thus

$$V=\exp\left[-(2D)^2\pi^2\bar{a}^2/4\log 2\right] \tag{7}$$

From (6), Δ corresponding to a definite value of V was obtained and used in (7) for the same value of V, viz. one-half the maximum V. This gave for the width

$$W=(v\lambda/c)\left(2\sqrt{\pi\log 2}\right) \tag{8}$$

or where the freedom of the source is modified by density

$$W=(v\lambda/c)\left(2\sqrt{\pi\log 2}+\lambda/\rho\right) \tag{9}$$

where ρ is the mean free path of the source which was assumed to be atomic in character.

If the source is atomic in character and depends on the atomic weight and the temperature, then (9) may be expressed

$$W=\frac{\lambda}{\sqrt{m}}\,\frac{v_0}{c}\sqrt{\frac{T}{T_0}}\left(2\sqrt{\pi\log 2}+\lambda/\rho\right) \tag{10}$$

In the application of these formulas Michelson made use of a comparator to compare the visibility for any path difference with the maximum visibility which he defined as unity.

Now the development of instruments for investigation in which multiple reflections exist and the use of methods of getting absolute measurement of visibility make it desirable that all these formulas should be reconsidered. Further since x-ray spectra are in general formed from numerous series of multiple reflections from a large number of parallel planes it is important that visibility expressions should also be obtained for transmitted and reflected x-radiation. The Lummer and Gehrcke and the Fabry and Perot interferometers are typical instruments where multiple reflections exist and will serve the purpose of illustration. These instruments may have their reflecting surfaces silvered or clear.

The following notation will be used: I_0 is the intensity of the incident light; $\partial = 4\pi D/\lambda$ where $2D$ is the path difference between two successive rays and λ is the wave-length; p is the number of reflections; b and e are the reflection coefficients and c and f are the transmission coefficients at the two reflecting surfaces, respectively.

The expressions for the intensity are as follows:

1. For silvered plates; (a) with transmitted light

$$I = I_0 \frac{(1-b^{2p})^2 + 4b^{2p}\sin^2(p\partial/2)c^2f^2}{1-2b^2\cos\partial + b^4} \tag{11}$$

which becomes simplified, when p is large to

$$I = I_0 \frac{c^2f^2}{1-2b^2\cos\partial + b^4} \tag{12}$$

(b) with reflected light and p large

$$I = I_0 \left[b^2 + \frac{2bcfe(\cos\partial - e^2) + c^2f^2e^2}{1-2e^2\cos\partial + e^4} \right] \tag{13}$$

2. For unsilvered plates where the principle of reversibility may be assumed to hold if the angle of incidence is suitable and consequently $b = -e$ and $1 - b^2 = cf$;

(a) with transmitted light

$$I = I_0 \frac{(1-b^2)^2[(1-b^{2p})^2 + 4b^{2p}\sin^2(p\partial/2)]}{1-2b^2\cos\partial + b^4} \tag{14}$$

(b) with reflected light

$$I = I_0 - \text{expression (14)} \tag{15}$$

these also may become simplified obviously if p is large. Eq. (12) may be taken as the result applicable to the Fabry and Perot interferometer if a monochromatic source emitting radiation of wave-length λ is used. Eqs. (14) and (15) represent the result obtained for the Lummer and Gehrcke plate where $\partial = \mu 4\pi D/\lambda$ in which μ is the index of refraction of the plate.

As in the case of the simple interferometer, if the source is complex (12) becomes

$$I = \int_{-x}^{x} F(x) \frac{c^2 f^2}{1 - 2b^2 \cos \partial + b^4} dx \tag{16}$$

where $\partial = 4\pi D(\overline{m} + x)$, and may be written in the form

$$I = I_0 \frac{c^2 f^2}{1 - b^4} \int_{-x}^{x} F(x)(1 + 2b^2 \cos \partial + 2b^4 \cos 2\partial + \cdots) dx \tag{17}$$

In the same way (12) may be written

$$I = I_0 \frac{c^2 f^2}{1 - b^4} (1 + 2b^2 \cos \partial + 2b^4 \cos 2\partial + \cdots) \tag{18}$$

and if $2b^4 \cos 2\partial + 2b^6 \cos 3\partial + \cdots$ may be neglected, then in (18)

$$I \doteqdot I_0 \frac{c^2 f^2}{1 - b^4} (1 + 2b^2 \cos \partial) \tag{19}$$

and in (17)

$$I \doteqdot \int_{-x}^{x} \frac{c^2 f^2}{1 - b^4} F(x)(1 + 2b^2 \cos \partial) dx \tag{20}$$

