# REFLECTION OF RADIATION FROM A FINITE NUMBER OF EQUALLY SPACED PARALLEL PLANES

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ABSTRACT

Equations are derived for the fractions of the incident energy which are absorbed, reflected and transmitted by $n+1$ equally spaced parallel planes, taking account of all possible internal reflections, in terms of the corresponding fractions for a single plane, $a$, $r$ and $t$ respectively. When $a$ and $r$ are very small, as in the case of x-rays incident on a crystal surface, they may be computed from measurements of the reflected fraction $R_N$ for $N$ large, and of the transmitted fraction $T_n$, for $n$ small.

1. *Introduction.* In dealing with the reflection, transmission and absorption of radiation in a crystalline medium, we assume the molecules of the medium to be situated in a number of equally spaced parallel planes (and to have the familiar lattice arrangement). We assume that when a ray of intensity $I$ strikes one of these planes, a part of it, which we denote by $rI$, is reflected, another part $tI$ transmitted, and the remaining part $aI$ absorbed, so that

$$r+t+a=1\,. \tag{1}$$

These constants $r$, $t$ and $a$ are assumed to be the same for all planes.

Let us consider $n+1$ of these planes, numbered 1, 2, . . . , $n$, $n+1$, and a ray of intensity $I$ striking the first plane; taking into account all possible modes of reflection at one or more of the planes, a certain part $T_nI$ will be transmitted across the $n+1$ plane, another part $R_nI$ will be reflected, that is, will emerge on the same side of the first plane as the incident ray $I$, and the remaining part $A_nI$ is absorbed by the $n+1$ planes; we evidently have

$$R_n+T_n+A_n=1\,. \tag{2}$$

Our problem is to express $R_n$ and $T_n$ (and hence, by (2), also $A_n$) in terms of $r$, $t$ and $n$, and it is obviously permissible to assume $I=1$ in the following discussion.

2. *Determination of $R_n$ and $T_n$ for $n=0$ and $n=1$.* In the case of a single plane ($n=0$) it follows at once from the definitions that

$$R_0=r,\qquad T_0=t\,. \tag{3}$$

In the case of two planes, numbered 1 and 2, a ray of unit intensity striking plane 1 gives a reflected ray $r$ and a ray $t$ penetrating plane 1, and when the latter ray strikes plane 2, it gives a transmitted ray $t \cdot t = t^2$ and a ray $tr$ reflected from plane 2 toward plane 1. At the latter plane, this $tr$ gives a ray $t \cdot tr = t^2r$ penetrating the plane toward the side of the incident ray, so that $t^2r$ is a component of the total reflected ray $R_1$, while a ray $r \cdot tr = tr^2$ is reflected toward plane 2. This $tr^2$ penetrates plane 2 in the amount $t \cdot tr^2 = t^2r^2$, which forms a component of the total transmitted ray $T_1$, while a ray $r \cdot tr^2 = tr^3$ is reflected toward plane 1, giving at the latter a component $t \cdot tr^3 = t^2r^3$ of $R_1$ and a ray $r \cdot tr^3 = tr^4$ reflected toward plane 2, where it gives a component of $T_1$ equal to $t \cdot tr^4 = t^2r^4$ and a ray $r \cdot tr^4 = tr^5$ reflected toward plane 1. Continuing in this manner, we find for the sums of the components of the transmitted and reflected rays the expressions

$$T_1 = t^2 + t^2r^2 + t^2r^4 + \cdots ,$$
$$R_1 = r + t^2r + t^2r^3 + t^2r^5 + \cdots ,$$

or summing the geometric series to the right

$$T_1 = t^2/(1-r^2) , \qquad R_1 = r + t^2r/(1-r^2) . \tag{4}$$

3. *The two difference equations for $R_n$ and $T_n$.* Proceeding to the general case, our first step consists in setting up two difference equations (that is, recurrent formulas) connecting $T_{n+1}$ and $R_{n+1}$ with $T_n$ and $R_n$. Consider the $n+2$ planes numbered $1, 2, \ldots, n+1, n+2$; these we divide in two layers, the first layer consisting of the planes numbered $1, 2, \ldots, n+1$, and the second of those numbered $n+1$ and $n+2$. The plane numbered $n+1$, common to both layers, is called the boundary plane between them. It will be convenient to separate the reflected ray $R_n$ in the ray $r$ reflected directly at plane 1 and the remainder $R'_n$ reflected at other planes after penetrating into the layer, so that

