A CORRESPONDENCE PRINCIPLE IN THE COMPTON EFFECT*

By G. Breit

Abstract

Correspondence theorem for frequencies.—It is well known that the frequency emitted by a hydrogen atom as it falls from one of its quantized states to another may be expressed as a mean value of the frequency of motion of the electron (or overtone thereof) when averaged in the proper manner over orbits intermediate between the initial and final states. In the present paper it is *proved* that, similarly, *the Compton shift is a properly taken mean of the classical Doppler shift.* The quantum frequency actually scattered is thus a properly taken average of the frequency which would be scattered on the classical theory as the electron is accelerated from its state of rest to its final recoil condition.

Correspondence principle for intensities.—In like manner, the amount of light scattered in various directions may be determined if it is *assumed* that the intensity in the quantum theory equals a proper average of the intensities scattered according to the classical theory. A comparison is made with observed data on the scattering of γ-rays.

The characteristic feature of the present paper is that *the corresponding classical electron is assumed to have the same direction of motion as the scattered quantum,* whereas an actual classical electron would from symmetry recoil straight forward in the direction of the incident beam, as in Compton's and Woo's theories of intensities. This new point of view eliminates the difficulty of a constant correction-factor which has been encountered by Compton and Woo in their explanation of intensity relations.

1. Introduction

Theories giving the dependence of the intensity of scattered radiation upon the angle have been given by Compton, Debye, and Woo. The theory of Debye is in poor agreement with experiment. The theories of Compton and Woo agree with experiment except for the occurrence of a factor $1+2\Lambda/\lambda_0$ which must be omitted in order to satisfy the experimental facts. The theory of Compton and Woo postulates that the intensity of the scattered radiation may be obtained by considering the classical scattering by an electron having the velocity $c\Lambda/(\Lambda+\lambda_0)$, where c is the velocity of light, λ_0 the wave-length of the incident radiation, and $\Lambda=h/m_0c$, h being Planck's constant and m_0 the rest mass of the electron. The reason for this assumption as given by Compton is that such an electron scatters radiation of the right frequency and that, therefore, it may also be expected to scatter it with the right intensity.

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A priori there is no objection to this reasoning. However, the calculations of the above authors show that their picture is not altogether satisfactory. The factor by which their result must be divided is just such as to give the classical intensity in the direction of the incident radiation. This fact in itself suggests another point of view.

The emitting atom and the scattering electron may be conceived as one complex atom. The scattering process is the emission process of that atom. In its initial state the electron is at rest, and in the final state the electron is in motion. According to Bohr's correspondence principle we should expect that both *the frequency of the emitted radiation and its intensity are given by proper averages of the corresponding quantities between the initial and final states.* It is clear without calculation that such a point of view leads us to expect the same intensity in the direction of incident radiation as is demanded by the classical theory, because in this case there is no recoil of the electron. We first show that the frequency actually scattered is a properly taken average of the frequency scattered classically in the initial and final states of the scattering electron. We then calculate the intensity of scattered radiation in the initial and final states and we show that a proper mean of these values is in agreement with experiment. Even though we apparently adhere in this to the prescriptions of the correspondence principle, we cannot altogether put this problem on the same footing as the application of the correspondence principle to the atomic frequency theorem of Bohr or to the selection rules. To do this it would be necessary to obtain a description of the possible atomic states by means of phase integrals.

2. Correspondence Theorem for Frequencies

The quantum theory of the Compton effect is governed by the relations

$$\begin{aligned} \frac{m_0 v}{\sqrt{1-\beta^2}} \cos \varphi + \frac{h\nu_\theta}{c} \cos \theta &= \frac{h\nu_0}{c} \\ -\frac{m_0 v}{\sqrt{1-\beta^2}} \sin \varphi + \frac{h\nu_\theta}{c} \sin \theta &= 0 \qquad \text{(I)} \\ m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) + h\nu_\theta &= h\nu_0 \end{aligned}$$