and maximum and minimum intensities are given by

$$I = \frac{c^2 f^2}{1 - b^4} (P \pm 2b^2 \sqrt{C^2 + S^2})$$

where $P = \int F(x) dx$, $C = \int F(x) \cos 4\pi Dx \, dx$, and $S = \int F(x) \sin 4\pi Dx \, dx$ and since $S = 0$ for a symmetrical source

$$I_{max} = \frac{c^2 f^2}{1 - b^4} (P + 2b^2 C) \text{ and } I_{min} = \frac{c^2 f^2}{1 - b^4} (P - 2b^2 C)$$

and therefore

$$V = 2b^2 C/P = 2b^2 \exp\left[-(2D)^2\pi^2/h^2\right]. \tag{21}$$

Using Michelson's form of expression

$$V = 2b^2 \exp\left[-(2D)^2\pi^2\bar{a}^2/4 \log 2\right] \tag{22}$$

and Rayleigh's form of expression

$$V = 2b^2 \exp\left[-\pi(\pi\Delta v/\lambda c)^2\right] \tag{23}$$

an expression which was obtained by Saha[1] and by A. J. Dempster[2] by somewhat different developments.

It may be observed that by increasing b^2 a large path difference Δ may be used and the visibility will persist, for from (23)

$$\Delta = \frac{\lambda c}{\pi v}\sqrt{\frac{1}{\pi}\log\,(2b^2/V)} \tag{24}$$

High values of b^2 may be obtained by the use of reflection at air-metal surfaces, e. g. by silvering the plates in the Fabry and Perot and Lummer and Gehrcke interferometers. Since, however, if b^2 is high, absorption is correspondingly great, the transmitted light may be so weak as to preclude the observation of fringes unless the source is made correspondingly bright. Now however, as b^2 becomes large it is necessary to determine if the approximate formula (19) may be used for (18) and obtain a measure of the error introduced thereby.

Examples may be taken showing the magnitude of the error in the expression for the intensity introduced by the use of $(1+2b^2\cos\partial)$ for $(1-b^4)/(1-2b^2\cos\partial+b^4)$. If $b^2=0.8$ and ∂ be given values successively such that $\cos\partial=1$, $1/2b^2$, 0.5, 0, -0.5, $-1/2b^2$, it will be seen that there are errors of excess and deficiency that are of very considerable magnitude indeed.

The maxima of the fractional excess and deficiency errors may be shown to occur when $\cos\partial=1$ or $\cos\partial=b^2/4$. In the first case the error is

$$I_0\frac{c^2f^2}{1-b^4}\left(\frac{2b^4}{1-b^2}\right)$$

and the fractional error is $2b^4/(1-b^2)$. If $b^2=0.8$ the fractional error is 0.71. In a similar manner it may be shown that if $\cos\partial=b^2/4$ the fractional error, when $b^2=0.8$, is 3.8.

The error in the expression for the intensity will be zero if $\cos\partial=(b^2\pm\sqrt{b^4+8})/4$ unless at the same time $\cos\partial=(1+b^4)/2b^2$, in which

[1] Saha, Phys. Rev. **10**, 782 (1917)

[2] Dempster, Ann. d. Physik, **47**, 791 (1915).

case $b^2=1$. But b^2 is always less than 1 so that it is possible to find a value for ∂ for which the use of the approximate formulas under consideration is justified.

The error introduced in the expression for the visibility by the substitution of (20) for (17) may also be determined. If (19) be used for intensity the expression for visibility is given by Eq. (21). If (17) is used, then

$$I=\frac{c^2f^2}{1-b^4}(P+2b^2C\cos\theta+2b^4C_1\cos 2\theta+2b^6C_2\cos 3\theta+\cdots)$$

where P and C have the same significance as before, $\theta=4\pi Dm$, and

$$C_1=\int_{-x}^{x}F(x)\cos 8\pi Dx\,dx\ ,\quad \text{etc.}$$

Maxima exist where $\theta=2\pi$, 4π, etc., and minima where $\theta=\pi$, 3π, etc. The visibility is therefore given by

$$V=\frac{I_{max}-I_{min}}{I_{max}+I_{min}}=\frac{2b^2C+2b^6C_2+2b^{10}C_4+\cdots}{P+2b^4C_1+2b^8C_3+\cdots}\doteq\frac{2b^2C}{P(1+2b^4C_1/P)}$$

and if it is assumed that $F(x)$ is $\exp[-h^2x^2]$

$$V=\frac{2b^2C}{P\Big(1+2b^4\exp\left[-(4\pi D/h)^2\right]\Big)} \tag{25}$$

Now for low values of $2D$, $\exp[-(4\pi D/h)^2]\doteq 1$. It is therefore only when b^2 is a small fraction that (21) may be used to represent visibility in spectra of low order. If one obtains the ratio of the visibilities for two values of $2D$, as was done by Michelson, using, however, values of some magnitude well separated but well towards the limit of visibility, the error which is introduced by the use of (21) for (25) is greatly reduced. It is therefore evident that in the use of formula (21) for visibility and the formulas (22), (23) and (24) which are derived from (21) if the value of V is obtained by means of a silvered Fabry and Perot interferometer or any multiple reflecting instrument where the coefficient of reflection of the intensity is high that errors of some magnitude will appear in the results unless measurements are made where suitable path differences exist.