$$R_n = r + R_n' , \qquad R_{n+1} = r + R'_{n+1} , \tag{5}$$

and (3) and (4) show that

$$R_0' = 0 , \qquad R_1' = t^2r/(1-r^2) . \tag{6}$$

Now consider a ray of unit intensity striking the first layer at plane 1; the ray $T_{n+1}$ transmitted through both layers is made up of components which we classify according to the number of times they cross the boundary plane between the first layer and the second. The component crossing the boundary plane once is obtained thus: let $t_n'$ be a ray which is transmitted into the second layer after a certain number of transmissions and reflections in the planes of the first layer. Since the intensity of a transmitted ray is multiplied by $t$ in the passage through a plane, it follows that the intensity of $t_n'$ immediately before passing through the boundary plane is $t_n' \cdot t^{-1}$. Passing through the second layer, which consists of two planes, the intensity becomes $t_n't^{-1} \cdot T_1$, and our component is the sum of $t_n't^{-1} \cdot T_1$ extended over all partial rays $t'_n$; the sum of the latter being $T_n$, the component of $T_{n+1}$ arising from all partial rays crossing the boundary plane once is $T_nt^{-1} \cdot T_1$. Any ray transmitted through both layers must evidently cross the boundary plane an odd number of times. Hence the next component of $T_{n+1}$ crosses the boundary plane three times and is obtained by taking the component $T_nt^{-1}$ arriving at the boundary plane and multiplying it by $R_1't^{-1}$, thus forming a component which has crossed the boundary plane and been reflected in the interior of the second layer, but has not yet crossed the boundary plane the second time. Observe that we multiply by $R_1't^{-1}$ and not by $R_1t^{-1}=(r+R_1')t^{-1}$, since we are not concerned with those rays that are reflected at the boundary plane without penetrating into the second layer. This component $T_nt^{-1} \cdot R_1't^{-1}$ we multiply by $R_n't^{-1}$, thus obtaining a component which has crossed the boundary plane twice and been reflected in the interior of the first layer, but has not yet crossed the boundary plane for the third time. The third crossing of the boundary plane and transmission through the second layer multiplies our component by $T_1$, so that the component of $T_{n+1}$ resulting from crossing the boundary plane three times will be

$$T_nt^{-1} \cdot R_1't^{-1} \cdot R_n't^{-1} \cdot T_1 ,$$

where each multiplication point stands at a crossing of the boundary plane. Similarly, the component which crosses the boundary plane five times is

$$T_nt^{-1} \cdot R_1't^{-1} \cdot R_n't^{-1} \cdot R_1't^{-1} \cdot R_n't^{-1} \cdot T_1 ,$$

and so on. Finally $T_{n+1}$ is the sum of all these components, so that

$$T_{n+1}=T_nt^{-1}T_1[1+R_1'R_n't^{-2}+(R_1'R_n't^{-2})^2+ \cdot \cdot \cdot ] ,$$

or by (4) and (6)

$$T_{n+1}=\frac{tT_n}{1-r^2}\left[1+\frac{rR_n'}{1-r^2}+\left(\frac{rR_n'}{1-r^2}\right)^2+\cdot \cdot \cdot\right].$$

The geometric series to the right converges, since $R_n<1$ by (2), so that $R_n'<1-r$ by (5) and $rR_n'/(1-r^2)\ <r/(1+r)<1$; summing the series, we obtain

$$T_{n+1}=\frac{t}{1-r^2}\cdot\frac{T_n}{1-\dfrac{r}{1-r^2}R_n'}\ . \tag{7}$$

Turning our attention to the reflected ray $R_{n+1}$ and reasoning in exactly the same manner, we have first the component $R_n$ which does not cross the boundary plane, then the component crossing the boundary plane twice which is $T_nt^{-1}\cdot R_1't^{-1}\cdot T_n$, next the component $T_nt^{-1}\cdot R_1't^{-1}\cdot R_n't^{-1}\cdot R_1't^{-1}\cdot T_n$ crossing the boundary plane four times, and so on, their sum being

$$R_{n+1}=R_n+T_n^2R_1't^{-2}[1+R_1'R_n't^{-2}+(R_1'R_n't^{-2})^2+\cdot\cdot\cdot]\ ,$$

whence, replacing $R_n$ and $R_{n+1}$ by $r+R_n'$ and $r+R_{n+1}'$ according to (6) and observing that the series to the right is the same as in the expression for $T_{n+1}$,

$$R'_{n+1}=R_n'+\frac{r}{1-r^2}\cdot\frac{T_n^2}{1-\dfrac{r}{1-r^2}R_n'}\ . \tag{8}$$

Equations (7) and (8) are the desired difference equations.