where the electron is supposed to be at rest before scattering. In these relations $h\nu_0$ and $h\nu_\theta$ are the magnitudes of the original and scattered quanta, respectively; θ, the angle between their directions; φ, the angle between the velocity v of the electron after scattering and the incident radiation; and β is v/c. As is known, when these equations are solved for ν_θ one finds

$$\nu_\theta = \frac{\nu_0}{1+\frac{\nu_0}{N}(1-\cos\theta)} \tag{II}$$

where

$$N = \frac{m_0 c^2}{h}\,. \tag{II'}$$

In order to determine the frequency which would be scattered by the electron in its final recoil state we must consider first of all what the incident frequency appears to be to an observer on the moving electron. We must next compute how the scattered radiation is transformed to the stationary frame of reference. The system of the recoil electron we refer to as K' and to that of the stationary one we refer as K. In the

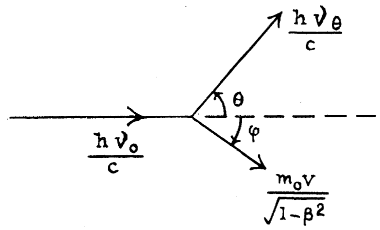

Fig. 1.

equations of the Lorentz transformation we take the X axis along the direction of recoil. The fundamental relation is

$$\frac{\nu}{\nu'} = \frac{1+\beta\cos\psi'}{\sqrt{1-\beta^2}} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\psi} \tag{III}$$

where Ψ is the angle with the X axis referred to K, Ψ' is the same angle referred to K' and ν, ν' are the frequencies of the radiation referred to K, K' respectively. We have, therefore, for the frequency scattered by the recoil electron on the basis of classical relativity

$$\frac{\nu_{\theta c}}{\nu_0} = \frac{1-\beta\cos\varphi}{1-\beta\cos(\theta+\varphi)} \tag{1}$$

It follows from (1) that

$$\beta\cos(\theta+\varphi) = \frac{\nu_0\cos\theta-\nu_\theta}{N+\nu_0-\nu_\theta} \tag{2}$$

$$\beta\cos\varphi = \frac{\nu_0-\nu_\theta\cos\theta}{N+\nu_0-\nu_\theta} \tag{3}$$

Whence using (II) we have

$$\frac{\nu_{\theta c}}{\nu_0}=\frac{N-\nu_\theta(1-\cos\theta)}{N+\nu_0(1-\cos\theta)}=\frac{1}{\left(1+\frac{\nu_0}{N}(1-\cos\theta)\right)^2}=\left(\frac{\nu_\theta}{\nu_0}\right)^2 \tag{4}$$

This means that

$$\log \nu_\theta=\frac{1}{2}[\log \nu_0+\log \nu_{\theta c}] \tag{5}$$

In other words the logarithm of the scattered frequency is the arithmetic mean of the frequency which would be scattered in the initial and in the final states on the basis of the classical theory. If $(\nu_\theta-\nu_0)/\nu_0<<1$ we also have approximately $\nu_\theta=\frac{1}{2}(\nu_0+\nu_{\theta c})$ as may be shown by a simple consideration of the vector diagram.

It is hardly necessary to point out that Eq. (5) is not altogether analogous to Bohr's Correspondence Theorem, for in that theorem attention is paid to the classical frequency in every intermediate step. In (5), however, we only use the mean of the end values of the classical frequency. It is perhaps dangerous to try to use a method entirely analogous to Bohr's because the Compton shift does not exist on the classical theory.

