INFLUENCE OF A BRANCH LINE UPON ACOUSTIC TRANSMISSION OF A CONDUIT

BY G. W. STEWART

ABSTRACT

General theory of transmission.—The ratio of the energy of the incident acoustic energy transmitted with the branch present to that with the branch absent is found to be

$$[(Z_1^2+Z_1\rho a/2S+Z_2^2)+(\rho a Z_2/2S)^2]\times[(Z_1+\rho a/2S)^2+Z_2^2]^{-2},$$

where the point impedance of the branch is Z_1+iZ_2, ρ is the density of the fluid, a the velocity of sound and S the area of the conduit. *Application to the Helmholtz resonator and the cylindrical resonator.* The transmission ratios found are respectively,

$$\left\{1+[4S^2(k/c-1/kv)^2]^{-1}\right\}^{-1} \text{ and } \left\{1+(4S^2)^{-1}[k/c-(\sigma \tan kl)^{-1}]^{-2}\right\}^{-1}$$

where k is 2π/wave-length, c is conductivity of the orifice, and V and σ are respectively the volume and area of the branch. The theory can be applied to ascertain the transmission, when the acoustic impedance of the side branch is known or it may be used to measure the acoustic impedance of the side branch.

Theory. This is a general treatment of the influence of the impedance in a branch line upon the transmission of acoustic energy through a conduit. Assume that the transmission occurs through a conduit of constant areas S, having one side branch and that the incident plane wave, or rather the source, is undisturbed by the action of the side branch.

The symbols used are defined as follows: k is 2π/(wave-length); a, the velocity of sound; t, the time elapsed; ρ, the density of the fluid; $P=P_0e^{ikat}$, the excess pressure of the incident wave; $P'=P_0'e^{ikat}$, the actual excess pressure at the opening into the side branch; $X=X_0e^{ikat}$, the volume current (rate of volume displacement) flowing into the side branch; Z, the point impedance of the side branch, equal to $P'/\dot{X}$. The values P_0, P_0', X_0 and Z are, in general, complex.

The pressure at the junction, P', is caused by two waves, one from the source and the other from the side branch. The pressure caused by the former in the conduit is P. The pressure caused by the side branch wave can be determined very readily. Assume that the incident wave in the conduit passes from left to right. The outward volume displacement $-X$ of the wave from the branch, will, on account of symmetry be divided equally between the conduits right and left. The magnitude of the volume displacement is $X/2$ and the particle displacement is $-X/2S$ on the right and $X/2S$ on the left. The particle velocities are correspondingly $-\dot{X}/2S$ and $\dot{X}/2S$. If the diameter of the conduit is assumed small in comparison with the wave-length, the wave produced in the conduit may be considered as plane; then the pressure produced in the conduit at the right by X is $-\rho a\dot{X}/2S$. The resulting pressure therefore is

$$P' = P - \rho a\dot{X}/2S. \tag{1}$$

Hence,

$$P'/P = 1 - \rho a\dot{X}/2PS. \tag{2}$$

Since $P'/Z = \dot{X}$, then from (1) $\dot{X}/P = 1/(Z+\rho a/2S)$, and (2) becomes

$$P'/P = 1 - \rho a/[(Z+\rho a/2S)2S]. \tag{3}$$

Let $Z = Z_1 + iZ_2$ where Z_1 and Z_2 are real. Then

$$P'/P = 1 - (\rho a/2S)[(Z_1+\rho a/2S)+iZ_2]^{-1} \tag{4}$$

and

$$(P'/P)^2 = [(Z_1^2+Z_1\rho\, a/2S+Z_2^2)^2+(\rho a Z_2/2S)^2]\,[(Z_1+\rho a/2S)^2+Z_2^2]^{-2}. \tag{5}$$

This equation, giving the value of the ratio of transmitted to incident energy in the conduit, is general and is independent of the nature of the branch. In the case of a branch in which $Z_1 = 0$, the transmission is, from Eq. (5)

$$(P'/P)^2 = [1+(\rho a/2S)^2(1/Z_2)^2]^{-1}. \tag{6}$$

Helmholtz resonator. In the case of a Helmholtz resonator as a branch, the point impedance Z is $(Mi\omega - i/\omega C)$, wherein M is the inertance of the opening and C is the capacitance of the chamber or $\rho a^2/V$, V being its volume.[1]

As shown previously,[1] the inertance of a channel is its mass divided by the square of its area. If inertance is expressed as ρ/c then c is the "conductivity." From this definition, for the channel of length L and radius R, $c = \pi R^2/L$. But "end" corrections must be made to L because the inertance is not included in the length L. Theoretically this correction is proportional to R if the channel can be regarded as in an infinite plane wall. Hence the well-known form, $c = \pi R^2/(L+\alpha R)$. With this definition of c, the point impedance is $i(\rho\omega/c - \rho a^2/V)$. Hence, $Z_1 = 0$ and Z_2 is the value in the parenthesis. Substituting in Eq. (6), the fraction transmitted is therefore

$$T = \left\{1+[4S^2(k/c-1/kV)^2]^{-1}\right\}^{-1}. \tag{7}$$

A closed tube. If a is the velocity of sound, X the volume displacement of a tube with axis in the x-direction and with constant cross-section, then the following equations hold for a simple harmonic vibration, e^{ikat},

$$\partial^2 p/\partial x^2 = -k^2 p;\quad \partial^2\dot{X}/\partial x^2 = -k^2\dot{X}$$

[1] For a more detailed discussion see Stewart, Phys. Rev. **20**, 535 (1922), Eqs. (16) and (22).

The solutions of these equations with the additional relationship

$$\rho\partial^2\xi/\partial t^2 = -\partial p/\partial x,$$

wherein ξ is the displacement, gives the following expression

$$\dot{X} = (i\sigma/\rho a)\,(A_1 \cos kx - A_2 \sin kx)\, e^{ikat}$$

wherein σ is the area of the tube and A_1 and A_2 are pressure amplitudes. If now the tube is closed at $x = l$, $\dot{X}$ is there zero and the following relationship can finally be obtained,

$$p/\dot{X} = Z = -i\rho a/(\sigma \tan kl).$$

Thus, for a closed tube attached as a branch, $Z_1 = 0$ and $Z_2 = -\rho a/\sigma \tan kl$.

But no consideration has been given to the conductivity of the opening into the branch, or c, which introduces inertance as in the orifice of the Helmholtz resonator and should be added to Z_2. We then have $Z_2 = \rho\omega/c - \rho a/(\sigma \tan kl)$ and Eq. (6), or the part of the energy transmitted therefore becomes

$$T = \left\{1 + (4S^2)^{-1}\left[k/c - (\sigma \tan kl)^{-1}\right]^{-2}\right\}^{-1} \tag{8}$$

Here $c = \pi R^2/(L + \alpha R/2)$, the factor 1/2 being introduced because there is for the end of the cylinder only one-half the correction needed in the case of the orifice.

Comparison with experiment. Eqs. (7) and (8) for the transmission have been tested experimentally and found to agree satisfactorily with theory. The results will be presented in a separate paper. Experimental and theoretical results will also be given for open tubes, orifices and other branches.

Applications. The theory can be used (1) for the computation of the transmission of a conduit when the acoustic impedance of the side branch is known and (2) in the experimental determination of the acoustic impedance of a side branch. The first use is simple; the second is more complicated in practice and will be the subject of a further communication.

Physical Laboratory,
University of Iowa,
July 30, 1925.