4. *Solution of the difference equations for $R_n$ and $T_n$.* We now introduce the notation

$$\rho_n=1-\frac{r}{1-r^2}R_n'\ , \tag{9}$$

whence, by (6),

$$\rho_0=1\ ,\qquad \rho_1=1-\frac{t^2r^2}{(1-r^2)^2}\ ; \tag{10}$$

since $0<R_n'<1-r$, it follows from (9) that

$$\frac{1}{1+r}<\rho_n<1\ . \tag{11}$$

Introducing $\rho_n$ in Eqs. (7) and (8), these take the form

$$T_{n+1}=\frac{t}{1-r^2}\cdot\frac{T_n}{\rho_n}\ , \tag{12}$$

$$\rho_n-\rho_{n+1}=\frac{r^2}{(1-r^2)^2}\cdot\frac{T_n^2}{\rho_n}\ , \tag{13}$$

From these, we form a difference equation of the second order for $\rho_n$ by eliminating $T_n$ in the following manner: in (13), replace $n$ by $n+1$ and in the result, substitute the value of $T_{n+1}$ from (12), whence

$$\rho_{n+1}-\rho_{n+2}=\frac{r^2}{(1-r^2)^2}\cdot\frac{T^2{}_{n+1}}{\rho_{n+1}}=\frac{t^2r^2}{(1-r^2)^4}\cdot\frac{T_n{}^2}{\rho_{n+1}\rho_n{}^2}\,.$$

In the last expression, we substitute the value of $T_n{}^2$ taken from (13) obtaining

$$\rho_{n+1}-\rho_{n+2}=\frac{t^2}{(1-r^2)^2}\cdot\frac{\rho_n-\rho_{n+1}}{\rho_{n+1}\rho_n}=\frac{t^2}{(1-r^2)^2}\left(\frac{1}{\rho_{n+1}}-\frac{1}{\rho_n}\right)\,.$$

Transposing terms, we obtain the difference equation of the second order

$$\rho_{n+2}+\frac{t^2}{(1-r^2)^2}\cdot\frac{1}{\rho_{n+1}}=\rho_{n+1}+\frac{t^2}{(1-r^2)^2}\cdot\frac{1}{\rho_n}\,, \tag{14}$$

which shows that the expression to the right remains unchanged in value when $n$ is increased by unity. Starting with $n=0$, and increasing it by a unit at a time, it follows that the right hand member of (14) has the same value for any $n$ as for $n=0$, so that

$$\rho_{n+1}+\frac{t^2}{(1-r^2)^2}\cdot\frac{1}{\rho_n}=\rho_1+\frac{t^2}{(1-r^2)^2}\cdot\frac{1}{\rho_0}\,,$$

and calculating the expression to the right by means of (10), we find

$$\rho_{n+1}+\frac{t^2}{(1-r^2)^2}\cdot\frac{1}{\rho_n}=1+\frac{t^2}{1-r^2}\,. \tag{15}$$

This difference equation of the first order is thus a first integral of (14), and is in its turn reduced to a directly integrable linear difference equation in the following manner. From (11) it is seen that $\rho_n$ is positive, and from (13) that $\rho_n>\rho_{n+1}$. The sequence $\rho_0, \rho_1, \ldots, \rho_n, \ldots$ is thus decreasing toward a limit $\rho$:

$$\lim_{n\to\infty}\rho_n=\rho\,, \tag{16}$$

and (11) shows that

$$\frac{1}{1+r}\leqq\rho<1\,. \tag{17}$$

From (15) and (16) it is seen that $\rho$ satisfies the equation

$$\rho+\frac{t^2}{(1-r^2)^2}\cdot\frac{1}{\rho}=1+\frac{t^2}{1-r^2}\,, \tag{18}$$

and solving this quadratic in $\rho$,

$$2(1-r^2)\rho=1+t^2-r^2\pm\sqrt{(1+t^2-r^2)^2-4t^2}\,.$$

To determine the sign of the radical, we observe that $t<1-r$ by (1), so that (17) gives

$$>\frac{t}{1-r^2} \; ; \tag{19}$$

on the other hand, the product of the roots of (18) equals $t^2/(1-r^2)^2$, and consequently (19) shows that the second root of (18) is less than $t/(1-r^2)$. Hence $\rho$ is the greater of the two roots of (18) and corresponds to the plus sign before the radical. Resolving the expression under the radical sign into factors, we thus obtain

$$2(1-r^2)\rho=1+t^2-r^2+\sqrt{(1+t+r)(1+t-r)(1-t+r)(1-t-r)} \cdot \tag{20}$$

As $n\rightarrow\infty$, $R_n$ tends toward a limit $R$, which equals $r+(1-r^2)(1-\rho)/r$ by (5) and (9), so that, according to (20),

$$R=\frac{1}{2r}\left[1-t^2+r^2-\sqrt{(1+t+r)(1+t-r)(1-t+r)(1-t-r)}\,\right] \cdot \tag{21}$$