In a purely speculative manner, however, we may try to look at the problem from the following point of view. Since the quantum emission process is entirely unidirectional, the "corresponding" classical picture we take to be also entirely unidirectional. The classical plane wave is incident on the scattering electron and gradually imparts to it its final momentum. At any intermediate stage in this scattering process the electron is executing a recoil motion which, however, is smaller than the final one. We shall attempt to justify presently the assumption that the classically scattered frequency must be averaged by regarding h in (4) as a variable having a range between 0 and 6.55×10^{-27}. The weights of equal intervals in this range we consider as equal. Letting

$$a=\frac{1-\cos\theta}{m_0c^2}, \tag{6}$$

the average frequency emitted is according to (5)

$$\bar{\nu}_{cl}=\frac{1}{h}\int_0^h\frac{\nu_0 dx}{(1+a\nu_0x)^2}=\frac{\nu_0}{1+a\nu_0h}=\nu_\theta, \tag{7}$$

as is seen from (II). This means that the scattered frequency is a properly taken average of the classically scattered frequency.

The justification for taking the average in the manner indicated by (7) may be seen in the fact that Bohr's correspondence theorem assigns equal weights to equal elements of length on the straight line in n-dimensional space joining $(J_1, J_2, \ldots .)$ to $(J_1+n_1h, J_2+n_2h, \ldots .)$. Considering for simplicity the one-dimensional case, the frequency emitted is $n(\partial W/\partial J)$ where W is the energy and where the average is taken between J and $J+nh$, equal weights being attached to all elements dJ. It is clear that equal weights are attached to all intervals corresponding to equal energy intervals as long as W increases uniformly with J. In general W does not increase uniformly with J, this being rigorously the case only for a linear oscillator. The averaging assumption included in (7) means that equal values of ΔW referred to the original emitting atom correspond to equal weights in the scattering mechanism. To this extent we suppose the emitting atom to be equivalent to a linear resonator at a large distance.

We think, therefore, of the emitting atom and the scattering electron as one complex atom. We have shown by means of II, (4), (7) that the frequency ν_0 which is actually scattered is the average of the frequency which would be scattered on the classical theory provided the *energy* extracted from the quantum incident on the electron determines the linear scale for averaging.

3. Correspondence Principle for Intensities

We may now pass on to the intensity relations. We need first of all to know what intensity the classical theory would give for the radiation scattered in the final recoil state. We adopt as a criterion the intensity scattered per unit solid angle in the direction θ of the actually scattered radiation. We shall simplify the calculation by showing that in the frame of reference of the recoil electron the incident and recoil quanta make with each other the same angle θ as in the frame of reference of the electron before scattering. This fact, as well as others useful in connection with the present problem, can be obtained most simply by using Pauli's very elegant treatment of the interaction between quanta and electrons.[1] Pauli denotes the energy of the quantum by E, its momentum by Γ, the energy of the electron by U and its momentum by G. He then shows that if a frame of reference be chosen such that in it Γ and G are equal and opposite before impact, then after impact Γ and G are equal and opposite,

[1] W. Pauli, Zeits. f. Physik **18**, 272 (1923).

[2] Laue, Die Relativitätstheorie, Vol. I, p. 124, formula 177; Vieweg, 1921.

and, further, the new values of Γ and G as well as the new values of E and U are all equal respectively to their old values. The only change in that frame which occurs at collision is a change in the direction of all the vectors without change in absolute magnitude. It is clear from this fact and the symmetry of Fig. 2 that a Lorentz transformation to a frame of reference moving with velocity v along G leads to the same result for the angle between Γ and Γ' as the Lorentz transformation to a frame moving with the same velocity v along G'. This proves that in the frame of the recoiling electron the angle between the incident and the reflected quantum is the same angle θ as that formed in the frame of the electron before collision.

It further proves that in the frame of the recoiling electron the reflected quantum has the magnitude $h\nu_0$ and the incident quantum has the magnitude $h\nu_\theta$, while in the frame of the electron before collision the reflected

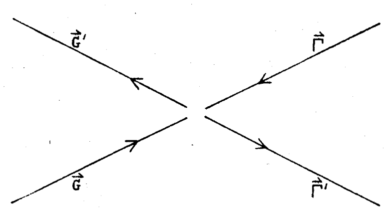

Fig. 2.

quantum has the magnitude $h\nu_\theta$ and the incident one is $h\nu_0$. (The magnitude of the quantum is the same if referred to the electron before and after collision.)