It should be observed also that the expressions for I_{max} and I_{min} were obtained by neglecting varitions of I with C and S. These were in general negligible for very narrow sources even for small differences in path. But for large differences in path they become quite important. However where visibility is measured near the limit, i. e. for very high values of $2D$, the variations in intensity will have both deficiency and excess values depending on $\cos 4\pi Dx$ and $\sin 4\pi Dx$ and the resultant integrated variation of I with C and S will be negligible.

From (6) and (24) it is clear that the path difference for any assigned value of the visibility is $\sqrt{1-\log 2b^2/\log V}$ times that for the same value of the visibility where multiple reflections do not exist. An increase or decrease exists according as b^2 is greater or lesser than .5. Further if .025 be taken as the limiting value of V the path difference corresponding to this value may be shown from (24) to be of the order of 4 cms. Path differences for the limiting visibility much larger than this are observable experimentally. In fact Fabry[3] obtained visible fringes for a path difference of 43 cm. and Lummer and Gehrcke are credited by H. A. Lorentz[4] with using path differences of over one meter. In these cases the sources which were used emitted intensely bright light.

In the interferometers of Fabry and Perot if the surfaces are not silvered and the light is incident normally $b^2 \doteqdot 0.05$ i. e. comparatively low. The maximal excess and deficiency errors in the use of the approximate expression for intensity are negligibly small and $V=2b^2C/P$ represents the visibility fairly well even for small path differences. In the interferometer of Lummer and Gehrcke the angle of incidence is very great and b^2 is very high even if the plate is unsilvered. If $i=85°$ $b^2>0.6$ and becomes larger rapidly as the angle of incidence increases and the errors which are associated with the intensity formula are of considerable magnitude. For a fairly thick plate the path difference is large and the expression for visibility is sufficiently accurate. Since, however, only a few orders of spectral fringes appear, it may be necessary to use two plates of the same glass but of different thicknesses in order to get the ratio of the visibilities for two suitable values of the path difference.

If interference is obtained by reflected light and $c^2f^2=(1-b^2)$ the expression for the intensity

$$I=I_0\left[1-\frac{(1-b^2)^2}{1-2b^2\cos\partial+b^4}\right]=I_0\left[\frac{2b^2(1-\cos\partial)}{1-2b^2\cos\partial+b^4}\right]$$

[3] Fabry, Comptes Rendus **128,** 1223 (1899).
[4] H. A. Lorentz, Phys. Zeits. **11,** 349 (1910).

Therefore

$$I=I_0\left[\frac{2b^2}{1-b^4}(1-\cos\partial)(1+2b^2\cos\partial+2b^4\cos 2\partial+++)\right]$$

and proceeding as in the case for transmitted light, $V=(1-b^2)C/P$.

The intensity formula for use with the Lummer and Gehrcke plate for transmitted light is more accurately that given in (14) which may be written

$$I=I_0\frac{(1-b^2)^2}{1-b^4}\left[(1-2b^{2p}\cos\partial+b^{4p})(1+2b^2\cos\partial+2b^4\cos 2\partial+)\right]$$

If the source has "width" the expression for visibility reduces to (21) if b is small and p is large. Similarly for reflected light

$$I=I_0\left[1-\frac{(1-b^2)^2}{1-b^4}\{(1-2b^{2p}\cos\partial+b^{4p})(1+2b^2\cos\partial+2b^4\cos 2\partial+++)\}\right]$$

and if b is small and p is large $V=(1-b^2)C/P$.

In the development of formulas (21) and (24) no account was taken of absorption between the planes of reflection or of loss by diffuse reflection or scattering. These factors may be ignored if the interferometers are used for the measurement of wave-lengths or for the direct observation of resolved spectral components. If, however, the visibility method is to be employed to determine the character of the source these should be taken into account. If a is the coefficient or loss of amplitude due to these causes the visibility formula (21) becomes

$$V=2b^2(1-a)^2C/P$$

With this modification of (21) formula (24) would become

$$v=\frac{\lambda c}{\pi\Delta}\sqrt{\frac{1}{\pi m}\log\frac{2b^2(1-a)^2}{V}} \tag{26}$$

if the atomic mass of the source is taken into consideration. Here a is the coefficient of loss of amplitude of the light for one passage of the light through the interferometer plate. Since the length of this path is given by $l=d/\cos\psi$ where d is the thickness of the plate and ψ is the angle of refraction or $l=\mu d/\sqrt{\mu^2-\sin^2 i}$ where μ is the index of refraction of the plate, it is seen that a varies with the angle of incidence. In the case of glass the values of both a and b are accurately known. From (26) the relationship between v and m may be obtained even for close components of spectral lines, since these components may first be obtained by the use of auxiliary spectroscopes of high resolving power.