An expression for this $R$, which is the reflection in an infinite number of equally spaced parallel planes, was found in an entirely different way by K. W. Lamson[1] in form of an infinite series, the sum of which was shown by the author to be (20).[2]

Returning to Eq. (15), we subtract from it Eq. (18), obtaining

$$\rho_{n+1}-\rho=\frac{t^2}{(1-r^2)^2}\left(\frac{1}{\rho}-\frac{1}{\rho_n}\right) \; ;$$

making the substitution

$$\rho_n=\rho\left(1+\frac{1}{\omega_n}\right) \; , \tag{22}$$

the preceding equation becomes

$$\frac{\rho}{\omega_{n+1}}=\frac{t^2}{(1-r^2)^2}\cdot\frac{1}{\rho}\cdot\frac{1}{\omega_n+1} \; ,$$

and writing

$$\mu=\frac{1-r^2}{t}\rho \; , \tag{23}$$

[1] K. W. Lamson, Physical Review, N. S. **17**, 624 (1921), Eqs. (2) and (3).

[2] Lamson, l.c. Eq. (5). This formula contains two misprints: the minus sign between $t^2$ and $r^2$ should read plus, and a minus sign should be inserted between $r^2$ and the square root. For the proof, see T. H. Gronwall, Annals of Mathematics, ser. 2, **23**, 282 (1922).

so that $\mu > 1$ by (19), this reduces to the linear difference equation in $\omega_n$:

$$\omega_{n+1} = \mu^2(\omega_n + 1) \ . \tag{24}$$

This may be written

$$\omega_{n+1} + \frac{\mu^2}{\mu^2-1} = \mu^2 \left(\omega_n + \frac{\mu^2}{\mu^2-1}\right) ,$$

and this last equation is integrated immediately, giving

$$\omega_n + \frac{\mu^2}{\mu^2-1} = \mu^{2n}\left(\omega_0 + \frac{\mu^2}{\mu^2-1}\right) .$$

Since $\rho_0 = 1$ by (10), we have $\omega_0 = \rho/(1-\rho)$ by (22), so that finally

$$\omega_n + \frac{\mu^2}{\mu^2-1} = \left(\frac{\rho}{1-\rho} + \frac{\mu^2}{\mu^2-1}\right)\mu^{2n}. \tag{25}$$

Having thus determined $\omega_n$ explicitly as a function of $r$, $t$ and $n$, we obtain $R_n$ by the combination of (5), (9), and (22):

$$R_n = \frac{1}{r} - (1-r^2)\rho \left(1 + \frac{1}{\omega_n}\right) . \tag{26}$$

For $T_n$, we find from (13) and (22)

$$\frac{r^2}{(1-r^2)^2} T_n^2 = \rho^2 \cdot \frac{1+\omega_n}{\omega_n} \cdot \frac{\omega_{n+1}-\omega_n}{\omega_n \omega_{n+1}} ,$$

or replacing $\omega_{n+1}$ by its expression from (24) and using (23),

$$T_n^2 = \frac{t^2}{r^2} \cdot \frac{(\mu^2-1)\omega_n + \mu^2}{\omega_n^2} . \tag{27}$$

5. *Approximate formulas for calculating $r$ and $a$ from experimental values of $R$ and $T_n$.* In experimental work, it is possible to determine the reflection in a section of the material of sufficient thickness to allow us to regard the number $N$ of planes contained in it as infinite, that is, $R$ is measured. Then the transmission $T_n$ is measured in a section as thin as possible ($n$ of the order $10^5$). Now both $r$ and $a$ are very small (of the order of magnitude $10^{-6}$), and $a$ is several times less than $r$. This suggests writing

$$a/r = \alpha \ , \tag{28}$$

so that $\alpha < 1$, and expanding $\rho$, $\mu$ etc. in powers of $r$. By (1) and (28) we have

$$t = 1 - (1+\alpha)r \ , \tag{29}$$

whence

$$t^{-1}=1+(1+\alpha)r+(1+\alpha)^2r^2+(1+\alpha)^3r^3+\cdots,$$
$$t^{-2}=1+2(1+\alpha)r+3(1+\alpha)^2r^2+4(1+\alpha)^3r^3+\cdots. \tag{30}$$