Suppose now in K' (the frame of the recoil electron) the amount of energy radiated into the angular domain $d\theta', d\varphi'$ is

$$f(\theta',\varphi') \sin\theta' \, d\theta' \, d\varphi' \tag{8}$$

then in the stationary frame of reference K this radiation appears to have an energy content greater than the above in the ratio $\nu^2:\nu'^2$ so that an amount of energy

$$\frac{\nu^2}{\nu'^2} f(\theta',\varphi') \sin\theta' \, d\theta' \, d\varphi' \tag{9}$$

is radiated into the corresponding angular domain $d\theta$, $d\varphi$.[2]

Since

$$\sin\theta' \, d\theta' \, d\varphi' = \frac{\nu^2}{\nu'^2} \sin\theta \, d\theta \, d\varphi \tag{10}$$

we have an amount of energy

$$\left(\frac{\nu}{\nu'}\right)^4 f(\theta',\varphi') \tag{11}$$

per unit solid angle in K. In this formula it is important to remember that ν/ν' refers to one particular direction of radiation, namely, that of the recoiling quantum. Therefore, here we must write

$$\frac{\nu}{\nu'} = \frac{\nu_\theta}{\nu_0} \tag{12}$$

making the energy distribution

$$\left(\frac{\nu_\theta}{\nu_0}\right)^4 f(\theta',\varphi') \tag{13}$$

The expression for $f(\theta', \varphi')$ is obtained by considering what becomes of the incident radiation when it is transformed to the frame of the recoil electron. If the intensity of this radiation in K is I_0 and if in K' it is I_0' the value of $f(\theta', \varphi')$ is given by the Thomson expression

$$f(\theta',\phi') = CI_0'\frac{1+\cos^2\theta}{2} \tag{14}$$

where

$$C = \frac{e^4}{c^4 m^2}$$

because the angle between the scattered and incident radiation we showed to be θ in K'. Further

$$I_0' = \left(\frac{\hat{\nu}'}{\hat{\nu}}\right)^2 I_0 \tag{15}$$

where $\hat{\nu}'/\hat{\nu}$ is the quotient of the frequencies computed by the Lorentz transformation for a ray in the direction of the incident quantum. Since in K this quantum is $h\nu_0$ and in K' it is $h\nu_\theta$

$$\frac{\hat{\nu}'}{\hat{\nu}} = \frac{\nu_\theta}{\nu_0} \tag{16}$$

and the amount of energy scattered per unit time by the recoil electron from a beam of radiation of intensity I_0 is

$$\mathscr{I}_\theta = CI_0\left(\frac{\nu_\theta}{\nu_0}\right)^6 \frac{1+\cos^2\theta}{2} = \frac{CI_0}{\left[1+\frac{\nu_0}{N}(1-\cos\theta)\right]^6}\,\frac{1+\cos^2\theta}{2} \tag{17}$$

where use has been made of II. We write this also as

$$\mathcal{J}_\theta = I_{\theta c}\left[1+\frac{\nu_0}{N}(1-\cos\theta)\right]^{-6} \tag{18}$$

where $I_{\theta c}$ is the classical expression of J. J. Thomson, which does not take into account the Compton shift.

The above expression $\mathcal{J}$ represents the rate at which energy is emitted by the electron during the scattering process. It does not represent, however, the amount of energy scattered in a given direction because if a train of finite length is incident the length of the train reemitted in the direction θ is not necessarily the same. Since the number of waves in the incident and the scattered rays is the same, the ratio of the two lengths is the same as the ratio of the wave-lengths as measured in K or the inverse ratio of the frequencies. Hence the amount of energy scattered is, using (4),

$$I_\theta = \frac{\nu_0}{\nu_{\theta c}}\mathcal{J}_\theta = \left(\frac{\nu_0}{\nu_\theta}\right)^2 \mathcal{J}_\theta\,. \tag{19}$$