If the definition of visibility chosen by the late Lord Rayleigh be used in connection with multiple reflection interferometers, viz., $V=I_{min}/I_{max}$ then the considerations of errors which are introduced by the use of the approximate formula above are still applicable. With this definition

$$V=(P-2b^2C)/(P+2b^2C)$$
$$=\frac{1-2b^2\exp\left[-(2D)^2\pi^2\bar{a}^2/4\log 2\right]}{1+2b^2\exp\left[-(2D)^2\pi^2\bar{a}^2/4\log 2\right]}$$

replaces (16) for transmitted light.
Or in terms of the velocity of the source

$$V=\frac{1-2b^2\ \exp\ \left[-\pi(\pi\Delta v/\lambda c)^2\right]}{1+2b^2\ \exp\ \left[-\pi(\pi\Delta v/\lambda c)^2\right]}$$

or where absorption is taken into account

$$V=\frac{1-2b^2(1-a)^2\ \exp\ \left[-\pi(\pi\Delta v/\lambda c)^2\right]}{1+2b^2(1-a)^2\ \exp\ \left[-\pi(\pi\Delta v/\lambda c)^2\right]}$$

which replaces (17).

The Lummer and Gehrcke plate offers some advantages for use with this method of analysis of a source of light.

(1) The angle of incidence is perfectly definite and the coefficients of reflection of intensity of light for air-glass surface at all angles of incidence are shown to a high degree of accuracy.

(2) The group of fringes obtained is for large differences of path and by choice of plates of suitable thickness groups of fringes with well graded visibility may be obtained and measurements may be made at many points suitable for obtaining an accurate visibility curve from which to deduce the conditions in the source.

(3) The plate may be crossed with another plate, with an echelon, or with other spectroscopes and visibility curves for the resolved components thereby obtained. Where components are not resolved they will make themselves evident by a periodic condition in the visibility curve for the fringes of the resolved components.

(4) To obtain measurements of the visibility experimentally fairly accurate results may be obtained from a photograph of the fringes if a uniform beam of light is sent through the photographic plate and the intensity measured for an unaffected part of the plate as well as for the various parts of the plate on which the fringes show. This may be

sufficiently accurate since it is the $\sqrt{\log(1/V)}$ which appears in the formulas (24) and (26). However, a greater degree of accuracy will probably be obtained by the use of a thermocouple, a photoelectric cell or a selenium or thalofide cell to measure the intensities at the maxima and minima.

(5) The existence of wide minima and narrow maxima of the transmitted light presents an opportunity of obtaining greater accuracy in the measurement of visibility than is the case with the simple interferometer since owing to the width of the minima they are not difficult to locate. Moreover, since I_{max} and I_{min} are small quantities, especially for large differences in path, it is imperative that the minima should be located very accurately if the measurement of I_{min} is to be at all accurate.

For a homogeneous source like that which gives the cadmium red line the visibility curve is simple and is obtainable with accuracy by the Michelson interferometer and may be readily interpreted.

For sources of more complex structure, e. g. those of hygroden and helium spectra, it is desirable not only to resolve the fine components but also to obtain the distribution of intensity in these components. Moreover it is now highly desirable to study the Zeeman and Stark effects not only in the production of components but also on the distribution of intensity in the components for use in the elucidation of spectral series and also for the determination of the character of the source. In this work multiple reflection interferometers should be of considerable assistance.

In the spectra of x-rays which are obtained from analyzing crystals of simple form only the first four or five orders are obtainable. Since, however, both the coefficients of absorption a and of reflection b are of very small magnitude, it is possible to develop expressions for intensity and visibility which may be reasonably accurate in these low orders and which may serve to compare crystal formations and perhaps give some information on the character of the source. In this connection there has been a very extensive consideration of the intensity of radiation from imperfect crystals by C. G. Darwin[5] and expressions for the intensity of the maxima for ideally monochromatic radiation when simple and almost perfect crystals are used have been obtained by K. W. Lamson.[6]

TORONTO,
August 5, 1925.

[5] C. G. Darwin, Phil. Mag. **43**, 800 (1922).
[6] K. W. Lawson, Phys. Rev. **17**, 624 (May, 1921).