It is convenient to introduce the notation

$$\beta=\sqrt{\alpha(2+\alpha)}\ , \qquad (1+\alpha)^2=\beta^2+1\ , \tag{31}$$

and from (29) we find

$$\sqrt{(1+t+r)(1+t-r)(1-t+r)(1-t-r)}=2\beta r\ \sqrt{1-[(1+\alpha)r-\tfrac{1}{4}\beta^2r^2]}\ ,$$

or expanding by the binomial theorem

$$\sqrt{(1+t+r)(1+t-r)(1-t+r)(1-t-r)}$$
$$=2\beta r\left[1-\frac{1}{2}(1+\alpha)r-\frac{1}{8}r^2-\frac{1}{16}(1+\alpha)r^3-\cdots\right], \tag{32}$$

whence by (20) and (21)

$$(1-r^2)\rho=1-(1+\alpha-\beta)r-\frac{\beta}{2}(1+\alpha-\beta)r^2-\frac{1}{8}\beta r^3-\cdots, \tag{33}$$

$$R=1+\alpha-\beta+\frac{1}{2}\beta(1+\alpha-\beta)r+\frac{1}{8}\beta r^2+\cdots. \tag{34}$$

In the last equation, it is sufficient to retain the constant term, so that $R=1+\alpha-\sqrt{\alpha(2+\alpha)}$, whence, solving for $\alpha$,

$$\alpha=(1-R)^2/2R\ , \qquad \beta=1+\alpha-R=(1-R^2)/2R\ . \tag{35}$$

This determines $\alpha$ in terms of the measured $R$.[3] To find $r$ from the measured $T_n$, we expand $\mu$ in powers of $r$ by means of (23), (30) and (33):

$$\mu=1+\beta r+\tfrac{1}{2}\beta(1+\alpha+\beta)r^2+\cdots, \tag{36}$$

whence

$$\frac{\mu^2}{\mu^2-1}=\frac{1}{2\beta r}\left[1-\frac{1}{2}(1+\alpha-2\beta)r+\cdots\right]. \tag{37}$$

From (33) and (37) it follows that

$$\frac{\rho}{1-\rho}+\frac{\mu^2}{\mu^2-1}=\frac{1}{r}\left(1+\alpha+\beta+\frac{1}{2\beta}+\cdots\right), \tag{38}$$

[3] Actually, (34) gives $\alpha$ as a power series in $r$, the constant term being (35), and the next term $(1-R^2)^2/8R^2\cdot r$ which is small enough to be neglected.

where the terms not written out contain the first and higher powers of $r$. Solving (27) for $\omega_n$, we find

$$\omega_n = \frac{t^2}{2r^2T_n^2}\left[\mu^2-1+\sqrt{(\mu^2-1)^2+4\mu^2r^2T_n^2t^{-2}}\,\right],$$

where the radical is taken with the plus sign, since $\omega_n$ is positive by (24). Using (36) and (30), we find that the lowest term in $r$ inside the bracket is $(2\beta+2\sqrt{\beta^2+T_n^2})r$, so that, with the aid of (37),

$$\omega_n+\frac{\mu^2}{\mu^2-1}=\frac{1}{r}\left(\frac{\beta+\sqrt{\beta^2+T_n^2}}{T_n^2}+\frac{1}{2\beta}+\cdots\right). \tag{39}$$

Dividing (39) by (38), taking the natural logarithm on both sides and expanding the logarithm of the right hand member in powers of $r$, we find

$$\log\frac{\omega_n+\dfrac{\mu^2}{\mu^2-1}}{\dfrac{\rho}{1-\rho}+\dfrac{\mu^2}{\mu^2-1}}=\log\frac{\dfrac{\beta+\sqrt{\beta^2+T_n^2}}{T_n^2}+\dfrac{1}{2\beta}}{1+\alpha+\beta+\dfrac{1}{2\beta}}+\cdots.$$

By (25), the expression to the left equals $2n\log\mu$, and since $\log\mu=\beta r+\cdots$ by (36), we finally find, retaining only the lowest powers of $r$ on either side,

$$r=\frac{1}{2n\beta}\cdot\log\frac{\dfrac{\beta+\sqrt{\beta^2+T_n^2}}{T_n^2}+\dfrac{1}{2\beta}}{1+\alpha+\beta+\dfrac{1}{2\beta}}. \tag{40}$$

Since $\alpha$ and $\beta$ are known by (35), this determines $r$ in terms of the measured $T_n$, and finally (28) and (29) give $a=\alpha r$, $t=1-(1+\alpha)r$.