We have, therefore, substituting (18),

$$I_\theta = I_{\theta c}\left[1+\frac{\nu_0}{N}(1-\cos\theta)\right]^{-4}. \tag{20}$$

Expression (20) represents the average rate at which energy is scattered by the electron if a number of trains of waves are incident, while (18) gives the rate of scattering during the incidence of one of the trains. Which of these expressions one must use depends upon the application. The important fact is that both I_θ and $\mathcal{J}_\theta$ are less than $I_{\theta c}[1+(\nu_0/N)(1-\cos\theta)]^{-3}$ and that the latter expression represents satisfactorily the experimental results. Expression (20) appears to be the correct one to use if the total amount of energy scattered is taken as the criterion. From that point of view very much more weight should be attached to the properties of the recoil electron than to those of the stationary one in calculations of intensity because the difference between $(1+y)^{-3}$ and $(1+y)^{-4}$ is smaller than between 1 and $(1+y)^{-3}$. Thus for small y

$$1-(1+y)^{-3} \cong 3y \text{ and } (1+y)^{-3}-(1+y)^{-4} \cong y$$

while for large y

$$1-(1+y)^{-3} \cong 1 \text{ and } (1+y)^{-3}-(1+y)^{-4} \cong 0$$

In justification of using expression (18), however, one may say that it deals with the stationary rate at which energy is scattered while a plane wave is continuously incident on the recoiling electron. Therefore, if the emitting atom and the recoiling electron are considered as a unit which is constantly radiating, expression (18) is the right one to use. It is a curious fact that the transition from (18) to the empirical formula

$$\bar{I}_\theta = I_{\theta c}\left[1+\frac{\nu_0}{N}(1-\cos\theta)\right]^{-3} \tag{21}$$

may be thought of as very analogous to the transition from $\nu_{\theta c}$ to ν_θ represented by (4). We may say in fact that (21) is obtained from (18) by using the same intermediate value of the averaging parameter x (see Eq. 7) as that which makes $\nu_{\theta c}$ go into ν_θ. In fact, this value is such that

$$\frac{1}{(1+a\nu_0 x)^2} = \frac{1}{1+a\nu_0 h}$$

and therefore

$$\frac{I_{\theta c}}{(1+a\nu_0 x)^6} = \frac{I_{\theta c}}{(1+a\nu_0 h)^3}$$

The meaning of this method of averaging is that the radiating ability of the atomic system is considered in such an intermediate stage that the classically computed frequency is the actual emitted frequency. It is of course known from the phenomena of dispersion that the reaction of an atomic system to a light wave depends on the properties of "virtual oscillators," and that the frequency of these oscillators is the quantum transition frequency. We attempt above to interpret this as meaning that approximately at least this is due to the action of an intermediate atomic system which is determined by that stage of the transition in which Bohr's corresponding frequency is equal to the emitted frequency.

It would be desirable, of course, to compare this method of averaging with others also in other cases. In view of the uncertainty of the atomic models, we cannot make a definite statement. However, it seems that the method used here is not worse than the customary ways of averaging and that it allows one to understand why in Lande's g-formula $j(j+1)$ takes place of j^2. A reason for this may be seen in the fact that different values of j correspond to different k's in the relativistic separation formula which is obeyed empirically with j substituted for k. This means that the dependence of the energy level on j is given by a term of the form A/n^3j (A a constant) and therefore the "corresponding" frequency is such that the value of j which corresponds to it is given by

$$\frac{1}{{}_c{}^2} = \frac{1}{j} - \frac{1}{j+1}$$

This means that $j_c{}^2 = j(j+1)$. Our assumption is here that the interaction between parts of an atom is not determined by the actual state of the atom but by a state intermediate between the state and a neighboring one, this intermediate state being chosen in such a way that the orbital frequency is equal to the frequency which would be emitted in the transition between the two states.

4. Comparison with Experiment

A comparison with experiment cannot be carried out satisfactorily at present on account of the scarcity of experimental material. The measurements on the Compton effect for x-rays are not covered by the above theory on account of the influence of atomic structure on the phenomenon of scattering. This question is treated in various papers by Jauncey.[3] We must rely, therefore, on the measurements of A. H. Compton[4] in the range of hard γ-rays. In these the wave-length of the scattered γ-rays has not been determined, and it is not known whether all of the scattered radiation was modified. Compton determined intensities of scattering in various directions relatively to scattering at 90°. He also found that $I_{90°}/I_0 = 0.037$. In the adjoining table the first column gives the angle of scattering, the second the average of Compton's experimental relative values, the third these values multiplied by 0.037, and the fourth values computed from our formula (21). The numbers in the fourth column are higher than those in the third. The fifth column gives ratios of numbers in the fourth to those in the third. The average of these is 1.3. It is striking that this is the ratio of 0.048 to 0.037. If, therefore, on account of a systematic error Compton's determination for 90° should be in error to that degree, formula (21) may be considered as representing experiments satisfactorily.

TABLE II

θ	Average Relative Value	$0.037\times$ Av. Rel. Value	Theoret. $I_{\theta c}(\frac{\nu_\theta^3}{\nu_0})$	$\frac{\text{Theor.}}{\text{Exp.}}$
30°	10.5	0.389	0.555	1.4
45	6.87	0.254	0.305	1.2
60	4.3	0.159	0.156	1.0
75	1.97	0.073	0.080	1.1
90	1.0	0.037	0.0476	1.3
120	0.7	0.0259	0.0288	1.1
135	0.45	0.0166	0.0267	1.6
150	0.47	0.0172	0.0260	1.5

[3] Jauncey, Phys. Rev. **25**, 314, 723 (1925); Phil. Mag. **49**, 427 (1925).
[4] Compton, Phil. Mag. **46**, 897 (1923).

It must be emphasized, however, that a definite conclusion may not be drawn at present because it is not known whether the unmodified radiation was entirely absent in Compton's experiments. Even a small amount of it would change the interpretation of the experiments because

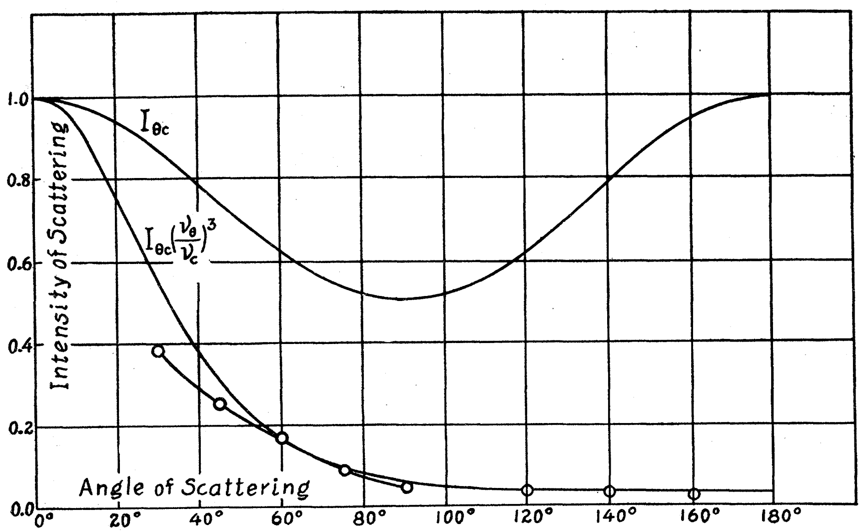

Fig. 3.

its dependence on the angle of scattering is very different from that given by (21). In Fig. 3 the J. J. Thomson formula is represented by $I_{\theta c}$, our (21) by $I_{\theta c}(\nu_\theta/\nu_0)^3$ and Compton's experimental values by the small circles.

Department of Terrestrial Magnetism,
Carnegie Institution of Washington,
Washington, D. C.
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